Phys 503

Homework # 1

Solution Set

1.1. Goldstein 1.8:

Lagrange's equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \dots, n$

$$L' = L + \underbrace{\frac{dF(q_1, \dots, q_n, t)}{dt}}$$

$$= \sum_{j=1}^{n} \frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial t}$$

fcn. of $q_1, \dots, q_n, t$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial F}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \underbrace{\frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right)}_{\sum_{k=1}^{n} \frac{\partial^2 F}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 F}{\partial t \partial q_j}}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 F}{\partial t \partial q_j}$$

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 F}{\partial q_j \partial t}$$

$$\Longrightarrow \quad 0 = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j}, \quad j = 1, \dots, N$$

This invariance is obvious from Hamilton's principle since the action satisfies

$$I'[q_1(t), \dots, q_n(t)] = \int_{t_1}^{t_2} dt\, L'$$

$$= \int_{t_1}^{t_2} dt \left(L + \frac{dF}{dt} \right)$$

$$= I[q_1(t), \dots, q_n(t)] + \underbrace{F(q_1(t), \dots, q_n(t), t) \Big|_{t_1}^{t_2}}$$

This is a constant under variations since the generalized coordinates are held constant at t_1 and t_2.

Thus $\delta I' = \delta I$, and the Lagrange equations must be the same.

1.2. Goldstein 1.9:

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

Under a gauge transformation,

$$\phi \longrightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla\psi$$

$$L' = \frac{1}{2}mv^2 - q\phi' + \frac{q}{c}\vec{v}\cdot\vec{A}'$$

$$= \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A} \quad \underbrace{+ \frac{q}{c}\frac{\partial \psi}{\partial t} + \frac{q}{c}\vec{v}\cdot\nabla\psi}$$

$$\frac{q}{c}\left(\sum_{j=1}^{3}\frac{\partial \psi}{\partial x_j}\frac{dx_j}{dt} + \frac{\partial \psi}{\partial t}\frac{dt}{dt}\right)$$

$$= \frac{q}{c}\frac{d}{dt}\psi(\vec{r},t)$$

Adding a total time derivative to the Lagrangian has no effect on the equations of motion.

1.3. $\nabla \times \vec{A} = \vec{e}_j \, \epsilon_{jkl} \, A_{l,k}$

$$\nabla \times (\nabla \times \vec{A}) = \vec{e}_j \, \epsilon_{jkl} \, \underbrace{(\nabla \times \vec{A})_{l,k}}_{= \epsilon_{lmn} A_{n,m}}$$

$$= \vec{e}_j \, \underbrace{\epsilon_{jkl} \, \epsilon_{lmn}}_{= \epsilon_{ljk}\epsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}} A_{n,mk}$$

$$= \underbrace{\vec{e}_j \, \delta_{jm} \, \delta_{kn} \, A_{n,mk}}_{= \vec{e}_j A_{k,jk}} - \underbrace{\vec{e}_j \, \delta_{jn} \, \delta_{km} \, A_{n,mk}}_{\vec{e}_j A_{j,kk} = \nabla^2 \vec{A}}$$

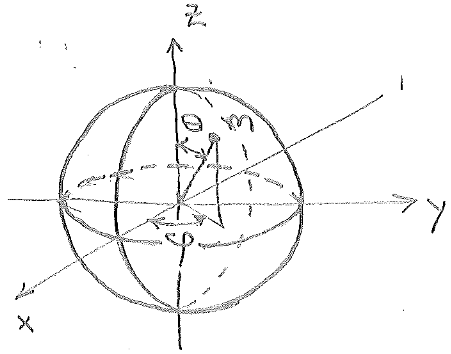
$$= \vec{e}_j \, A_{k,kj}$$

$$= \vec{e}_j \, (\nabla \cdot \vec{A})_{,j}$$

$$= \nabla (\nabla \cdot \vec{A})$$

$$\therefore \; \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

1.4. Goldstein 1.19



The mass m is constrained to move on a sphere of radius l. Use spherical coordinates θ and φ.

Distances on the sphere are governed by the metric

$$ds^2 = l^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right),$$

so the squared velocity is

$$v^2 = \left(\frac{ds}{dt}\right)^2 = l^2\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2\right)$$

You can also get this from

$$\vec r = \underbrace{l\sin\theta\cos\varphi}_{x}\,\vec e_x + \underbrace{l\sin\theta\sin\varphi}_{y}\,\vec e_y + \underbrace{l\cos\theta}_{z}\,\vec e_z$$

$$\vec v = \dot{\vec r} = l\Big[\dot\theta\big(\underbrace{\cos\theta\cos\varphi\,\vec e_x + \cos\theta\sin\varphi\,\vec e_y - \sin\theta\,\vec e_z}_{\vec A}\big) + \dot\varphi\big(\underbrace{-\sin\theta\sin\varphi\,\vec e_x + \sin\theta\cos\varphi\,\vec e_y}_{\vec B}\big)\Big]$$

$$\vec A\cdot\vec A = \cos^2\theta\left(\cos^2\varphi + \sin^2\varphi\right) + \sin^2\theta = 1$$

$$\vec B\cdot\vec B = \sin^2\theta\left(\sin^2\varphi + \cos^2\varphi\right) = \sin^2\theta$$

$$\vec A\cdot\vec B = -\cos\theta\sin\theta\cos\varphi\sin\varphi + \cos\theta\sin\theta\sin\varphi\cos\varphi = 0$$

$$\therefore\ v^2 = l^2\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2\right)$$

$$L = T - V = \frac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\varphi^2) - mgl\cos\theta$$

$T \downarrow \frac{1}{2} m v^2$

$V \rightarrow mgz = mgl\cos\theta$

We get the same equations using

$$; L' = L/ml = \frac{1}{2} l (\dot\theta^2 + \sin^2\theta\, \dot\varphi^2) - g\cos\theta$$

$$\frac{\partial L'}{\partial \theta} = l \sin\theta \cos\theta\, \dot\varphi^2 + g\sin\theta$$

$$\frac{\partial L}{\partial \dot\theta} = l\dot\theta \;, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = l\ddot\theta$$

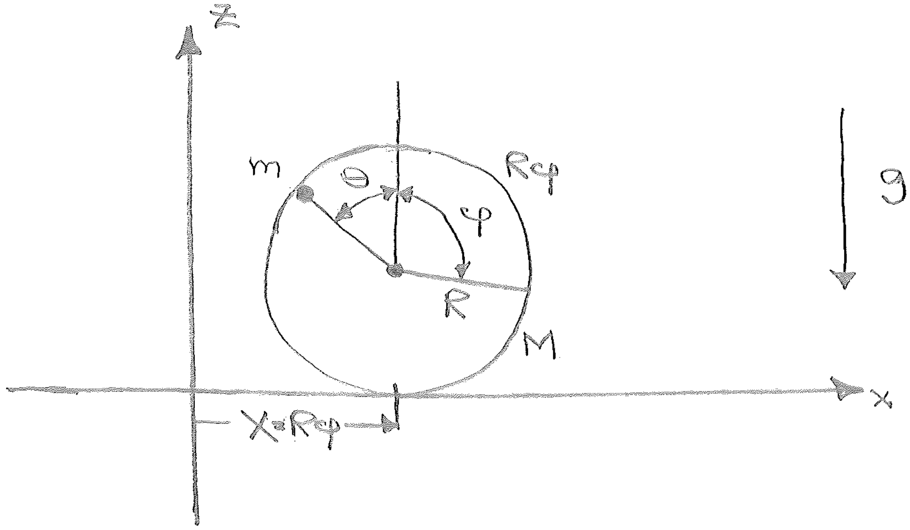
$$\Longrightarrow \boxed{l\ddot\theta - \underbrace{l\sin\theta\cos\theta\,\dot\varphi^2}_{\text{component of centripetal acceleration along } \theta\text{-direction}} = g\sin\theta}$$

$$\frac{\partial L'}{\partial \varphi} = 0$$

$$\frac{\partial L'}{\partial \dot\varphi} = l\sin^2\theta\,\dot\varphi \;, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\varphi}\right) = 2l\sin\theta\cos\theta\,\dot\theta\dot\varphi + l\sin^2\theta\,\ddot\varphi$$

$$\Longrightarrow \boxed{0 = \frac{d}{dt}\big(\underbrace{l\sin^2\theta\,\dot\varphi}_{\text{(conserved angular momentum)}}\big) = 2l\sin\theta\cos\theta\,\dot\theta\dot\varphi + l\sin^2\theta\,\ddot\varphi}$$

1.5



Hollow cylinder of mass M and radius R. Rolling constraint implies rolls without slipping. $R\varphi = X$. Particle of mass m is attached to the interior of the cylinder and slides without friction. Its inertial coordinates are $x = X - R\sin\theta$ and $z = R + R\cos\theta$, which means that the components of the inertial velocity are $\dot{x} = \dot{X} - R\dot{\theta}\cos\theta$ and $\dot{z} = -R\dot{\theta}\sin\theta$.

(a) $T_{cylinder} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{\varphi}^2 = M\dot{X}^2$

$$T_{particle} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2$$

$$= \frac{1}{2}m\left(\dot{X} - R\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin^2\theta$$

$$= \frac{1}{2}m\dot{X}^2 - mR\dot{\theta}\dot{X}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2$$

$$T = T_{cylinder} + T_{particle}$$

$$= \frac{1}{2}\underbrace{(2M+m)}_{M'}\dot{X}^2 - mR\dot{\theta}\dot{X}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2$$

$$V = mg(z - R) = mgR\cos\theta$$

$$\boxed{L = T - V = \frac{1}{2}M'\dot{X}^2 - mR\dot{\theta}\dot{X}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta \qquad \boxed{M' = 2M + m}}$$

(b) X is cyclic, so

$$\text{(1)} \quad \boxed{p_X = \frac{\partial L}{\partial \dot{X}} = M'\dot{X} - mR\dot{\theta}\cos\theta = 2M\dot{X} + m\dot{x}}$$

is conserved. Notice that the x-component of the total momentum,

$$P_x = M\dot{X} + m\dot{x} = p_X - M\dot{X},$$

is not conserved, because there is a force on the system — the frictional force that causes the cylinder to roll. The other conserved quantity is the energy:

$$\text{(2)} \quad \boxed{\begin{aligned} E &= p_X\dot{X} + p_\theta\dot{\theta} - L = \left(\begin{matrix}\text{Jacobi}\\ \text{integral}\end{matrix}\right) \\ &= T + V \\ &= \frac{1}{2}M'\dot{X}^2 - mR\dot{\theta}\dot{X}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta \\ &= M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2 + mgR\cos\theta \end{aligned}}$$

We imagine the particle is given a tiny positive $\dot{\theta}$ to get it away from the unstable equilibrium.

(c) $t=0:\quad \theta=0,\ \dot{\theta}=0,\ \dot{X}=0$

$$0 = P_X = M'\dot{X} - mR\dot{\theta}\cos\theta = 2M\dot{X} + m\dot{x}$$

$\Longrightarrow$

$$mgR = E = \tfrac{1}{2}M'\dot{X}^2 - mR\dot{\theta}\dot{X}\cos\theta + \tfrac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

$$= M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{z}^2 + mgR\cos\theta$$

Inertial velocity v of m at $\underline{\theta=\pi}$: At $\theta=\pi$, $\dot{z}=0$, so

$$v = |\dot{x}| = |\dot{X} + R\dot{\theta}|$$

The $\dot{\theta}$ route

$$\dot{X} = \frac{m}{M'}R\dot{\theta}\cos\theta \underset{\substack{\uparrow \\ \theta=\pi}}{=} -\frac{m}{M'}R\dot{\theta}$$

$$\underset{\substack{\uparrow \\ \theta=0}}{\cancel{m}gR} = \underbrace{\boxed{\frac{1}{2}\frac{m^2R^2\dot{\theta}^2}{M'} - \frac{m^2R^2\dot{\theta}^2}{M'} + \frac{1}{2}\cancel{m}R^2\dot{\theta}^2 - \cancel{m}gR}}_{\theta=\pi}$$

$$2g = \tfrac{1}{2}R\dot{\theta}^2 - \tfrac{1}{2}\frac{mR\dot{\theta}^2}{M'}$$

$$= \tfrac{1}{2}R\dot{\theta}^2\left(1 - \frac{m}{M'}\right)$$

$$R\dot{\theta}^2 = \frac{4g}{1-m/M'}$$

$$R\dot{\theta} = \frac{2\sqrt{gR}}{\sqrt{1-m/M'}} \geq 0$$

given the initial push

$$v = R\dot{\theta}\left(1 - \frac{m}{M'}\right)$$

$$\boxed{v = 2\sqrt{gR}\sqrt{1-\frac{m}{M'}} = \frac{2\sqrt{gR}}{\sqrt{1+m/2M}}}$$

$$1 - \frac{m}{M'} = \frac{2M}{2M+m} = \frac{1}{1+m/2M}$$

The $\dot{x}$ route

$$\dot{X} = -\frac{m}{2M}\dot{x} \quad \text{at all times}$$

$$\underbrace{\cancel{m}gR}_{E \text{ at } \theta=0} = \underbrace{\frac{1}{4}\frac{m^2\dot{x}^2}{M} + \frac{1}{2}\cancel{m}\dot{x}^2 - \cancel{m}gR}_{E \text{ at } \theta=\pi}$$

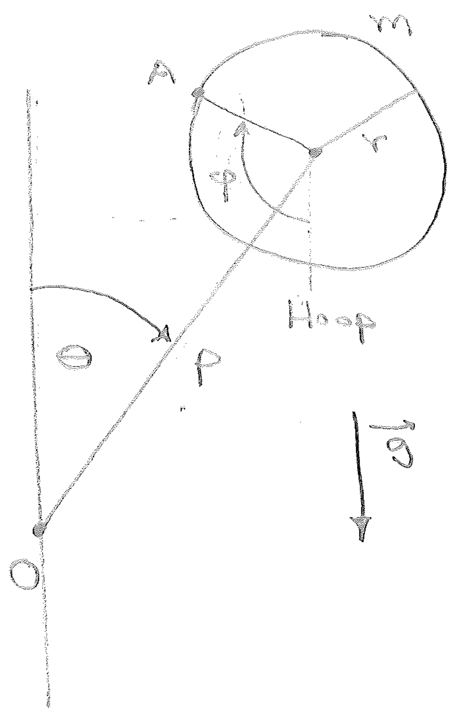
$$2gR = \tfrac{1}{2}v^2\left(1 + \frac{m}{2M}\right)$$

$$\boxed{v = \frac{2\sqrt{gR}}{\sqrt{1+m/2M}} \geq 0}$$

Notice that we also get

$$V = \dot{X} = -\frac{m}{2M}v = -\sqrt{\frac{m}{2M}}\,\frac{2\sqrt{gR}}{\sqrt{1+2M/m}}$$

1.6. Goldstein 2.14



Generalized coordinates

(1) ρ and θ of center of mass

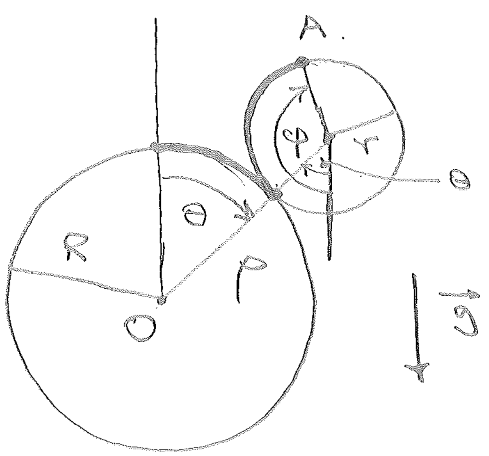
(2) angle φ of rotation of some point A

$$T_{CM} = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\theta}^2)$$

$$T_{rot} = \frac{1}{2} m r^2 \dot{\varphi}^2$$

$$V = mgz = mg\rho\cos\theta$$

Put the hoop on cylinder of radius R. Let A be the point that is in contact with the cylinder when $\theta = 0$.



Constraints:

(1) Rolling without slipping

$$R\theta = r(\varphi - \theta) \Leftrightarrow (R+r)\theta = r\varphi$$

$$r\dot{\varphi} = (R+r)\dot{\theta}$$

(2) $\rho = R + r$

We can elimate φ using constraint (1), but we handle constraint (2) with a Lagrange multiplier in order to get the radial force of constraint

$$T_{rot} = \frac{1}{2} m r^2 \dot{\varphi}^2 = \frac{1}{2} m (R+r)^2 \dot{\theta}^2$$

$$T = T_{cm} + T_{rot} = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \left(\rho^2 + (R+r)^2 \right) \dot{\theta}^2$$

$$L = T - V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \left[\rho^2 + (R+r)^2 \right] \dot{\theta}^2 - m g \rho \cos\theta$$

To enforce constraint 2, we use an effective Lagrangian

$$L' = L + \lambda(t) \left(\rho - r - R \right)$$

Notice that if we eliminate $\rho = R+r$,

$$L = m (R+r)^2 \dot{\theta}^2 - m g (R+r) \cos\theta$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = 2m (R+r)^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = + m g (R+r) \sin\theta$$

$$\Rightarrow 2\cancel{m} (R+r)^{2} \ddot{\theta} = \cancel{m} g \cancel{(R+r)} \sin\theta$$

$$(R+r) \ddot{\theta} = \frac{1}{2} g \sin\theta$$

Jacobi integral

$$h = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L$$

$$= m (R+r)^2 \dot{\theta}^2 + m g (R+r) \cos\theta$$

$$= T + V$$

a) $$\frac{\partial L'}{\partial \dot{\rho}} = m \dot{\rho}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\rho}} \right) = m \ddot{\rho}$$

$$\frac{\partial L'}{\partial \rho} = m \rho \dot{\theta}^2 - m g \cos\theta + \lambda(t)$$

$$\Rightarrow \boxed{ m \ddot{\rho} = \underbrace{m \rho \dot{\theta}^2}_{\text{centrifugal force}} - \underbrace{m g \cos\theta}_{\text{gravity}} + \underbrace{\lambda(t)}_{\text{normal (constraint) force}} }$$

b) $$\frac{\partial L'}{\partial \dot{\theta}} = m \left[\rho^2 + (R+r)^2 \right] \dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}} \right) = m \left[\rho^2 + (R+r)^2 \right] \ddot{\theta} + 2 m \rho \dot{\rho} \dot{\theta}$$

$$\frac{\partial L'}{\partial \theta} = m g \rho \sin\theta$$

$$\boxed{ \cancel{m} \left[\rho^2 + (R+r)^2 \right] \ddot{\theta} + 2 \cancel{m} \rho \dot{\rho} \dot{\theta} = \cancel{m} g \rho \sin\theta }$$

c $\rho = R + r \Rightarrow \dot{\rho} = 0 \Rightarrow \ddot{\rho} = 0$

Substitute c into a and b

b $2(R+r)^{\cancel{2}}\ddot{\theta} = g\cancel{(R+r)}\sin\theta \Longrightarrow (R+r)\ddot{\theta} = \frac{1}{2} g \sin\theta$

a $\lambda(t) = mg\cos\theta - m(R+r)\dot{\theta}^2$

The conserved energy is

$$E = T + V = m(R+r)^2\dot{\theta}^2 + mg(R+r)\cos\theta = mg(R+r)$$

Hoop at rest ($\dot{\theta} = 0$)
at top ($\theta = 0$)

$$\Rightarrow m(R+r)\dot{\theta}^2 + mg\cos\theta = mg$$

$$\lambda(t) = mg\cos\theta - \underbrace{m(R+r)\dot{\theta}^2}_{mg(1-\cos\theta)} = mg(2\cos\theta - 1)$$

The force of constraint, $\lambda(t)$, must be positive. When it is negative, the hoop loses contact with the cylinder. The critical θ is

$$\cos\theta = 1/2 \Longrightarrow \theta = 60^\circ$$

What if one used Lagrange multipliers for both constraints?

$$L'' = L + \lambda(t)(\rho - r - R) + \gamma(t)((R+r)\theta - r\varphi)$$

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}mr^2\dot{\varphi}^2 - mg\rho\cos\theta$$

$$\frac{\partial L''}{\partial \dot{\rho}} = m\dot{\rho} \qquad \frac{d}{dt}\left(\frac{\partial L''}{\partial \dot{\rho}}\right) = m\ddot{\rho}$$

$$\frac{\partial L''}{\partial \rho} = m\rho\dot{\theta}^2 - mg\cos\theta + \lambda(t)$$

$$\boxed{m\ddot{\rho} = \underbrace{m\rho\dot{\theta}^2}_{\text{centrifugal force}} - \underbrace{mg\cos\theta}_{\text{gravity}} + \underbrace{\lambda(t)}_{\text{normal force}}}$$

$$\frac{\partial L''}{\partial \dot{\theta}} = m\rho^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L''}{\partial \dot{\theta}}\right) = 2m\rho\dot{\rho}\dot{\theta} + m\rho^2\ddot{\theta}$$

$$\frac{\partial L''}{\partial \theta} = + mg\rho\sin\theta + (R+r)\gamma(t)$$

$$\boxed{m\rho^2\ddot{\theta} = -2m\rho\dot{\rho}\dot{\theta} + mg\rho\sin\theta + (R+r)\gamma(t)}$$

frictional force — removes translational kinetic energy and turns it into rotational kinetic energy

$$\frac{\partial L''}{\partial \dot{\varphi}} = mr^2 \dot{\varphi} \qquad \frac{d}{dt}\left(\frac{\partial L''}{\partial \dot{\varphi}}\right) = mr^2 \ddot{\varphi}$$

$$\frac{\partial L''}{\partial \varphi} = -r\eta(t)$$

$$\boxed{mr^2 \ddot{\varphi} = \underbrace{-r\eta(t)}_{\text{frictional torque}}}$$

Enforce constraints:

$$\lambda(t) = mg\cos\theta - m\rho\dot{\theta}^2 = mg(2\cos\theta - 1)$$

$$\eta(t) = -mr\ddot{\varphi} = -m(R+r)\ddot{\theta} = -\frac{1}{2}mg\sin\theta$$

$$2\cancel{m}(R+r)^{\cancel{2}}\ddot{\theta} = \cancel{m}g(\cancel{R+r})\sin\theta$$

$$(R+r)\ddot{\theta} = \frac{1}{2}g\sin\theta$$

$$\frac{|\eta(t)|}{\lambda(t)} = \frac{\frac{1}{2}\cancel{mg}\sin\theta}{\cancel{mg}(2\cos\theta + 1)} = \frac{1}{2}\frac{\sin\theta}{2\cos\theta - 1} \leq \underset{\substack{\uparrow \\ \text{coefficient of} \\ \text{static friction}}}{\mu}$$

We need an infinite μ to keep the thing from slipping before it loses contact.

1.7. Goldstein 2.16

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)$$

$$\frac{\partial L}{\partial \dot{q}} = e^{\gamma t} m\dot{q}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \gamma e^{\gamma t} m\dot{q} + e^{\gamma t} m\ddot{q}$$
$$= e^{\gamma t}(m\ddot{q} + m\gamma\dot{q})$$

$$\frac{\partial L}{\partial q} = -e^{\gamma t} kq$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \Longrightarrow \boxed{m\ddot{q} = -m\gamma\dot{q} + kq}$$

This is a damped harmonic oscillator — e.g., a particle of mass m subjected to a restoring force $-kq$ and a frictional force $-m\gamma\dot{q}$.

There are no (apparent) constants of motion since $\partial L/\partial t \neq 0$

This must be what textbook intended

$$s = e^{\gamma t/2} q$$

$$q = e^{-\gamma t/2} s, \quad \dot{q} = e^{-\gamma t/2}\left(\dot{s} - \frac{1}{2}\gamma s\right)$$

$$L = \frac{1}{2}m\left(\dot{s} - \frac{1}{2}\gamma s\right)^2 - \frac{1}{2}ks^2$$

$$= \frac{1}{2}m\dot{s}^2 - \frac{1}{2}m\gamma\dot{s}s - \frac{1}{2}\left(k - \frac{1}{4}m\gamma^2\right)s^2$$

$$\frac{\partial L}{\partial \dot{s}} = m\left(\dot{s} - \frac{1}{2}\gamma s\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} - \frac{1}{2}m\gamma\dot{s}$$

$$\frac{\partial L}{\partial s} = +\frac{1}{2}m\gamma\dot{s} - \left(k - \frac{1}{4}m\gamma^2\right)s$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{\partial L}{\partial s} \Rightarrow \boxed{m\ddot{s} = -\left(k - \frac{1}{4}m\gamma^2\right)s}$$

Since L now has no explicit time dependence, the Jacobi integral is conserved!

$$h = \dot{s}\frac{\partial L}{\partial \dot{s}} - L = m\dot{s}\left(\dot{s} - \frac{1}{2}\gamma s\right) - \frac{1}{2}m\dot{s}^2 + \frac{1}{2}m\gamma\dot{s}s + \frac{1}{2}\left(k - \frac{1}{4}m\gamma^2\right)s^2$$

$$\boxed{h = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\left(k - \frac{1}{4}m\gamma^2\right)s^2}$$

One can easily verify that this is the 1st integral of the equation of motion.

How to think about it:

$$s = e^{\gamma t/2}q, \quad \dot{s} = e^{\gamma t/2}\left(\dot{q} + \frac{\gamma}{2}q\right)$$

$$h = e^{\gamma t}\left(\underbrace{\frac{1}{2}m\left(\dot{q} + \frac{\gamma}{2}q\right)^2}_{\frac{1}{2}m\dot{q}^2 + \frac{\gamma}{2}m\dot{q}q + \frac{1}{2}m\frac{\gamma^2}{4}q^2} + \frac{1}{2}\left(k - \frac{1}{4}m\gamma^2\right)q^2\right)$$

$$= e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 + \frac{m\gamma}{2}q\dot{q}\right)$$

$$= e^{\gamma t}\left(E + \frac{1}{2}m\gamma q\dot{q}\right)$$

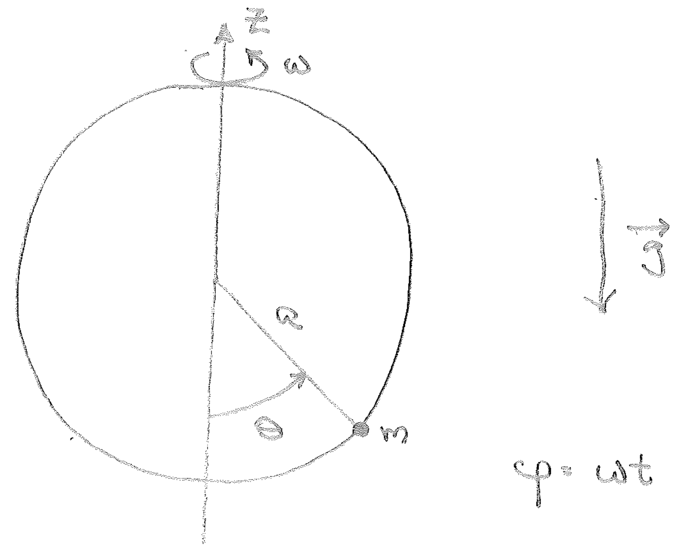
$$\Rightarrow E = e^{-\gamma t} h - \underbrace{\tfrac{1}{2} m \gamma q \dot{q}}$$

Secular damping of the energy

This term is here because the energy is not damped uniformly during a cycle. It is preferentially damped when the oscillator is moving fastest, i.e., near the equilibrium position.

Ignoring for the moment the $q\dot{q}$ term, h is conserved because it is the initial energy; i.e., it is the energy multiplied by $e^{\gamma t}$ to get back to the initial energy.

1.8. Goldstein 2.18



$\varphi = \omega t$

$$v^2 = a^2\dot\theta^2 + a^2\sin^2\theta\,\dot\varphi^2 = a^2\dot\theta^2 + a^2\omega^2\sin^2\theta$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta$$

$$V = mgz = -mga\cos\theta$$

$$L = T - V = \frac{1}{2}ma^2\dot\theta^2 + \underbrace{\frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta}_{-V_{eff}}$$

$$\frac{\partial L}{\partial\dot\theta} = ma^2\dot\theta, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = ma^2\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = -\frac{\partial V_{eff}}{\partial\theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$$

$$= ma\sin\theta\left(a\omega^2\cos\theta - g\right)$$

$$\boxed{a\ddot\theta = \sin\theta\left(a\omega^2\cos\theta - g\right) = -\frac{1}{ma}\frac{\partial V_{eff}}{\partial\theta}}$$

The Jacobi integral is conserved:

$$h = \dot{\theta}\underbrace{\frac{\partial L}{\partial \dot{\theta}}}_{= ma^2\dot{\theta}^2} - L = \boxed{\frac{1}{2}ma^2\dot{\theta}^2 + V_{eff} = h}$$

Equilibrium positions: $\dot{\theta} = \ddot{\theta} \Rightarrow 0 = \sin\theta\,(a\omega^2\cos\theta - g)$

(1) $\sin\theta = 0 \Rightarrow \theta = 0$ or $\theta = \pi$

Stability: $\frac{\partial}{\partial\theta}\Big(\sin\theta\,(a\omega^2\cos\theta - g)\Big) = -\frac{1}{ma}\frac{\partial^2 V_{eff}}{\partial\theta^2}$

$$= \cos\theta\,(a\omega^2\cos\theta - g) - a\omega^2\sin^2\theta$$

$$= \begin{cases} a\omega^2 - g, & \theta = 0 \\ a\omega^2 + g, & \theta = \pi \end{cases}$$

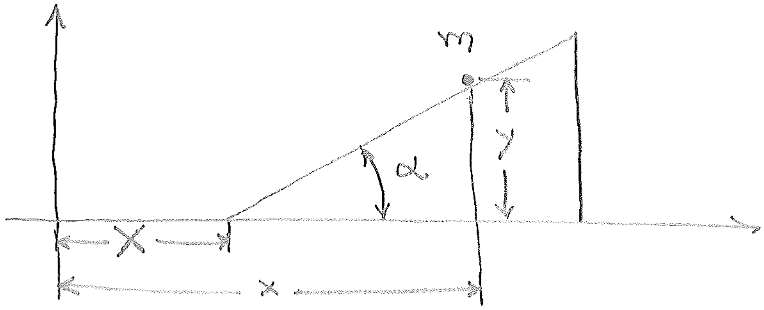
The $\theta = 0$ equilibrium position is stable if $a\omega^2 < g$ and unstable if $a\omega^2 > g$; The $\theta = \pi$ equilibrium position is always unstable because $a\omega^2 + g > 0$.

(2) $\cos\theta = g/a\omega^2$; this equilibrium exists only if $g/a\omega^2 \leq 1$, i.e., $\omega \geq \omega_0 = \sqrt{g/a}$.

Stability: $\frac{\partial}{\partial\theta}\Big(\sin\theta\,(a\omega^2\cos\theta - g)\Big)\Big|_{\cos\theta = g/a\omega^2} = -a\omega^2\sin^2\theta < 0$.

This position is always stable if it exists

1.9. Goldstein 2.20



Constraint: $\tan\alpha = \dfrac{y}{x - X}$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$V = mgy$$

1 Use the constraint to eliminate $x = X + y/\tan\alpha$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\underbrace{\left(\dot{X} + \dot{y}/\tan\alpha\right)^2}_{\dot{X}^2 + 2\dot{X}\dot{y}/\tan\alpha + \dot{y}^2/\tan^2\alpha} + \frac{1}{2}m\dot{y}^2$$

$$= \frac{1}{2}(M+m)\dot{X}^2 + \frac{m}{\tan\alpha}\dot{X}\dot{y} + \frac{1}{2}m\dot{y}^2\underbrace{\left(1 + \frac{1}{\tan^2\alpha}\right)}_{\frac{1}{\sin^2\alpha}}$$

$$L = T - V = \frac{1}{2}(M+m)\dot{X}^2 + \frac{m}{\tan\alpha}\dot{X}\dot{y} + \frac{1}{2}\frac{m}{\sin^2\alpha}\dot{y}^2 - mgy$$

(a) $\dfrac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + \dfrac{m}{\tan\alpha}\dot{y}$

$\dfrac{\partial L}{\partial X} = 0$ $\Rightarrow$ $\dfrac{d}{dt}\Big[\underbrace{(M+m)\dot{X} + \dfrac{m}{\tan\alpha}\dot{y}}\Big] = 0$

$\boxed{= M\dot{X} + m\dot{x} = \text{constant} \quad (\text{conservation of } x\text{-momentum})}$

ⓑ $\dfrac{\partial L}{\partial \dot{y}} = \dfrac{m}{\tan\alpha}\dot{X} + \dfrac{m}{\sin^2\alpha}\dot{y}$

$$\Rightarrow \underbrace{\frac{m}{\tan\alpha}\ddot{X} + \frac{m}{\sin^2\alpha}\ddot{y}}_{} = -mg$$

$\dfrac{\partial L}{\partial y} = -mg$

$$-\frac{m}{\tan\alpha}\frac{m}{M+m}\frac{\ddot{y}}{\tan\alpha} + \frac{m\ddot{y}}{\sin^2\alpha}$$

$$= \boxed{\frac{m}{\sin^2\alpha}\ddot{y}\left(1 - \frac{m}{m+M}\cos^2\alpha\right) = -mg}$$

The Jacobi integral is conserved:

$$h = \dot{X}\frac{\partial L}{\partial \dot{X}} + \dot{y}\frac{\partial L}{\partial \dot{y}} - L$$

$$= \dot{X}\left[(M+m)\dot{X} + \frac{m}{\tan\alpha}\dot{y}\right] + \dot{y}\left[\frac{m}{\tan\alpha}\dot{X} + \frac{m}{\sin^2\alpha}\dot{y}\right]$$

$$- L$$

$$= \frac{1}{2}(M+m)\dot{X}^2 + \frac{m}{\tan\alpha}\dot{X}\dot{y} + \frac{1}{2}\frac{m}{\sin^2\alpha}\dot{y}^2 + mgy$$

$$\boxed{h = T + V}$$

Now use a Lagrange multiplier to get the constraint forces:

$$L = T - V = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

$$L' = L + \lambda(t)\left((x - X)\tan\alpha - y\right)$$

ⓐ $\dfrac{\partial L}{\partial \dot{X}} = M\dot{X}$

$$\Rightarrow \boxed{M\ddot{X} = \underbrace{-\lambda(t)\tan\alpha}_{\text{constraint force}}}$$

$\dfrac{\partial L}{\partial X} = -\lambda(t)\tan\alpha$

(b) $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\frac{\partial L}{\partial x} = \lambda(t) \tan\alpha$

$$\Longrightarrow \boxed{m\ddot{x} = \underbrace{\lambda(t)\tan\alpha}_{\text{constraint force}}}$$

(c) $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{\partial L}{\partial y} = -mg - \lambda(t)$

$$\Longrightarrow \boxed{m\ddot{y} = -mg - \underbrace{\lambda(t)}_{\text{constraint force}}}$$

(d) $(x - X)\tan\alpha = y$

(a) + (b) $\Longrightarrow$ $M\ddot{X} + m\ddot{x} = 0$ (conservation of x-momentum)

(d) $\Longrightarrow$ $\ddot{y} = (\ddot{x} - \underbrace{\ddot{X}}_{= -\frac{m}{M}\ddot{x}})\tan\alpha = \left(\ddot{x} + \frac{m}{M}\ddot{x}\right)\tan\alpha = \left(1 + \frac{m}{M}\right)\tan\alpha\, \ddot{x}$

$$m\ddot{y} = \frac{m+M}{M}\tan\alpha\, m\ddot{x} = \lambda(t)\frac{m+M}{M}\tan^2\alpha = -mg - \lambda(t)$$

$$\lambda(t)\left(1 + \frac{m+M}{M}\tan^2\alpha\right) = -mg$$

$$\boxed{\lambda = -\frac{mg}{1 + \frac{m+M}{M}\tan^2\alpha}} \longleftarrow \text{independent of time}$$

The work done on m by the constraint forces during dt is

$$\lambda \tan\alpha \, dx - \lambda \, dy = \lambda \underbrace{(\tan\alpha \, dx - dy)}_{\tan\alpha \, dX} = \lambda \tan\alpha \, dX$$

The work done on M by the constraint forces during dt is

$$-\lambda \tan\alpha \, dX$$

The total work done by the constraints is zero.