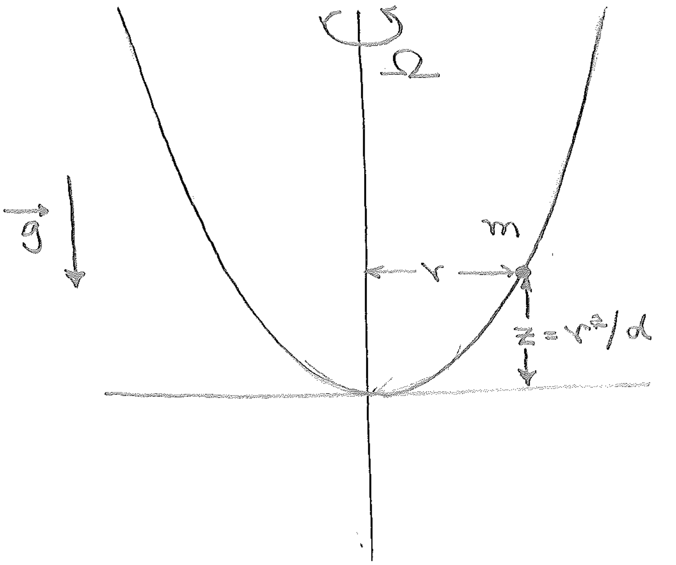
Phys 503

Midterm Exam #3

Solutions

1.



(a) Cartesian coördinates

$x = r\cos\Omega t$

$y = r\sin\Omega t$

$z = r^2/d$

$\dot{x} = \dot{r}\cos\Omega t - r\Omega\sin\Omega t$

$\dot{y} = \dot{r}\sin\Omega t + r\Omega\cos\Omega t$

$\dot{z} = 2r\dot{r}/d$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2 + 4r^2\dot{r}^2/d^2)$

Cylindrical coördinates

$r = r(t)$

$\phi = \Omega t$

$z = r^2/d$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$

$= \frac{1}{2}m(\dot{r}^2 + r^2\Omega^2 + 4r^2\dot{r}^2/d^2)$

Notice that this is dimensionless, as it must be

$T = \frac{1}{2}m\left(1 + \frac{4r^2}{d^2}\right)\dot{r}^2 + \frac{1}{2}mr^2\Omega^2$

$V = mgz = mgr^2/d$

$$\boxed{L = T - V = \frac{1}{2}\underbrace{m\left(1+\frac{4r^2}{d^2}\right)}_{m(r)}\dot{r}^2 - \underbrace{\frac{1}{2}m\left(\frac{2g}{d} - \Omega^2\right)r^2}_{V_{eff}(r)}}$$

Equivalent to a particle of mass $m(r)$ moving in a potential $V_{eff}(r)$.

(b) $$p_r = \frac{\partial L}{\partial \dot{r}} = m(r)\dot{r}$$

$$H = \dot{r}p_r - L = \frac{1}{2}m(r)\dot{r}^2 + V_{eff}(r)$$

$$\boxed{H = \frac{p_r^2}{2m(r)} + V_{eff}(r) = \frac{p_r^2}{2m(1+4r^2/d^2)} + \frac{1}{2}m\left(\frac{2g}{d} - \Omega^2\right)r^2}$$

(c) We can judge the stability in the small-oscillations approximation, in which $m(r) = m$ and

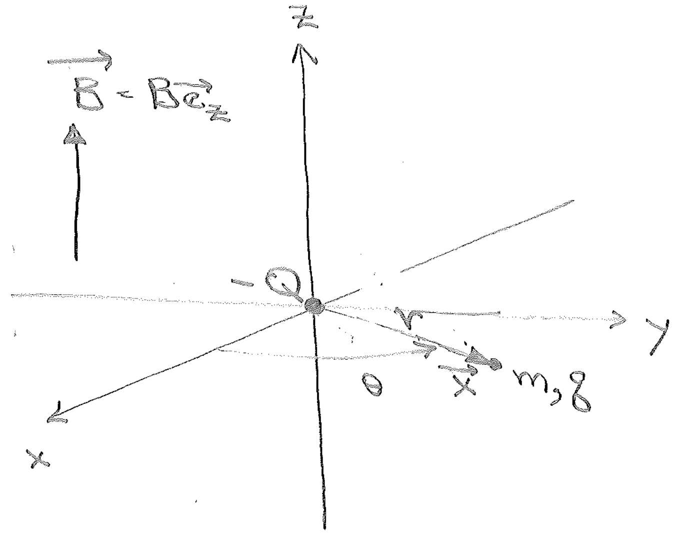
$$L = \frac{1}{2}m\dot{r}^2 - \underbrace{\frac{1}{2}m\left(\frac{2g}{d} - \Omega^2\right)r^2}_{V_{eff}(r)}$$

Stability $\Leftrightarrow$ V_{eff} has a <u>minimum</u> at $r=0$

$$\Leftrightarrow \frac{2g}{d} > \Omega^2 \Leftrightarrow \boxed{|\Omega| < \sqrt{2g/d}}$$

The frequency of small oscillations is $\omega = \left(\frac{2g}{d} - \Omega^2\right)^{1/2}$

2.



$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{x} = \frac{1}{2}Br\,\vec{e}_\theta$$

$$\phi = -\frac{Q}{r}$$

(a) $$L = T + \frac{q}{c}\underbrace{\dot{\vec{x}}\cdot\vec{A}}_{v_\theta \frac{1}{2}Br \,=\, \frac{1}{2}Br^2\dot\theta} - q\phi$$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{qB}{2c}r^2\dot\theta + \frac{qQ}{r}$$

$$p_r = m\dot r$$

$$p_\theta = mr^2\dot\theta + \frac{qB}{2c}r^2 = mr^2\Big(\dot\theta + \underbrace{\frac{qB}{2mc}}_{\Omega}\Big)$$

$$H = p_r\dot r + p_\theta\dot\theta - L$$

$$= m\dot r^2 + mr^2\dot\theta^2 + \frac{qB}{2c}r^2\dot\theta - \frac{1}{2}m\dot r^2 - \frac{1}{2}mr^2\dot\theta^2 - \frac{qB}{2c}r^2\dot\theta - \frac{qQ}{r}$$

0

$$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 - \frac{qQ}{r}$$

$$H = \frac{p_r^2}{2m} + \frac{1}{2}mr^2\underbrace{\left(\frac{p_\theta}{mr^2} - \Omega\right)^2}_{\frac{p_\theta^2}{(mr^2)^2} - 2\frac{p_\theta\Omega}{mr^2} + \Omega^2} - \frac{g\varphi}{r}$$

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \Omega p_\theta + \frac{1}{2}m\Omega^2 r^2 - \frac{g\varphi}{r}}$$

(b) Coördinate transformation

$$\phi = \theta + \Omega t \quad \text{and} \quad p_\phi = p_\theta$$

Generating function: $\boxed{F_2(\theta, p_\phi) = (\theta + \Omega t)p_\phi}$

$$p_\theta = \frac{\partial F_2}{\partial \theta} = p_\phi$$

$$\phi = \frac{\partial F_2}{\partial p_\phi} = \theta + \Omega t$$

Transformed Hamiltonian:

$$K = H + \frac{\partial F_2}{\partial t} \quad \Omega p_\phi$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} \cancel{- \Omega p_\phi} + \frac{1}{2}m\Omega^2 r^2 - \frac{g\varphi}{r} \cancel{+ \Omega p_\phi}$$

$$\boxed{K = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2}m\Omega^2 r^2 - \frac{g\varphi}{r}}$$

$$\boxed{\begin{aligned}&\text{(c) (1) } K \text{ is conserved}\\ &\text{(2) } p_\phi = m r^2 \dot\phi \text{ is conserved}\\ &\qquad = m r^2(\dot\theta + \Omega) = p_\theta\end{aligned}}$$

$$\left(\dot\phi = \frac{\partial K}{\partial p_\phi} = \frac{p_\phi}{m r^2}\right)$$

H is also conserved

(d) The radial motion is governed by

$$K = \frac{1}{2} m \dot r^2 + \boxed{\underbrace{\frac{p_\phi^2}{2 m r^2} + \frac{1}{2} m \Omega^2 r^2 - \frac{g Q}{r}}_{= V(r) = \left(\substack{\text{effective}\\ \text{potential}}\right)}}$$

$$-f(r) = \frac{\partial V}{\partial r} = -\frac{p_\phi^2}{m r^3} + m\Omega^2 r + \frac{gQ}{r^2}$$

Circular orbit: $f(r_0) = 0$

$$\frac{p_\phi^2}{m r_0^3} = m\Omega^2 r_0 + \frac{gQ}{r_0^2}$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} = +\frac{3 p_\phi^2}{m r_0^4} + m\Omega^2 - \frac{2gQ}{r_0^3}$$

$$= \frac{3}{r_0}\left(m\Omega^2 r_0 + \frac{gQ}{r_0^2}\right) + m\Omega^2 - \frac{2gQ}{r_0^3}$$

$$= 4 m\Omega^2 + \frac{gQ}{r_0^3}$$

Expand $V(r)$ about $r = r_0$:

$$V(r) = \underbrace{V(r_0)}_{V_0} + \cancel{\frac{\partial V}{\partial r}}\bigg|_{r=r_0}^{\to 0} \underbrace{(r-r_0)}_{y} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\bigg|_{r=r_0} \underbrace{(r-r_0)^2}_{y^2}$$

$$= V_0 + \frac{1}{2}\left(4m\Omega^2 + \frac{qQ}{r_0^3}\right) y^2$$

$$= V_0 + \frac{1}{2}m\left(4\Omega^2 + \frac{qQ}{m r_0^3}\right) y^2$$

$$K - V_0 = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\underbrace{\left(4\Omega^2 + \frac{qQ}{m r_0^3}\right)}_{\omega^2} y^2$$

$$\boxed{\omega^2 = 4\Omega^2 + \frac{qQ}{m r_0^3}}$$

3.

(a) $H = \frac{p^2}{2mq^2} + \frac{1}{2} m g q^2$

$$\dot{q} = \frac{\partial H}{\partial p} = \boxed{\frac{p}{mq^2} = \dot{q}}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = \boxed{+\frac{p^2}{mq^3} - mgq = \dot{p}}$$

(b) $p_z = \frac{p}{q}$, $z = \frac{1}{2} q^2$

① Find a generating function $F_2(q, p_z)$:

$$\frac{\partial F_2}{\partial q} = p = q\, p_z \Rightarrow F_2(q,p) = \frac{1}{2} q^2 p_z + f(p_z)$$

$$z = \frac{\partial F_2}{\partial p_z} = \frac{1}{2} q^2 + f'(p_z) \Longrightarrow \boxed{F_2(q,p) = \frac{1}{2} q^2 p_z}$$

$f' = 0 \Rightarrow f = \text{constant} = 0$ ← choose

② Preservation of fundamental Poisson bracket:

$$[z, p_z] = \underset{= q}{\frac{\partial z}{\partial q}} \underset{= q^{-1}}{\frac{\partial p_z}{\partial p}} - \underset{= 0}{\frac{\partial z}{\partial p}} \underset{= -p/q^2}{\frac{\partial p_z}{\partial q}} = \boxed{1 = [z, p_z]}$$

$$\boxed{H = \frac{p_z^2}{2m} + mgz}$$

particle in a uniform gravitational field

$$\dot z = \frac{\partial H}{\partial p_z} = \boxed{\frac{p_z}{m} = \dot z}$$

Solution:

$$\boxed{z = -\frac{1}{2} g t^2 + \frac{p_{z0}}{m} t + z_0}$$

$$\dot p_z = -\frac{\partial H}{\partial z} = \boxed{-mg = \dot p_z}$$

$$\boxed{p_z = -mgt + p_{z0}}$$

(c) $P = H = \dfrac{p_z^2}{2m} + mgz$

Find q, Q:

Generating function $F_2(z, P)$:

$$\frac{\partial F_2}{\partial z} = p_z = \sqrt{2m}\,(P - mgz)^{1/2}$$

$$\Rightarrow F_2(z, P) = -\frac{2}{3}\frac{\sqrt{2m}}{mg}(P - mgz)^{3/2} + f(P)$$

$$= -\frac{2}{3g}\sqrt{\frac{2}{m}}\,(P - mgz)^{3/2} + f(P)$$

$$Q = \frac{\partial F_2}{\partial P} = -\frac{1}{g}\sqrt{\frac{2}{m}}\,\underbrace{(P - mgz)^{1/2}}_{\sqrt{\frac{p_z^2}{2m}} = \frac{p_z}{\sqrt{2m}}} + f'(P)$$

$$= -\frac{p_z}{mg} + \underbrace{f'\left(\frac{p_z^2}{2m} + mgz\right)}_{\text{can choose this to be zero}}$$

$$\boxed{Q = -\frac{p_z}{mg}}$$

Verify: $[Q, P] = -\left[\frac{p_z}{mg}, mgz\right] = +[z, p_z] = 1$

$H = P$

$$\boxed{\begin{aligned} \dot{Q} &= \frac{\partial H}{\partial P} = 1 \\ \dot{P} &= -\frac{\partial H}{\partial Q} = 0 \end{aligned}}$$

Solution

$$\boxed{\begin{aligned} Q &= t + Q_0 \\ P &= P_0 \end{aligned}}$$