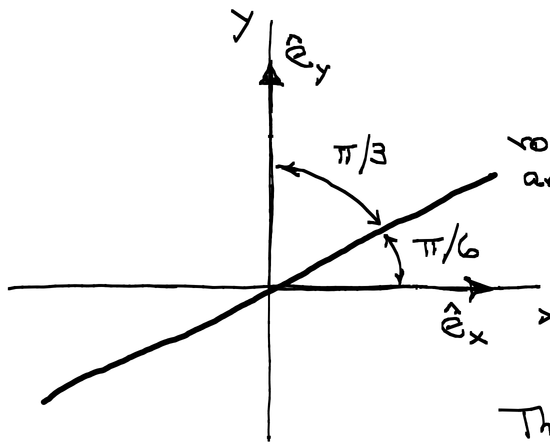
Problem 1

$$\cos\frac{\pi}{6} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$$



rod of mass M and length ℓ

$$I_{jk} = \int dm\,(r^2\delta_{jk} - x_j x_k)$$

The linear mass density of the rod is $\lambda = M/\ell$

(a) $$\int dm\, r^2 = \int \lambda\, dr\, r^2 = \lambda \int_{-\ell/2}^{\ell/2} dr\, r^2 = 2\lambda \underbrace{\int_0^{\ell/2} r^2\, dr}_{\frac{1}{3}(\ell/2)^3 = \frac{1}{24}\ell^3} = \frac{1}{12}\lambda\ell^3 = \frac{1}{12}M\ell^2$$

$$\int dm\, x^2 = \frac{3}{4}\int dr\,\lambda r^2 = \frac{1}{16}M\ell^2$$

$$x = r\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}r$$

$$\int dm\, y^2 = \frac{1}{4}\int dr\,\lambda r^2 = \frac{1}{48}M\ell^2$$

$$y = r\sin\frac{\pi}{6} = \frac{1}{2}r$$

$$\int dm\, xy = \frac{\sqrt{3}}{4}\int dr\,\lambda r^2 = \frac{\sqrt{3}}{48}M\ell^2$$

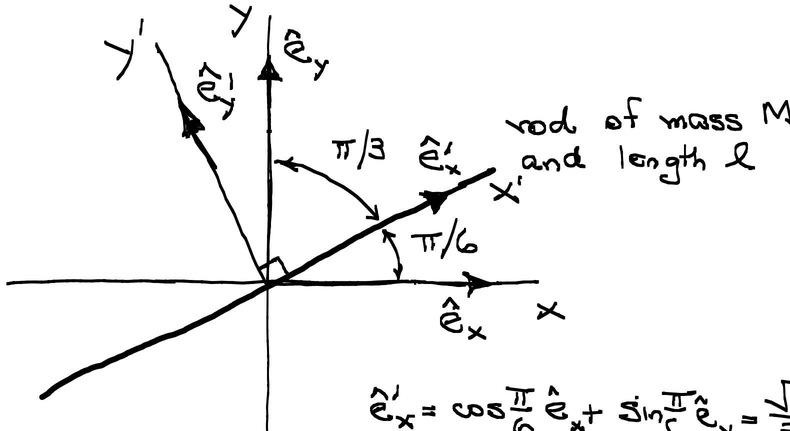
$$\int dm\, z^2 = \int dm\, xz = \int dm\, yz = 0 \longleftarrow$$ all mass is at $z=0$

$$\boxed{\begin{aligned} I_{zz} &= \int dm\, r^2 = \frac{1}{12}M\ell^2 \\ I_{xz} &= I_{yz} = 0 \\ I_{xx} &= \int dm\,(r^2 - x^2) = \frac{1}{48}M\ell^2 \\ I_{yy} &= \int dm\,(r^2 - y^2) = \frac{3}{48}M\ell^2 \\ I_{xy} &= -\int dm\, xy = -\frac{\sqrt{3}}{48}M\ell^2 \end{aligned}}$$

$$\overleftrightarrow{I} = \frac{1}{12} M\ell^2 \left(\frac{1}{4} \hat{e}_x \otimes \hat{e}_x + \frac{3}{4} \hat{e}_y \otimes \hat{e}_y - \frac{\sqrt{3}}{4} \hat{e}_x \otimes \hat{e}_y - \frac{\sqrt{3}}{4} \hat{e}_y \otimes \hat{e}_x + \hat{e}_z \otimes \hat{e}_z \right)$$

$$\longleftrightarrow \frac{1}{12} M\ell^2 \begin{pmatrix} 1/4 & -\sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)



$$\hat{e}'_x = \cos\frac{\pi}{6}\,\hat{e}_x + \sin\frac{\pi}{6}\,\hat{e}_y = \frac{\sqrt{3}}{2}\,\hat{e}_x + \frac{1}{2}\,\hat{e}_y \longleftrightarrow \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\hat{e}'_y = -\sin\frac{\pi}{6}\,\hat{e}_x + \cos\frac{\pi}{6}\,\hat{e}_y = -\frac{1}{2}\,\hat{e}_x + \frac{\sqrt{3}}{2}\,\hat{e}_y \longleftrightarrow \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$$

The inertia tensor is already diagonal in the z direction, so $\hat{e}_z = \hat{e}'_z$ is an eigenvector with eigenvalue $\frac{1}{12} M\ell^2$. The mass distribution is symmetric under reflection through the z'-y' plane; this reflection must take an eigenvector to another eigenvector with the same eigenvalue.
This implies that $\hat{e}'_x$ is an eigenvector, and it is obvious that its eigenvalue is 0, since all the mass lies on the x' axis. It also implies that the other two eigenvectors lie in the z'-y' plane, and since one such eigenvector is $\hat{e}_z$, the third is $\hat{e}'_y$. Actually, since the mass distribution is symmetric under rotations about x', any such rotation must take an eigenvector to another eigenvector with the same eigenvalue, which is $\frac{1}{12} M\ell^2$.
We can choose the two orthogonal eigenvectors in this plane to be $\hat{e}'_y$ and $\hat{e}_z$, both with eigenvalue $\frac{1}{12} M\ell^2$ (this is called a two-fold degeneracy and allows us to

conclude that any linear combination of $\hat{e}_y'$ and $\hat{e}_z$ is an eigenvector with eigenvalue $\frac{1}{12}M\ell^2$). Of course, one can see the degeneracy from the fact spinning the rod about any axis in the z'-y' plane is the same.

Eigenvectors (principal axes)	Eigenvalues (principal moments)
$\hat{e}_x' = \frac{\sqrt{3}}{2}\hat{e}_x + \frac{1}{2}\hat{e}_y$	0
$\hat{e}_y' = -\frac{1}{2}\hat{e}_x + \frac{\sqrt{3}}{2}\hat{e}_y$	$\frac{1}{12}M\ell^2$
$\hat{e}_z' = \hat{e}_z$	$\frac{1}{12}M\ell^2$

Trust, but verify:

① $$\overleftrightarrow{I}\cdot\hat{e}_x' = \frac{\sqrt{3}}{2}\underbrace{\overleftrightarrow{I}\cdot\hat{e}_x}_{\frac{1}{12}M\ell^2\left(\frac{1}{4}\hat{e}_x - \frac{\sqrt{3}}{4}\hat{e}_y\right)} + \frac{1}{2}\underbrace{\overleftrightarrow{I}\cdot\hat{e}_y}_{\frac{3}{4}\hat{e}_y - \frac{\sqrt{3}}{4}\hat{e}_x} = \frac{1}{12}M\ell^2\left[\hat{e}_x\left(\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}\right) + \hat{e}_y\left(-\frac{3}{8}+\frac{3}{8}\right)\right] = 0$$

$$\longleftrightarrow \frac{1}{12}M\ell^2\begin{pmatrix} 1/4 & -\sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 \\ 1/2 \\ 0\end{pmatrix} = \frac{1}{12}M\ell^2\begin{pmatrix}\sqrt{3}/8 - \sqrt{3}/8 \\ -3/8 + 3/8 \\ 0\end{pmatrix} = 0$$

② $$\overleftrightarrow{I}\cdot\hat{e}_y' = -\frac{1}{2}\overleftrightarrow{I}\cdot\hat{e}_x + \frac{\sqrt{3}}{2}\overleftrightarrow{I}\cdot\hat{e}_y = \frac{1}{12}M\ell^2\underbrace{\left[\hat{e}_x\underbrace{\left(-\frac{1}{8}-\frac{3}{8}\right)}_{-1/2} + \hat{e}_y\underbrace{\left(\frac{\sqrt{3}}{8}+\frac{3\sqrt{3}}{8}\right)}_{\sqrt{3}/2}\right]}_{\hat{e}_y'} = \frac{1}{12}M\ell^2\hat{e}_y'$$

$$\longleftrightarrow \frac{1}{12}M\ell^2\begin{pmatrix} 1/4 & -\sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1/2 \\ \sqrt{3}/2 \\ 0\end{pmatrix} = \frac{1}{12}M\ell^2\begin{pmatrix}-1/8 - 3/8 \\ \sqrt{3}/8 + 3\sqrt{3}/8 \\ 0\end{pmatrix} = \frac{1}{12}M\ell^2\begin{pmatrix}-1/2 \\ \sqrt{3}/2 \\ 0\end{pmatrix}$$

③ $$\overleftrightarrow{I}\cdot\hat{e}_z = \frac{1}{12}M\ell^2\,\hat{e}_z$$

(c)

$$\hat{e}_x' = \frac{\sqrt{3}}{2}\hat{e}_x + \frac{1}{2}\hat{e}_y \longleftrightarrow \begin{pmatrix}\sqrt{3}/2 \\ 1/2 \\ 0\end{pmatrix} \qquad \hat{e}_x = \frac{\sqrt{3}}{2}\hat{e}_x' - \frac{1}{2}\hat{e}_y'$$

$$\hat{e}_y' = -\frac{1}{2}\hat{e}_x + \frac{\sqrt{3}}{2}\hat{e}_y \longleftrightarrow \begin{pmatrix}-1/2 \\ \sqrt{3}/2 \\ 0\end{pmatrix} \qquad \hat{e}_y = \frac{1}{2}\hat{e}_x' + \frac{\sqrt{3}}{2}\hat{e}_y'$$

The orthogonal matrix that transforms between the two basis is the matrix of inner products $M_{jk} = \hat{e}'_j \cdot \hat{e}_k$:

$$M = \| M_{jk} \| = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Method 1:

$$\overleftrightarrow{I} = \frac{1}{12} M\ell^2 \left(\frac{1}{4} \hat{e}_x \otimes \hat{e}_x + \frac{3}{4} \hat{e}_y \otimes \hat{e}_y - \frac{\sqrt{3}}{4} (\hat{e}_x \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_x) + \hat{e}_z \otimes \hat{e}_z \right)$$

$$\hat{e}_x \otimes \hat{e}_x = \left(\frac{\sqrt{3}}{2} \hat{e}'_x - \frac{1}{2} \hat{e}'_y \right) \otimes \left(\frac{\sqrt{3}}{2} \hat{e}'_x - \frac{1}{2} \hat{e}'_y \right) = \frac{3}{4} \hat{e}'_x \otimes \hat{e}'_x + \frac{1}{4} \hat{e}'_y \otimes \hat{e}'_y - \frac{\sqrt{3}}{4} (\hat{e}'_x \otimes \hat{e}'_y + \hat{e}'_y \otimes \hat{e}'_x)$$

$$\hat{e}_y \otimes \hat{e}_y = \left(\frac{1}{2} \hat{e}'_x + \frac{\sqrt{3}}{2} \hat{e}'_y \right) \otimes \left(\frac{1}{2} \hat{e}'_x + \frac{\sqrt{3}}{2} \hat{e}'_y \right) = \frac{1}{4} \hat{e}'_x \otimes \hat{e}'_x + \frac{3}{4} \hat{e}'_y \otimes \hat{e}'_y + \frac{\sqrt{3}}{4} (\hat{e}'_x \otimes \hat{e}'_y + \hat{e}'_y \otimes \hat{e}'_x)$$

$$\hat{e}_x \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_x = \left(\frac{\sqrt{3}}{2} \hat{e}'_x - \frac{1}{2} \hat{e}'_y \right) \otimes \left(\frac{1}{2} \hat{e}'_x + \frac{\sqrt{3}}{2} \hat{e}'_y \right) + \left(\frac{1}{2} \hat{e}'_x + \frac{\sqrt{3}}{2} \hat{e}'_y \right) \otimes \left(\frac{\sqrt{3}}{2} \hat{e}'_x - \frac{1}{2} \hat{e}'_y \right)$$

$$= \frac{\sqrt{3}}{2} \hat{e}'_x \otimes \hat{e}'_x - \frac{\sqrt{3}}{2} \hat{e}'_y \otimes \hat{e}'_y + \frac{1}{2} (\hat{e}'_x \otimes \hat{e}'_y + \hat{e}'_y \otimes \hat{e}'_x)$$

$$\overleftrightarrow{I} = \frac{1}{12} M\ell^2 \Big[\hat{e}'_x \otimes \hat{e}'_x \underbrace{\left(\frac{3}{16} + \frac{3}{16} - \frac{3}{8} \right)}_{0} + \hat{e}'_y \otimes \hat{e}'_y \underbrace{\left(\frac{1}{16} + \frac{9}{16} + \frac{3}{8} \right)}_{1} + (\hat{e}'_x \otimes \hat{e}'_y + \hat{e}'_y \otimes \hat{e}'_x) \underbrace{\left(-\frac{\sqrt{3}}{16} + \frac{3\sqrt{3}}{16} - \frac{\sqrt{3}}{8} \right)}_{0}$$
$$+ \hat{e}'_z \otimes \hat{e}'_z \Big]$$

$$\overleftrightarrow{I} = \frac{1}{12} M\ell^2 \left(\hat{e}'_y \otimes \hat{e}'_y + \hat{e}'_z \otimes \hat{e}'_z \right)$$

Method 2: $I'_{jk} = \hat{e}'_j \cdot \overleftrightarrow{I} \cdot \hat{e}'_k$

$$= \sum_{\ell, m} \underbrace{\hat{e}'_j \cdot \hat{e}_\ell}_{M_{j\ell}} I_{\ell m} \underbrace{\hat{e}_m \cdot \hat{e}'_k}_{M_{km} = M^T_{mk}}$$

$$= \sum_{\ell, m} M_{j\ell} I_{\ell m} M^T_{mk}$$

$$\begin{pmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{pmatrix} = \frac{1}{12} M\ell^2 \underbrace{\overbrace{\begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{M} \underbrace{\begin{pmatrix} 1/4 & -\sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{M^T}}_{\begin{pmatrix} 0 & -1/2 & 0 \\ 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}_{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{pmatrix} = \frac{1}{12} M\ell^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$