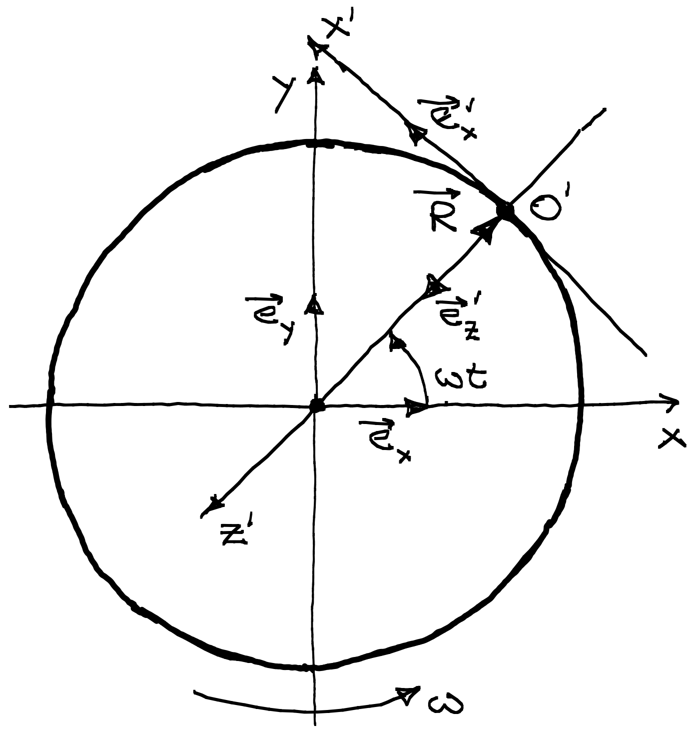
Phys 503

Midterm # 2

Solution Set

1



$\vec{e}_z$ out of page

$\vec{e}_y'$ into page

We will need

$$\vec{e}_x' = -\vec{e}_x \sin\omega t + \vec{e}_y \cos\omega t$$
$$\vec{e}_y' = -\vec{e}_z$$
$$\vec{e}_z' = -\vec{e}_x \cos\omega t - \vec{e}_y \sin\omega t$$

and
$$x' = -x\sin\omega t + y\cos\omega t$$
$$y' = -z$$
$$z' = R - x\cos\omega t - y\sin\omega t$$

$$\vec{e}_j' = A_{jk}\vec{e}_k \qquad A = \begin{pmatrix} -s & c & 0 \\ 0 & 0 & -1 \\ -c & -s & 0 \end{pmatrix}$$

$$x_k\vec{e}_k = \vec{R} + x_j'\vec{e}_j' = (x_j' - R\delta_{jz})\vec{e}_{j'} = (x_j' - R\delta_{jz})A_{jk}\vec{e}_k$$

$$\Rightarrow x_k = (x_j' - R\delta_{jz})A_{jk}$$

or $(x \;\; y \;\; z) = (x' \;\; y' \;\; z'-R)\,A$

or $(x' \;\; y' \;\; z'-R) = (x \;\; y \;\; z)\,A^T$

$$\begin{pmatrix} x' \\ y' \\ z'-R \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a)
$$\boxed{\begin{aligned} \vec{\omega} &= \omega\vec{e}_z = -\omega\vec{e}_y' \\ \vec{R} &= -R\vec{e}_z' = R(\cos\omega t\,\vec{e}_x + \sin\omega t\,\vec{e}_y) \\ \dot{\vec{R}} &= \omega R(-\sin\omega t\,\vec{e}_x + \cos\omega t\,\vec{e}_y) = \vec{\omega}\times\vec{R} = \omega R\vec{e}_x' \\ \ddot{\vec{R}} &= \omega^2 R(-\cos\omega t\,\vec{e}_x - \sin\omega t\,\vec{e}_y) = \vec{\omega}\times(\vec{\omega}\times\vec{R}) = \omega^2 R\vec{e}_z' \end{aligned}}$$

(b) Method 1: Use the equations for a rotating frame

$$\vec{a}_r = \underbrace{-\vec{\omega}\times(\vec{\omega}\times\vec{R})}_{-\omega^2 R\,\vec{e}_z^{\,\prime}} \underbrace{-\,\vec{\omega}\times(\vec{\omega}\times\vec{r})} \underbrace{-\,2\vec{\omega}\times\vec{v}_r}$$

main centrifugal acceleration

$$2\omega\vec{e}_{y'}\times\vec{v}_r = -2\omega\dot{x}'\vec{e}_{z'} + 2\omega\dot{z}'\vec{e}_{x'}$$

Coriolis force

$$-\omega^2\vec{e}_y^{\,\prime}\times(\vec{e}_y^{\,\prime}\times\vec{r}) = -\omega^2\vec{e}_y^{\,\prime}\times(-x'\vec{e}_z^{\,\prime} + z'\vec{e}_{x'})$$

$$= \omega^2 x'\vec{e}_x^{\,\prime} + \omega^2 z'\vec{e}_z^{\,\prime}$$

additional centrifugal acceleration when not at O'

$$\boxed{\begin{aligned}\ddot{x}' &= \omega^2 x' + 2\omega\dot{z}' \\ \ddot{z}' &= -\omega^2 R + \omega^2 z' - 2\omega\dot{x}' \\ \ddot{y}' &= 0\end{aligned}}$$

Method 2. Write the primed coördinates in terms of the inertial coördinates.

$$x' = -x\sin\omega t + y\cos\omega t$$

$$y' = -z$$

$$z' = R - x\cos\omega t - y\sin\omega t$$

The inertial coördinates satisfy

$$\ddot{x} = \ddot{y} = \ddot{z} = 0,$$

so

$$\dot{x}' = -\dot{x}s + \dot{y}c - \omega\overbrace{(xc+ys)}^{R-z'}$$

$$\ddot{x}' = -2\omega\underbrace{(\dot{x}c+\dot{y}s)}_{-\omega x' - \dot{z}'} + \omega^2\underbrace{(xs-yc)}_{-x'}$$

$$\dot{z}' = -\dot{x}c - \dot{y}s + \omega\overbrace{(xs-yc)}^{-x'}$$

$$\ddot{z}' = 2\omega\underbrace{(\dot{x}s-\dot{y}c)}_{-\omega(R-z')-\dot{x}'} + \omega^2\underbrace{(xc+ys)}_{R-z'}$$

$$\Longrightarrow \boxed{\begin{aligned}\ddot{x}' &= \omega^2 x' + 2\omega\dot{z}' \\ \ddot{z}' &= -\omega^2(R-z') - 2\omega\dot{x}' \\ \ddot{y}' &= 0\end{aligned}}$$

(c) Initial conditions: $x'(0)=0$, $\dot{x}'(0)=0$, $z'(0)=h$, $\dot{z}'(0)=0$

Method 1. One neglects the tiny centrifugal corrections $\omega^2 x'$ and $\omega^2 z'$. In addition, one neglects the Coriolis term in the z' equation. So

$$\ddot{z}' = -\omega^2 R \implies \dot{z}' = -\omega^2 R t \quad \text{and} \quad z' = h - \tfrac{1}{2}\omega^2 R t^2$$

fall under effective gravity $g = \omega^2 R$

$$\ddot{x}' = 2\omega \dot{z}' = -2\omega^3 R t \implies x' = -\tfrac{1}{3} R(\omega t)^3$$

$$\boxed{\begin{aligned} z'(t) &= h - \tfrac{1}{2} R(\omega t)^2 \\ x' &= -\tfrac{1}{3} R(\omega t)^3 \end{aligned}}$$

Method 2. One can easily find the exact solution in the inertial coördinates.

Initial conditions: $x(0) = R-h$, $\dot{x}(0)=0$, $y(0)=0$, $\dot{y}(0) = \omega(R-h)$

$$\implies x(t) = R-h \quad \text{and} \quad y(t) = \omega t(R-h)$$

Exact: $$x'(t) = (R-h)\underbrace{\left(-\sin\omega t + \omega t\cos\omega t\right)}_{\omega t \ll 1:\ -\omega t + \frac{1}{6}(\omega t)^3 + \omega t - \frac{1}{2}(\omega t)^3 = -\frac{1}{3}(\omega t)^3}$$

Exact: $$z'(t) = R - (R-h)\underbrace{\left(\cos\omega t + \omega t \sin\omega t\right)}_{\omega t \ll 1:\ 1 - \frac{1}{2}(\omega t)^2 + (\omega t)^2 = 1 + \frac{1}{2}(\omega t)^2}$$

$\omega t \ll 1$: $$z' = h - \tfrac{1}{2}(R-h)(\omega t)^2 = h - \tfrac{1}{2}R(\omega t)^2$$

neglect

$$x' = -\tfrac{1}{3}(R-h)(\omega t)^3 = -\tfrac{1}{3}R(\omega t)^3$$

$$\boxed{\begin{aligned} z'(t) &= h - \tfrac{1}{2} R(\omega t)^2 \\ x' &= -\tfrac{1}{3} R(\omega t)^3 \end{aligned}}$$

The particle hits the ground at $t = \sqrt{2h/g}$, $g = \omega^2 R$, when

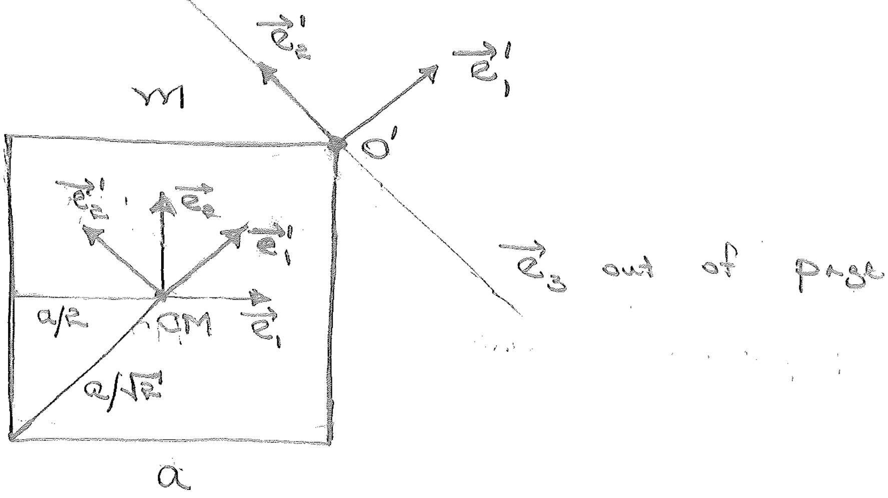
$$x' = -\frac{1}{3} R\omega^3 t^3 = -\frac{1}{3} g \sqrt{\frac{g}{R}}\, \frac{2h}{g} \sqrt{\frac{2h}{g}} = -\frac{2\sqrt{2}}{3} \sqrt{\frac{h}{R}}\, h,$$

so

$$\frac{x'}{h} = -\frac{2\sqrt{2}}{3} \underbrace{\sqrt{\frac{h}{R}}}_{\simeq \frac{\sqrt{2}}{100}} \simeq -\frac{4}{3} \times 0.01 \implies \boxed{\text{$|x'|$ would be about 2 cm, so you could see this in a careful measurement.}}$$

$h \simeq 2\,\text{m}$
$R = 10^4\,\text{m}$

2.



Surface density $\sigma = m/a^2$

(a) Inertia tensor about CM. $\overleftrightarrow{I}_{CM} = \int \sigma\, dA \left(r^2 \overleftrightarrow{1} - \vec{r}\otimes\vec{r}\right)$

Principal axes: $\boxed{\vec{e}_1, \vec{e}_2, \vec{e}_3 \text{ shown}}$

$$I_{13} = -\int \sigma\, dA\, xz = 0$$

$$I_{23} = -\int \sigma\, dA\, yz = 0$$

because square is thin ($z=0$)

$$I_{12} = -\int \sigma\, dA\, xy = 0$$

because of reflection symmetry $x \to -x$ or $y \to -y$.

$$I_{11} = \int \sigma\, dA (y^2 + z^2) = \int \sigma\, dA\, y^2 =$$

$$= \frac{m}{a^2} \underbrace{\int_{-a/2}^{+a/2} dx}_{a} \underbrace{\int_{-a/2}^{+a/2} y^2\, dy}_{2\frac{1}{3}\left(\frac{a}{2}\right)^3 = \frac{1}{12}a^3}$$

$$\boxed{I_{11} = \frac{1}{12} m a^2}$$

$$I_{22} = \int \sigma\, dA\,(x^2+z^2) = \int \sigma\, dA\; x^2 = I_{11}$$

$$\boxed{I_{22} = \frac{1}{12} m a^2}$$

$$I_{33} = \int \sigma\, dA\,(x^2+y^2) = I_{11} + I_{22} = \boxed{\frac{1}{6} m a^2 = I_{33}}$$

Since $I_{11} = I_{22}$, any pair of orthogonal vectors in the plane of the square are principal axes.

In particular, $\vec{e}_1{}'$ and $\vec{e}_2{}'$ are principal axes with $I'_{11} = I'_{22} = \frac{1}{12} m a^2$.

(b) Inertia tensor about O': $\overleftrightarrow{I}_{O'} = \int \sigma\, dA \left(r'^2 \overleftrightarrow{1} - \vec{r}\,' \otimes \vec{r}\,' \right)$

Principal axes: $\boxed{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3 \text{ shown}}$

These axes are <u>unique</u> because the 3 principal moments of inertia are different

Use $\overleftrightarrow{I}_{O'} = \overleftrightarrow{I}_{CM} + m\left(R^2 \overleftrightarrow{1} - \vec{R} \otimes \vec{R} \right)$

$\vec{R} = -\frac{a}{\sqrt{2}}\, \vec{e}_1{}'$ is the vector from O' to CM.

(R)

$$\therefore\; \overleftrightarrow{I}_{O'} = \overleftrightarrow{I}_{CM} + \frac{1}{2} m a^2 \underbrace{\left(\overleftrightarrow{1} - \vec{e}_1{}' \otimes \vec{e}_1{}' \right)}_{\vec{e}_2{}' \otimes \vec{e}_2{}' + \vec{e}_3 \otimes \vec{e}_3}$$

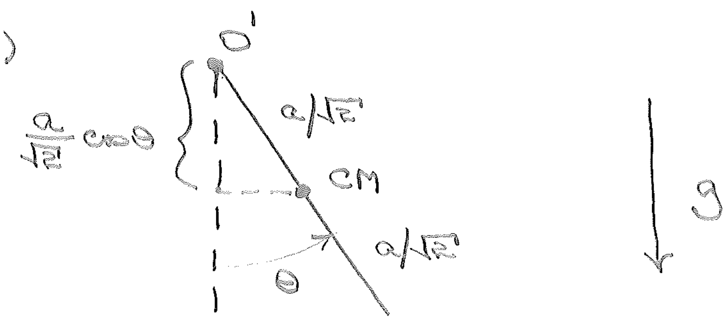
This shows $\vec{e}_1{}', \vec{e}_2{}'$, and $\vec{e}_3$ are principal axes

$$(I_{O'})'_{11} = I'_{11} = \frac{1}{12} m a^2$$

$$(I_{O'})'_{22} = I'_{22} + \frac{1}{2} m a^2 = \frac{7}{12} m a^2$$

$$(I_{O'})'_{33} = I'_{33} + \frac{1}{2} m a^2 = \frac{2}{3} m a^2$$

(c)



Rotation about $\vec{e}_z^{\,\prime}$ axis

Method 1

$$T = T_{CM} + T_{\text{rot about CM}}$$

$$T_{CM} = \frac{1}{2}m\left(\frac{a}{\sqrt{2}}\dot{\theta}\right)^2 = \frac{1}{4}ma^2\dot{\theta}^2$$

$$T_{\text{rot about CM}} = \frac{1}{2}I'_{zz}\dot{\theta}^2 = \frac{1}{24}ma^2\dot{\theta}^2$$

$$\boxed{T = \frac{7}{24}ma^2\dot{\theta}^2}$$

Method 2: $T = T_{\text{rot about } O'} = \frac{1}{2}(I_{O'})'_{zz}\dot{\theta}^2 = \boxed{\frac{7}{24}ma^2\dot{\theta}^2 = T}$

$$V = mg\left(-\frac{a}{\sqrt{2}}\cos\theta\right) = \boxed{-\frac{1}{\sqrt{2}}mga\cos\theta = V}$$

$$L = T - V = \boxed{\frac{7}{24}ma^2\dot{\theta}^2 + \frac{1}{\sqrt{2}}mga\cos\theta = L}$$

Small oscillations: $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

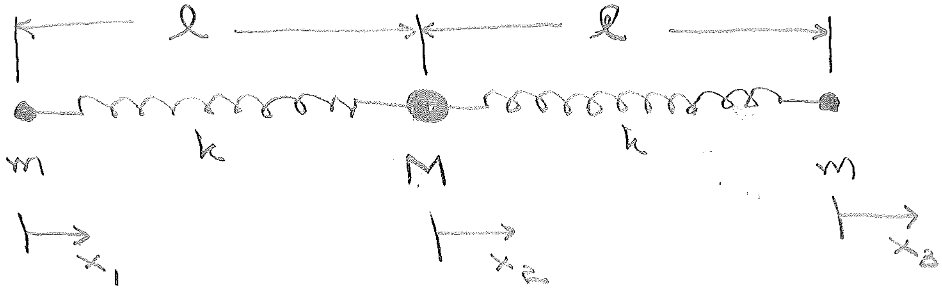
$$L = \frac{7}{24}ma^2\dot{\theta}^2 - \frac{1}{2\sqrt{2}}mga\theta^2 + \frac{1}{\sqrt{2}}mga$$

$$= \frac{7}{24}ma^2\Big(\dot{\theta}^2 - \underbrace{\frac{12}{7\sqrt{2}}\frac{g}{a}}_{\omega^2}\theta^2\Big) + \frac{1}{\sqrt{2}}mga$$

$$\boxed{\omega = \sqrt{\frac{12}{7\sqrt{2}}}\sqrt{\frac{g}{a}}}$$

3. Linear triatomic molecule

Exam: specialize to $M = 2m$



(a) $T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_3^2 \right) + \frac{1}{2} M \dot{x}_2^2$

$$= \frac{1}{2} (\dot{x}_1 \ \dot{x}_2 \ \dot{x}_3) \underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}}_{\mathbf{T}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

$$V = \frac{1}{2} k \underbrace{\left[\overbrace{(x_2 - x_1)^2}^{(x_2 + l - x_1 - l)^2} + \overbrace{(x_3 - x_2)^2}^{(x_3 + l - x_2 - l)^2} \right]}_{x_1^2 + 2x_2^2 + x_3^2 - 2x_1 x_2 - 2x_2 x_3}$$

$$= \frac{1}{2} (x_1 \ x_2 \ x_3) \underbrace{k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{\mathbf{V}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\boxed{\mathbf{T} = m \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & M/m & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{t} = mt}$$

$$\boxed{\mathbf{V} = k \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{v} = kv}$$

(b-e) Eigenvalue problem: $0 = (V - \omega^2 T)a = k\Big(v - \underbrace{\frac{\omega^2}{k/m}}_{\equiv \alpha^2} t\Big)$

$$(v - \alpha^2 t)a = 0 \quad , \quad \alpha^2 = \frac{\omega^2}{k/m}$$

<u>Guess method</u>: There are two obvious normal modes: (i) all masses in uniform motion, with no restoring force,

$$\boxed{a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \Rightarrow \omega_1 = 0}$$

$$v a_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_1^2 = 0$$

(ii) middle mass at rest, outer two masses out of phase

$$\boxed{a_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \omega_2 = \sqrt{k/m}}$$

$$v a_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$t a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & M/m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v a_2 = t a_2 \Rightarrow \alpha_2 = 1$$

The third mode must be orthogonal to a_1 and a_2, relative to t:

$$a_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

$$0 = \tilde{a}_1 t a_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & M/m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

$$= a_{13} + \frac{M}{m} a_{23} + a_{33}$$

$$0 = \tilde{a}_2 t a_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & M/m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

$$= a_{13} - a_{33}$$

$$\Rightarrow \quad a_{13} = a_{33} = 1 \quad \text{choose}$$

$$a_{23} = -\frac{2m}{M} a_{13} = -\frac{2m}{M}$$

$$a_3 = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

Check: $$v a_3 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2m/M \\ -2 - 4m/M \\ 1 + 2m/M \end{pmatrix} = \left(1 + \frac{2m}{M}\right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$t a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & M/m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad v a_3 = \underbrace{\left(1 + \frac{2m}{M}\right)}_{\alpha_3^2} t a_3$$

$$\boxed{a_3 = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}, \quad \omega_3 = \sqrt{1 + 2m/M}\,\sqrt{k/m}}$$

<u>Official</u> <u>method</u>:

$$(v - \alpha^2 t) a = 0$$

Secular equation: $$0 = \det(v - \alpha^2 t)$$

$$= \det \begin{pmatrix} 1 - \alpha^2 & -1 & 0 \\ -1 & 2 - \alpha^2 M/m & -1 \\ 0 & -1 & 1 - \alpha^2 \end{pmatrix}$$

$$= (1 - \alpha^2)\left(\left(2 - \alpha^2 \frac{M}{m}\right)(1 - \alpha^2) - 1\right)$$
$$- (1 - \alpha^2)$$

$$0 = (1-\alpha^2)\left(2 - \alpha^2\frac{M}{m} - 2\alpha^2 + \alpha^4\frac{M}{m} - 1 - 1\right)$$

$$= (1-\alpha^2)\alpha^2\left(\frac{M}{m}\alpha^2 - \left(2 + \frac{M}{m}\right)\right)$$

$$0 = (1-\alpha^2)\alpha^2\left(\alpha^2 - \left(1 + \frac{2m}{M}\right)\right)$$

$$\Longrightarrow \quad \alpha_1^2 = 0, \quad \alpha_2^2 = 1, \quad \alpha_3^2 = 1 + \frac{2m}{M}$$

$$\boxed{\omega_1 = 0, \quad \omega_2 = \sqrt{k/m}, \quad \omega_3 = \sqrt{1 + 2m/M}\sqrt{k/m}}$$

Mode 1: $0 = V a_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$

$$\Longrightarrow \begin{matrix} 0 = a_{11} - a_{21} \\ 0 = a_{31} - a_{21} \end{matrix} \Longrightarrow a_{31} = a_{21} = a_{11}$$

$$\boxed{a_1 \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Mode 2: $0 = (V - T) a_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 - \frac{M}{m} & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$

$$\Longrightarrow a_{22} = 0, \quad -a_{12} - a_{32} = 0 \Longrightarrow \begin{matrix} a_{22} = 0 \\ a_{32} = -a_{12} \end{matrix}$$

$$a_2 \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

C

Mode 3:

$$0 = \left(v - \left(1 + \frac{2m}{M}\right)t\right)Q_3 = \begin{pmatrix} 2m/M & -1 & 0 \\ -1 & -M/m & -1 \\ 0 & -1 & 2m/M \end{pmatrix}\begin{pmatrix} Q_{13} \\ Q_{23} \\ Q_{33} \end{pmatrix}$$

$$0 = \frac{2m}{M} Q_{13} - Q_{23} = \frac{2m}{M} Q_{33} - Q_{23} \implies Q_{13} = Q_{33}$$

$$0 = -Q_{13} - Q_{33} - \frac{M}{m} Q_{23} = -2Q_{12} - \frac{M}{m} Q_{23}$$

$$\implies Q_{23} = -\frac{2m}{M} Q_{13}$$

$$Q_3 \propto \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$