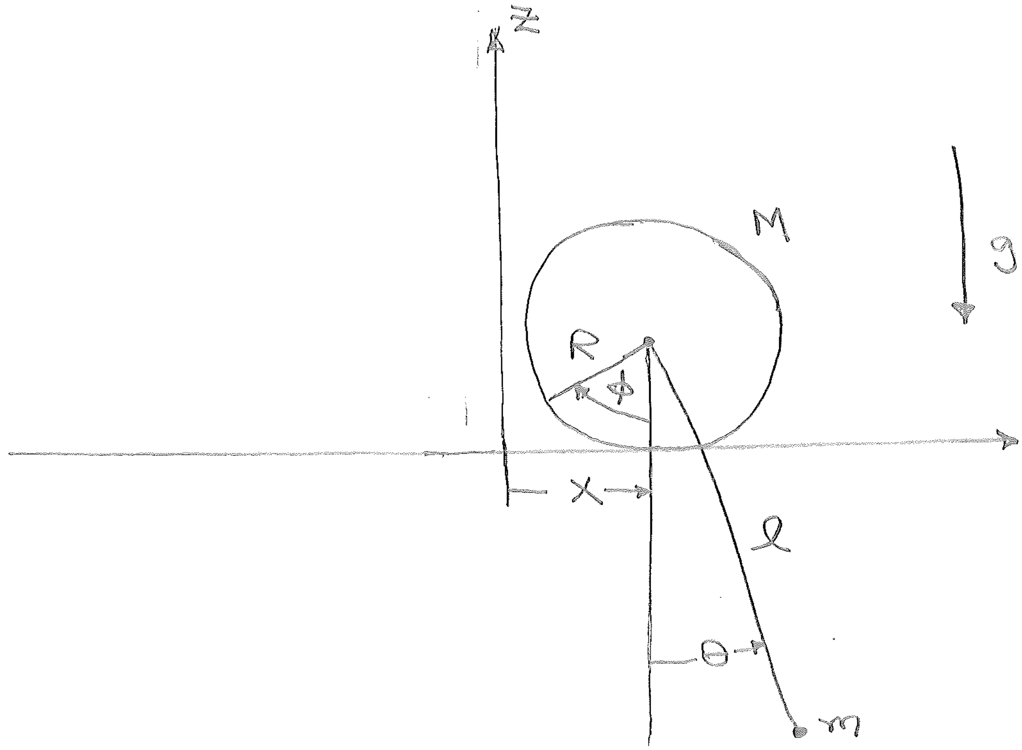
Phys 503

Midterm Exam #1

Solutions

1.



(a) Generalized coördinates: X, θ

X is the displacement of the CM of cylinder. The rolling constraint implies that $X = R\phi$, where ϕ is the angle through which the cylinder has rolled.

θ is angle of the pendulum relative to vertical. The Cartesian (inertial) coördinates of m are

$$x = X + l\sin\theta \qquad \dot{x} = \dot{X} + l\dot{\theta}\cos\theta$$
$$z = R - l\cos\theta \quad \Longrightarrow \quad \dot{z} = l\dot{\theta}\sin\theta$$

(b) Lagrangian:

$$T = T_{M,\,trans} + T_{M,\,rot} + T_m$$

$$T_{M,\,trans} = \frac{1}{2} M \dot{X}^2$$

$$T_{M,\,rot} = \frac{1}{2} M R^2 \dot{\phi}^2 = \frac{1}{2} M \dot{X}^2$$

$$T_m = \frac{1}{2} m \left(\dot{x}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2} m \left(\dot{X}^2 + 2l\dot{X}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right)$$

$$= \frac{1}{2} m \left(\dot{X}^2 + 2l\dot{X}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)$$

$$\boxed{T = \frac{1}{2}(2M + m)\dot{X}^2 + ml\dot{X}\dot{\theta}\cos\theta + \frac{1}{2} m l^2 \dot{\theta}^2}$$

$$\boxed{V = mgz + (\text{const}) = -mgl\cos\theta}$$

$$\boxed{\begin{aligned} L &= T - V \\ &= \frac{1}{2}(2M + m)\dot{X}^2 + ml\dot{X}\dot{\theta}\cos\theta + \frac{1}{2} m l^2 \dot{\theta}^2 \\ &\quad + mgl\cos\theta \end{aligned}}$$

(c) Conserved quantities:

(1) X is cyclic $\Rightarrow$ $p_X = \frac{\partial L}{\partial \dot{X}}$ is conserved

$$\boxed{p_X = \frac{\partial L}{\partial \dot{X}} = (2M+m)\dot{X} + m\ell\dot{\theta}\cos\theta}$$

$$= M\dot{X} + \underbrace{M\dot{X} + m\dot{x}}_{\text{x-component of linear momentum}}$$

(2) $E = h =$ Jacobi integral

$$\boxed{E = T + V = \frac{1}{2}(2M+m)\dot{X}^2 + m\ell\dot{X}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta}$$

2. Arbitrary Kepler orbit: $\frac{1}{r} = \frac{mk}{\ell^2}(1 + e\cos\theta)$

$$e = \sqrt{1 + \frac{2\ell^2 E}{mk^2}}$$

(a) Initial orbit is parabolic: $E = 0 \Leftrightarrow e = 1$

$$\frac{1}{r} = \frac{mk}{\ell^2}(1 + \cos\theta)$$

Impulse at periastron ($\theta = 0$), where $r = \frac{\ell^2}{2mk}$

Final circular orbit: $e = 0$

$$r = \frac{\ell'^2}{mk} = \frac{\ell^2}{2mk} \Longrightarrow \boxed{\ell' = \ell/\sqrt{2}}$$

$$E' = -\frac{mk^2}{2\ell'^2} \Longrightarrow \boxed{E' = -\frac{mk^2}{\ell^2}}$$

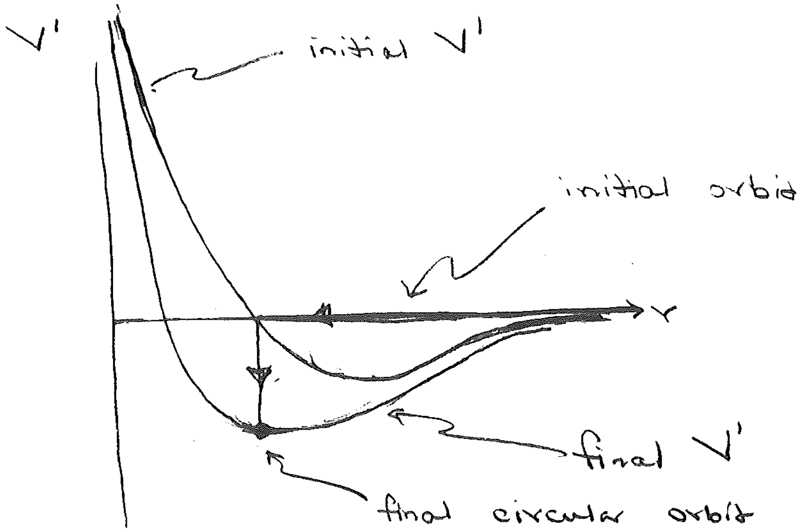
$$\left(E' = -\frac{k}{2r} = -\frac{mk^2}{2\ell'^2}\right)$$

Impulse: Since $\dot{r} = 0$ at periastron, there is no radial impulse. The required impulse is purely angular, and it is in the $-\theta$ direction to remove the excess angular momentum.

$$S_\theta = \int F_\theta \, dt = \Delta P_\theta = mr\Delta\dot{\theta} = \frac{\Delta \ell}{r}$$

$$\therefore S_\theta = \frac{2mk}{\ell^2}(\ell' - \ell) = \frac{2mk}{\ell} \underbrace{\frac{\ell' - \ell}{\ell}}_{\left(\frac{1}{\sqrt{2}} - 1\right)} = \boxed{\frac{mk}{\ell}(\sqrt{2} - 2) = S_\theta}$$

negative as required



(b) Initial orbit is arbitrary elliptical:

$$\frac{1}{r} = \frac{mk}{l^2}(1 + e\cos\theta) \quad , \quad e < 1$$

Impulse at $\theta = \pi/2$, where $r = \frac{l^2}{mk}$

Final circular orbit: $e = 0$

$$r = \frac{l'^2}{mk} = \frac{l^2}{mk} \Longrightarrow \boxed{l' = l, \quad \Delta l = 0}$$

$$E' = -\frac{mk^2}{2l'^2} = \boxed{-\frac{mk^2}{2l^2} = E'}$$

Impulse: Since $\Delta l = 0$, there is no angular impulse. The impulse is purely radial, and it is in the $-r$ direction to remove the radial kinetic energy.

$$S_r = \int F_r \, dt = \Delta p_r$$

Initial orbit: $E = \frac{p_r^2}{2m} - \frac{k}{r} + \frac{l^2}{2mr^2}$

Final orbit: $E' = -\frac{k}{r} + \frac{l^2}{2mr^2}$

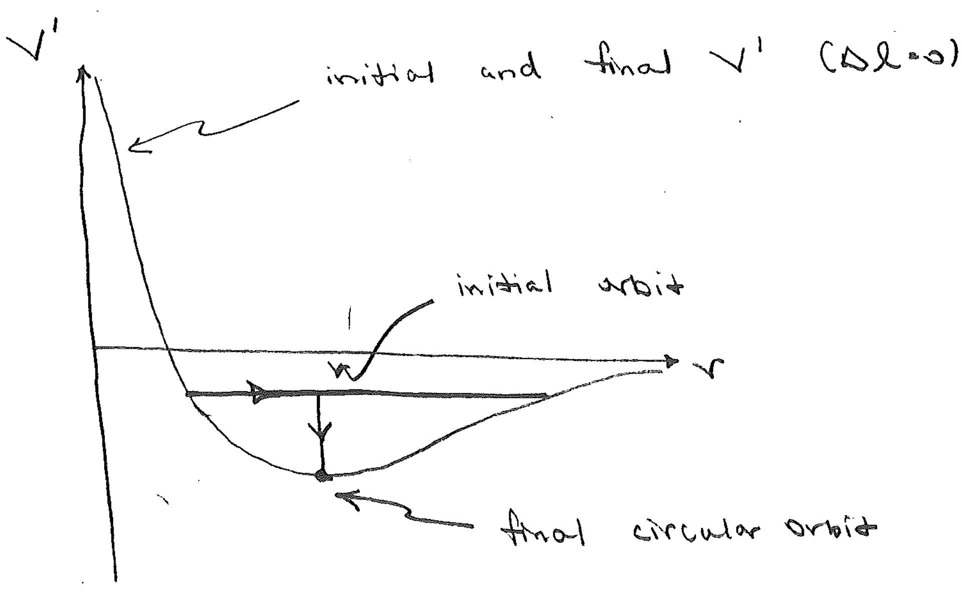
$$\Rightarrow \quad \Delta E = E' - E = -\frac{p_r^2}{2m} = -\frac{(\Delta p_r)^2}{2m} = -\frac{S_r^2}{2m}$$

negative impulse

$$S_r^2 = -2m\,\Delta E \quad \Rightarrow \quad S_r = -\sqrt{\underbrace{2m(E-E')}_{2mE + \frac{m^2k^2}{l^2}}}$$

$$\boxed{S_r = -\frac{mk}{l}\sqrt{1 + \frac{2l^2E}{mk^2}} = -\frac{mk}{l}\,e}$$

This actually works for all e, not just e < 1.



$$E' = -\frac{mk^2}{2l^2}$$

$$E = -\frac{mk^2}{2l^2}(1-e^2)$$

$$\Rightarrow E - E' = \frac{mk^2}{2l^2}e^2 \Rightarrow S_r = -\frac{mk}{l}\,e$$

Alternative method to get Δp_r:

$$S_r = \Delta p_r = m \, \Delta \dot{r} = -p_r = -\sqrt{2m}\,\underbrace{\left[E + \frac{k}{r} - \frac{l^2}{2mr^2}\right]}^{1/2}$$

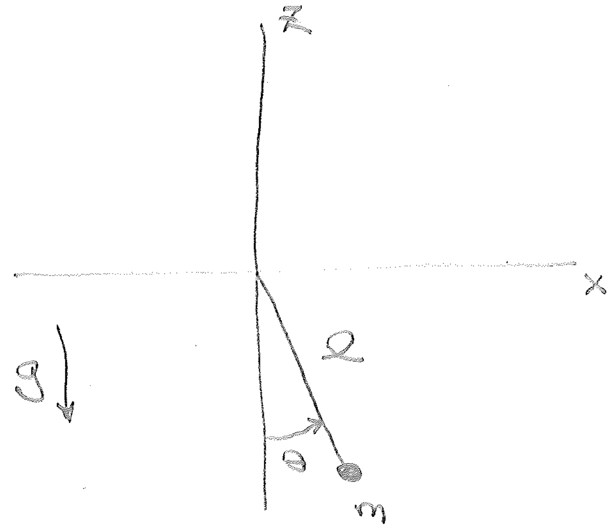
$$E + \frac{mk^2}{l^2} - \frac{l^2 m^2 k^2}{2\cancel{m}\, l^{4}}$$

$$= E + \frac{mk^2}{2l^2}$$

$$= \frac{mk^2}{2l^2}\left(1 + \frac{2l^2E}{mk^2}\right)$$

$$\boxed{S_r = -\frac{mk}{l}\sqrt{1+\frac{2l^2E}{mk^2}} = -\frac{mk}{l}\,e}$$

Problem 3



$x = r\sin\theta$

$z = -r\cos\theta$

$V = mgz = -mgr\cos\theta$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2)$

$\quad = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

(a)
$$\boxed{\begin{aligned} L' &= T - V - \lambda(r - l) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta - \lambda(r - l) \end{aligned}}$$

$$\frac{\partial L'}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\Longrightarrow m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - \lambda$$

$$\frac{\partial L'}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - \lambda$$

$$\frac{\partial L'}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{\partial L'}{\partial \theta} = -mgr\sin\theta \qquad \Longrightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta$$

Apply constraint: $r = l$

$$0 = \overbrace{ml\dot{\theta}^2}^{\text{centrifugal force}} + \underbrace{mg\cos\theta}_{\text{gravity}} - \lambda \qquad \lambda = J = \text{tension}$$

J is positive inward

$$ml\ddot{\theta} = -mg\sin\theta$$

(b) The impulse gives the mass a momentum S and a kinetic energy $S^2/2m$. The conserved energy is

$$E = T + V = \underbrace{\frac{S^2}{2m} - mgl}_{\text{initial value}} = \underbrace{\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta}_{\text{value at any time}}$$

If the string were a rigid rod, it would get to (...) a maximal angle θ_m ($\dot{\theta}=0$) given by

$$mgl\cos\theta_m = mgl - \frac{S^2}{2m}$$

$$\boxed{\cos\theta_m = 1 - \frac{S^2}{2m^2gl}}$$

The tension $\mathcal{T} = \lambda$ is given by

$$\mathcal{T} = ml\dot{\theta}^2 + mg\cos\theta$$

$$\frac{1}{2}l\mathcal{T} = \underbrace{\frac{1}{2}ml^2\dot{\theta}^2}_{\frac{S^2}{2m} + mgl(\cos\theta - 1)} + \frac{1}{2}mgl\cos\theta$$

$$= \frac{S^2}{2m} - mgl + \frac{3}{2}mgl\cos\theta$$

Writing this in dimensionless form, we have

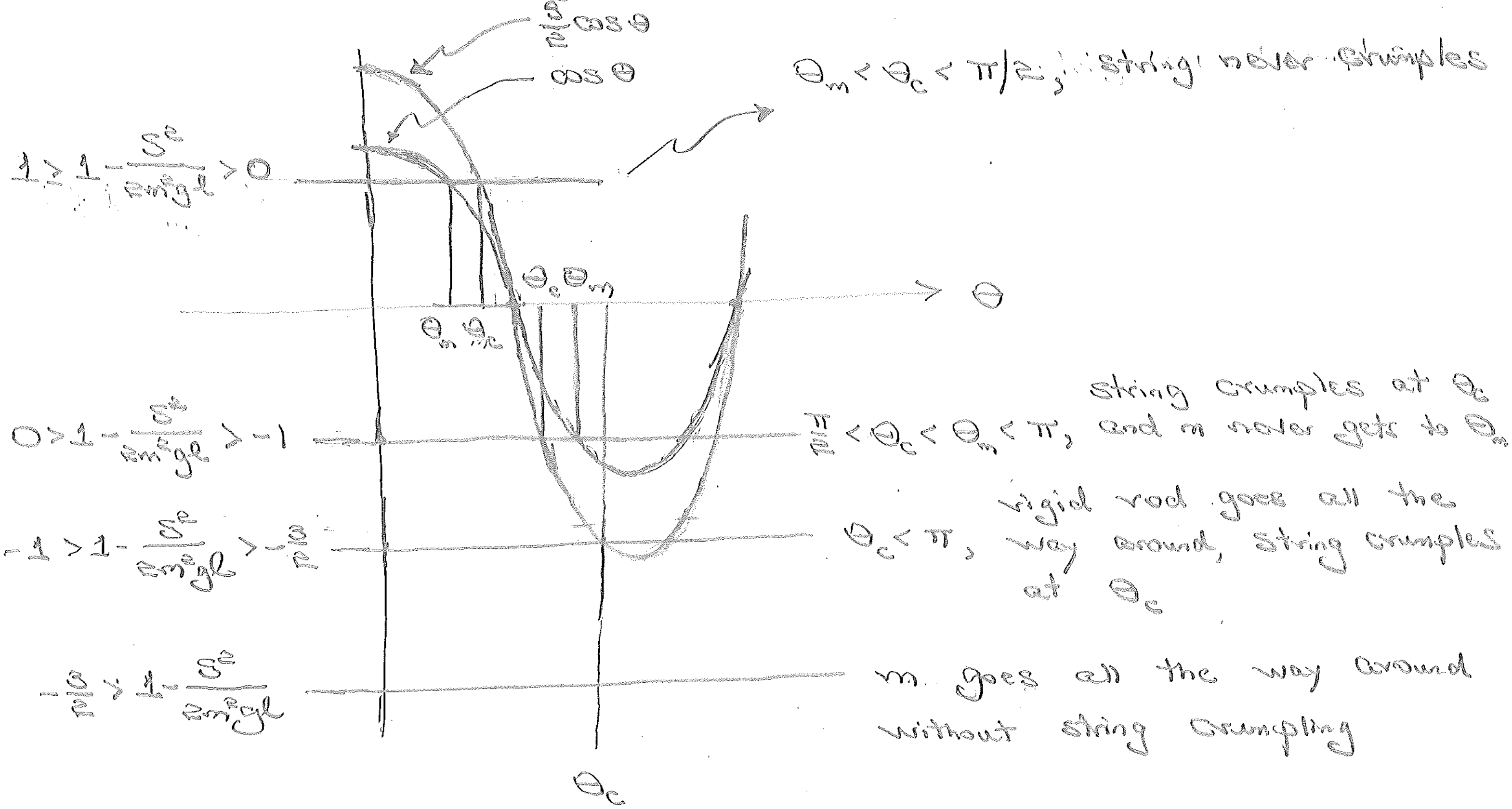
$$\frac{\mathcal{T}}{2mg} = \frac{S^2}{2m^2gl} - 1 + \frac{3}{2}\cos\theta$$

The string can only pull inward ($\mathcal{T} > 0$), not push outward ($\mathcal{T} > 0$), so the string crumples when $J=0$, i.e., at angle θ_c given by

$$0 = \frac{S^2}{2m^2gl} - 1 + \frac{3}{2}\cos\theta_c$$

$$\Longrightarrow \boxed{\frac{3}{2}\cos\theta_c = 1 - \frac{S^2}{2m^2gl} \equiv \cos\theta_m}$$

The whole thing can be captured in a diagram:



<u>Summary:</u> String crumples at angle θ_c given by

$$\cos\theta_c = \frac{2}{3}\left(1 - \frac{S^2}{2m^2gl}\right), \quad \frac{\pi}{2} < \theta_c < \pi$$

when

$$1 < \frac{S^2}{2m^2gl} < \frac{5}{2}.$$