1. $Y = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_1| \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

This operator is called the <u>lowering operator</u> because it decreases the index by 1 cyclically, i.e.,

$$Y|e_3\rangle = |e_2\rangle, \quad Y|e_2\rangle = |e_1\rangle, \quad Y|e_1\rangle = |e_3\rangle$$

$$\begin{tikzcd} |e_3\rangle \arrow[r, "Y"] & |e_2\rangle \arrow[r, "Y"] & |e_1\rangle \arrow[ll, bend left, "Y"] \end{tikzcd}$$

Since it maps an orthonormal basis to an orthonormal basis, it has to be unitary.

(a) $Y^\dagger = |e_2\rangle\langle e_1| + |e_3\rangle\langle e_2| + |e_1\rangle\langle e_3| \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$Y^\dagger$ is called the raising operator.

$$YY^\dagger = \left(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_1|\right)\left(|e_2\rangle\langle e_1| + |e_3\rangle\langle e_2| + |e_1\rangle\langle e_3|\right)$$

These three products are the only ones that survive

$$= |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|$$

$$= I$$

(b) Eigenvectors: $Y|\lambda\rangle = \lambda|\lambda\rangle$

Characteristic equation: $0 = \det(Y - \lambda I)$

$$= \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{pmatrix}$$

$$= -\lambda(\lambda^2 - 0) - 1(0 - 1)$$

$$= -\lambda^3 + 1$$

$$\Rightarrow \lambda^3 = 1$$

The eigenvalues are the 3 cube roots of unity:

$\lambda = 1,\ e^{i2\pi/3},\ e^{-i2\pi/3}$.

① $\lambda = 1$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longrightarrow \begin{matrix} b = a \\ c = b \\ a = c \end{matrix} \Rightarrow a = b = c \Longrightarrow |1\rangle = a(|e_1\rangle + |e_2\rangle + |e_3\rangle)$

Normalization $\Longrightarrow |a| = \frac{1}{\sqrt{3}}$

Choose a real and positive

$$\boxed{|1\rangle = \frac{1}{\sqrt{3}}(|e_1\rangle + |e_2\rangle + |e_3\rangle) \longleftrightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

eigenvalue $\lambda = 1$

② $\lambda = e^{2\pi i/3}$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = e^{2\pi i/3}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longrightarrow \begin{matrix} b = e^{2\pi i/3}a \\ c = e^{2\pi i/3}b \\ a = e^{2\pi i/3}c \end{matrix} \Longrightarrow \begin{matrix} b = e^{2\pi i/3}a \\ c = e^{4\pi i/3}a = e^{-2\pi i/3}a \end{matrix}$

$\Rightarrow |e^{2\pi i/3}\rangle = a(|e_1\rangle + e^{2\pi i/3}|e_2\rangle + e^{-2\pi i/3}|e_3\rangle)$

Normalization $\Longrightarrow |a| = \frac{1}{\sqrt{3}}$

Choose a real and positive

$$\boxed{|e^{2\pi i/3}\rangle = \frac{1}{\sqrt{3}}(|e_1\rangle + e^{2\pi i/3}|e_2\rangle + e^{-2\pi i/3}|e_3\rangle \longleftrightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{2\pi i/3} \\ e^{-2\pi i/3} \end{pmatrix}}$$

eigenvalue $\lambda = e^{2\pi i/3}$

③ $\lambda = e^{-2\pi i/3}$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = e^{-2\pi i/3}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Longrightarrow \begin{matrix} b = e^{-2\pi i/3}a \\ c = e^{-2\pi i/3}b \\ a = e^{-2\pi i/3}c \end{matrix} \Longrightarrow \begin{matrix} b = e^{-2\pi i/3}a \\ c = e^{-4\pi i/3}a = e^{2\pi i/3}a \end{matrix}$

$\Rightarrow |e^{-2\pi i/3}\rangle = a(|e_1\rangle + e^{-2\pi i/3}|e_2\rangle + e^{2\pi i/3}|e_3\rangle)$

Normalization $\Longrightarrow |a| = \frac{1}{\sqrt{3}}$

Choose a real and positive

$$\boxed{|e^{-2\pi i/3}\rangle = \frac{1}{\sqrt{3}}(|e_1\rangle + e^{-2\pi i/3}|e_2\rangle + e^{2\pi i/3}|e_3\rangle \longleftrightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-2\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}}$$

eigenvalue $\lambda = e^{-2\pi i/3}$

(c) $Y^2 = \left(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_1|\right)\left(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_1|\right)$

These three products are the only ones that survive

$$Y^2 = |e_1\rangle\langle e_3| + |e_2\rangle\langle e_1| + |e_3\rangle\langle e_2| = {}^{\dagger} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This is equivalent $Y^3 = I$, which is obvious.

The eigenvectors of Y^2 are the same as the eigenvectors of Y, and the eigenvalues are the squares of the eigenvalues of Y:

$$Y^2|\lambda\rangle = Y(\underbrace{Y|\lambda\rangle}_{\lambda|\lambda\rangle}) = \lambda \underbrace{Y|\lambda\rangle}_{\lambda|\lambda\rangle} = \lambda^2|\lambda\rangle .$$

Since $Y^2 = Y^\dagger$, we can also say that the eigenvectors of Y^2 are the same as the eigenvectors of Y, and the eigenvalues are the complex conjugates of the eigenvalues of Y.