Phys 503

Homework #2

Solution Set

2.1. Goldstein 3.14

(a) Parabolic orbit ($e = 1$):

$$\frac{1}{r} = \frac{1}{r_0}(1 + \cos\theta) \quad , \quad r_0 = \frac{l^2}{mk} = \begin{pmatrix} \text{radius of circular} \\ \text{orbit with angular} \\ \text{momentum } l \end{pmatrix}$$

$$\boxed{r_{min} = r(\theta = 0) = \frac{r_0}{2}}$$

(b) Parabolic orbit: $0 = E = \frac{1}{2}mv^2 - \frac{k}{r} \Rightarrow v_p = \sqrt{\frac{2k}{mr}}$

Circular orbit : $r = \frac{l^2}{mk}$

$$\frac{1}{2}mv^2 = \frac{l^2}{2mr^2} = \frac{k}{2r}$$

direct consequence of the virial theorem

$$\Rightarrow \quad v_c = \sqrt{\frac{k}{mr}}$$

$$\boxed{v_p = \sqrt{2}\, v_c}$$

2.2. Goldstein 3.18.

Initial orbit $r_i(\theta)$: $\frac{1}{r_i} = \frac{1}{r_0}(1 + e_i \cos\theta)$

$$e_i = \sqrt{1 + \frac{2l^2 E_i}{mk^2}} \qquad r_0 = l^2/mk$$

A radial impulse changes the radial momentum by $\Delta p_r = \int F_r \, dt \equiv S$, leaving the particle's position and angular velocity (and, hence, the angular momentum) fixed.

Final orbit $r_f(\theta)$: $\frac{1}{r_f} = \frac{1}{r_0}\left(1 + e_f \cos(\theta - \underbrace{\theta_f}_{\text{new periastron angle}})\right)$

$$e_f = \sqrt{1 + \frac{2l^2 E_f}{mk^2}} \qquad r_0 = l^2/mk \text{ is unchanged}$$

The impulse occurs at periastron of initial orbit $(\theta = 0)$, so we have

$$\frac{1+e_i}{r_0} = \frac{1}{r_i(\theta=0)} = \frac{1}{r_f(\theta=0)} = \frac{1 + e_f \cos\theta_f}{r_0}$$

$$\Longrightarrow \boxed{e_i = e_f \cos\theta_f}$$

Orbital energy: $E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{\text{unchanged}}$

At impulse $\longrightarrow$ $\frac{1}{2}m\dot{r}^2$ $= 0$ for initial orbit (since periastron)

$= \frac{p_r^2}{2m} = \frac{(\Delta p_r)^2}{2m} = \frac{S^2}{2m}$ for final orbit

②

$$\therefore \quad E_f = E_i + \underbrace{\frac{S^2}{2m}}_{\Delta E} \implies e_f^2 = e_i^2 + \frac{2l^2 \cdot \Delta E}{mk^2}$$

$$\boxed{e_f^2 = e_i^2 + \left(\frac{lS}{mk}\right)^2}$$

Solution: $\boxed{e_f^2 = e_i^2 + \left(\frac{lS}{mk}\right)^2 \geq e_i^2}$

$$e_i^2 \underbrace{\left(\frac{1}{\cos^2\theta_f} - 1\right)}_{\tan^2\theta_f} = \left(\frac{lS}{mk}\right)^2$$

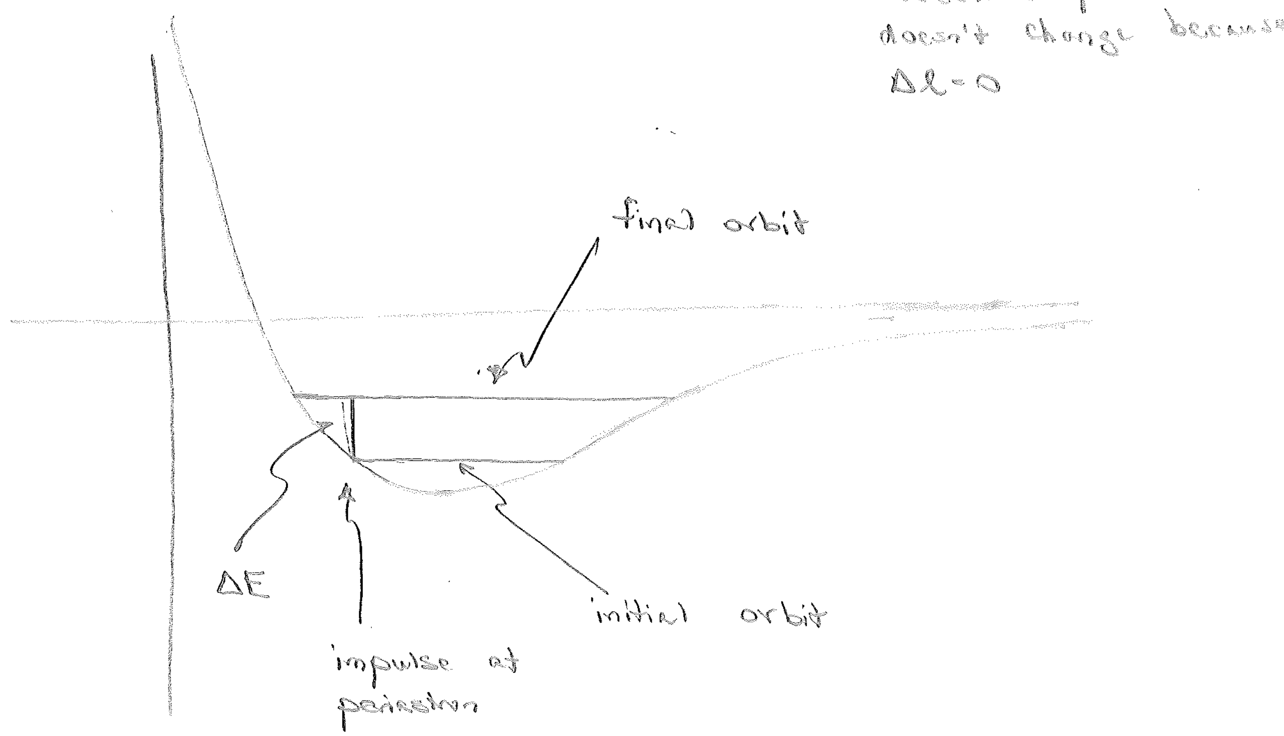
$$\boxed{\tan^2\theta_f = \frac{1}{e_i^2}\left(\frac{lS}{mk}\right)^2}$$

$$\boxed{a_f = \frac{r_0}{1 - e_f^2} = \frac{l^2/mk}{1 - e_i^2 - (lS/mk)^2} \geq a_i}$$

one can also see this from $a = k/2|E|$

$$(r_{min})_f \equiv a_f(1 - e_f) = \frac{r_0}{1 + e_f} \leq (r_{min})_i$$

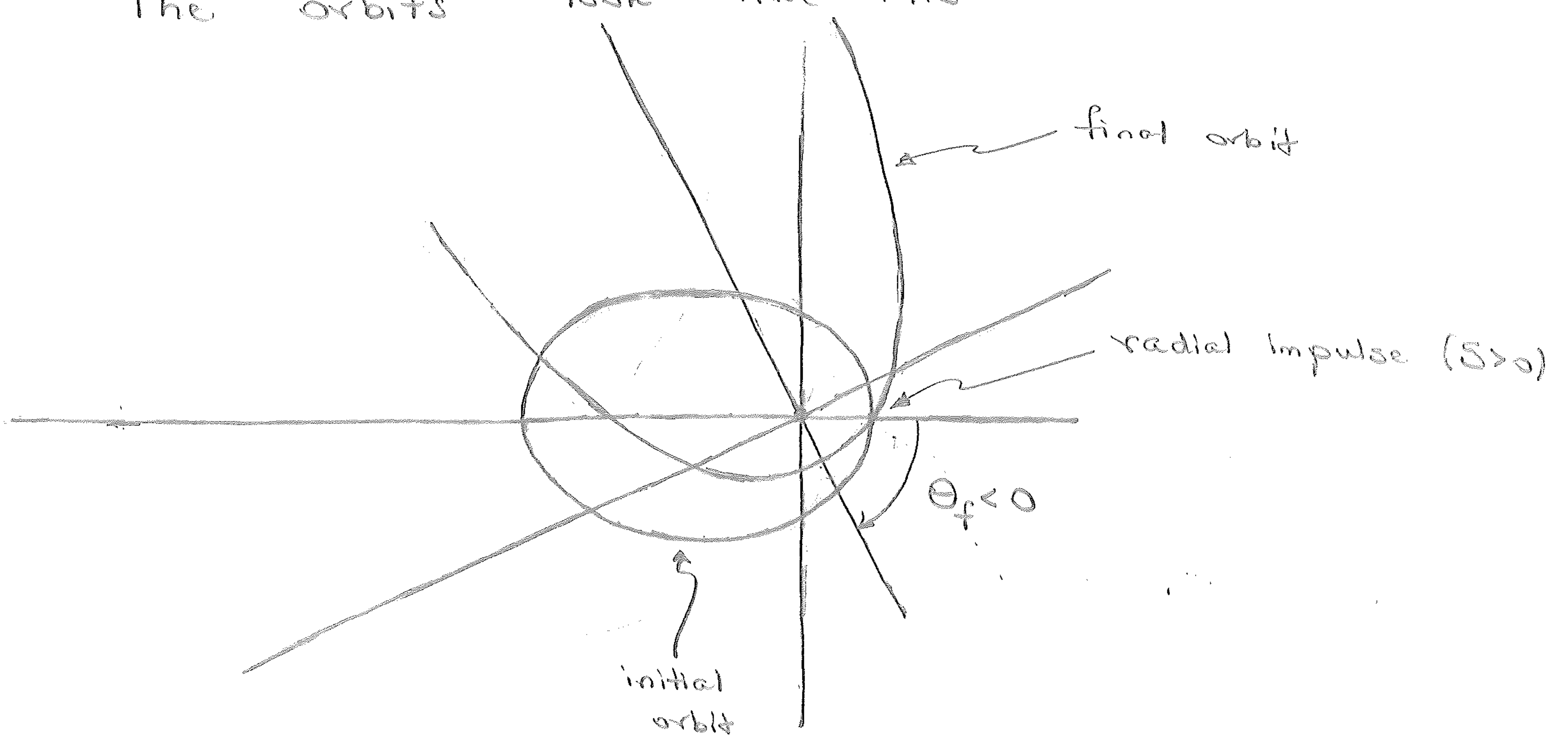
Notice that this solution does not give the sign of θ_f, which depends on the sign of S. We can get that from an effective-potential diagram.



If $S > 0$, then $\dot{r} > 0$ after impulse, so particle moves outward toward apiastron, which means that the new periastron would have occurred earlier, so $\theta_f < 0$.

$$\therefore \boxed{\tan\theta_f = -\frac{lS}{e_i m k}}$$

The orbits look like this:



Notice that if we go back to the 2-body problem, where

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{p} = \frac{m_1\vec{p}_2 - m_2\vec{p}_1}{m_1+m_2},$$

An impulse $\vec{S}$ to m_2 results in

$$\Delta\vec{p}_2 = \vec{S} \Rightarrow \Delta\vec{P} = \vec{S},$$

$$\Delta\vec{p} = \frac{m_1}{m_1+m_2}\vec{S} = \frac{\mu}{m_2}\vec{S}$$

If $m_1 \gg m_2$, then $\Delta\vec{p} = \vec{S}$, but if $m_1 = m_2$, then $\Delta\vec{p} = \frac{1}{2}\vec{S}$.

2.3. Goldstein 3.21 and 3.22

3.21

The equations of motion are

$$\dot{\theta} = \frac{l}{mr^2}$$

$$E = \frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} + \frac{h}{r^2}$$

Define $l'^2 = l^2 + 2mh$, so that

$$\boxed{E = \frac{1}{2} m\dot{r}^2 + \frac{l'^2}{2mr^2} - \frac{k}{r}.}$$

Let $\varphi = \theta - \Omega$, where Ω is a time-dependent precession angle, chosen so that

$$\boxed{\dot{\varphi} = \frac{l'}{mr^2}.}$$

This requires

$$\boxed{\dot{\Omega} = \dot{\theta} - \dot{\varphi} = \frac{l - l'}{mr^2}}$$

These equations are identical to the equations for the Kepler problem, so the orbit is an ellipse in r and φ. Of course, the $r\theta$ motion won't look much like a precessing ellipse unless $\dot{\Omega}$ is small (h/r^2 is a weak perturbation) or is nearly constant (nearly circular orbit).

The <u>average</u> precession rate is

$$\dot{\Omega}_{av} = \underset{\substack{\uparrow \\ \text{period of Keplerian orbit}}}{\frac{1}{\tau}} \int_0^\tau dt\, \dot{\Omega} = \frac{1}{\tau}\int_0^\tau dt \underbrace{\frac{\ell - \ell'}{mr^2}}_{\dot{\varphi}\frac{\ell-\ell'}{\ell'}} = \frac{\ell/\ell' - 1}{\tau}\underbrace{\int_0^\tau dt\, \dot{\varphi}}_{2\pi}$$

$$\boxed{\dot{\Omega}_{av} = \frac{2\pi}{\tau}\left(\frac{\ell}{\ell'} - 1\right)}$$

For a weak perturbation, $2mh = \ell'^2 - \ell^2 = (\ell' - \ell)\overbrace{(\ell' + \ell)}^{\simeq 2\ell}$,

so $\ell - \ell' \simeq -mh/\ell$ and $\ell/\ell' - 1 = \frac{\ell - \ell'}{\ell'} \simeq -\frac{mh}{\ell^2}$ and

$$\boxed{\dot{\Omega}_{av} = -\frac{2\pi}{\tau}\frac{mh}{\ell^2}}$$

The small dimensionless parameter here is

$$\boxed{\frac{|\dot{\Omega}_{av}|}{2\pi/\tau} = \frac{mh}{\ell^2} = \frac{h}{kr_0} \equiv \eta \ll 1}$$

Numbers for Mercury: I leave that to you.

3.22. Turning on h>0 affects the radial motion just like an increase in l, but the angular motion is unaffected, i.e., still has the original smaller l, so it lags behind the radial motion, thus producing a backward precession of the orbit,

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Alternative approach to 3.14: $f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2} + \frac{2h}{r^3}$

Orbital equation: $\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f(u) = \frac{mk}{l^2} - \frac{2mh}{l^2} u$

$$\frac{d^2u}{d\theta^2} + \underbrace{\left(1 + \frac{2mh}{l^2}\right)}_{\beta^2} u = \frac{mk}{l^2}$$

$$\frac{d^2}{d\theta^2}\left(u - \frac{mk}{\beta^2 l^2}\right) + \beta^2\left(u - \frac{mk}{\beta^2 l^2}\right) = 0$$

$$\Longrightarrow \quad \frac{1}{r} = u = \frac{mk}{\beta^2 l^2}\left(1 + e\cos\beta\theta\right)$$

Periastron at $\beta\theta = 2n\pi$, so the periastron shift per orbit is $\Delta\theta = \frac{2\pi}{\beta} - 2\pi = 2\pi(\beta^{-1} - 1)$, and if the perturbation is weak, the periastron shift per unit time is

$$\dot{\Omega} = \Delta\theta / \tau = \frac{2\pi}{\tau}\left(\frac{1}{\beta} - 1\right) = -\frac{2\pi}{\tau}\frac{mh}{l^2}$$

$$\beta = 1 + \frac{mh}{l^2} \; ; \quad \beta^{-1} = 1 - \frac{mh}{l^2}$$

24. $V(r) = -\dfrac{k e^{-\alpha r}}{r}$

(a) $f(r) = -\dfrac{\partial V}{\partial r} = k\left(-\dfrac{e^{-\alpha r}}{r^2} - \dfrac{\alpha e^{-\alpha r}}{r}\right)$

$$= -\frac{k e^{-\alpha r}}{r^2}(1+\alpha r)$$

$$\boxed{f(r) = -\frac{k e^{-\alpha r}}{r^2}(1+\alpha r)}$$

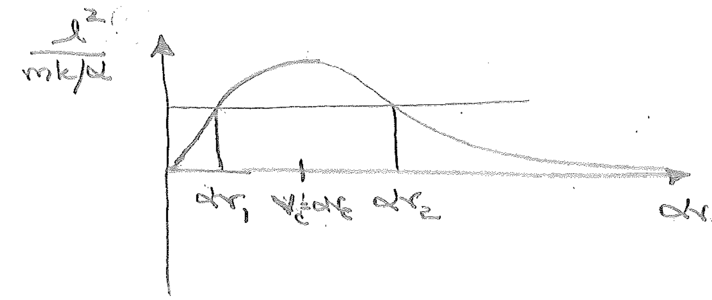
(b) Circular orbit condition: $f(r_0) = -\dfrac{l^2}{m r_0^3} = -\dfrac{k e^{-\alpha r_0}}{r_0^2}(1+\alpha r_0)$

$$\Longrightarrow l^2 = m k r_0 (1+\alpha r_0)\, e^{-\alpha r_0} = \frac{mk}{\alpha}\alpha r_0 (1+\alpha r_0)\, e^{-\alpha r_0}$$

$$= \frac{mk}{\alpha} v(1+v)\, e^{-v}$$

$v = \alpha r_0$

Plot $l^2/(mk/\alpha)$ vs. αr_0



There is a maximum value of $l = l_c$, corresponding to $r = r_c$ ($v = v_c$), for which there are circular orbits:

$$0 = \frac{dl^2}{dv} = \frac{mk}{\alpha}\Big(\underbrace{(1+2v) - (v+v^2)}_{1+v-v^2}\Big)e^{-v} \Rightarrow 0 = v_c^2 - v_c - 1$$

$$\Rightarrow v_c = \frac{+1 \pm \sqrt{1+4}}{2} = \boxed{\frac{1+\sqrt{5}}{2} = 1.62 = v_c = \alpha r_c}$$

lower sign gives a negative v_c.

$$\ell_c^2 = \frac{mk}{\alpha}\underbrace{r_c(1+r_c)}e^{-r_c} = \frac{mk}{\alpha}(2+\sqrt{5})\,e^{-r_c}$$

$$r_c^2 + r_c = r_c^2 - r_c + 2r_c = 1 + 2r_c = 2+\sqrt{5}$$

$$\boxed{\ell_c = r_c(1+r_c)e^{-r_c}\sqrt{\frac{mk}{\alpha}} = 0.84\sqrt{\frac{mk}{\alpha}}}$$

For $\ell > \ell_c$, there are no circular orbits.

For $0 < \ell < \ell_c$, there are <u>two</u> circular orbits, at the two roots, $r_1 < r_c$ and $r_2 > r_c$, of the circular-orbit condition. $\ell^2 = \frac{mk}{\alpha}\alpha r_0(1+\alpha r_0)\,e^{-\alpha r_0}$.

(c) Stability: r_1 is the inner orbit; r_2 is the outer orbit.

Method 1: ℓ is increasing at $r_1 \Rightarrow$ r_1 is stable

ℓ is decreasing at $r_2 \Rightarrow$ r_2 is unstable

(Not advisable) Method 2: Textbook criterion for stability: $\frac{df}{dr} < -3\frac{f}{r} \Leftrightarrow \frac{df}{dr} < -3\frac{f}{r}$

$\Leftrightarrow \frac{r}{f}\frac{df}{dr} > -3$

$\Leftrightarrow \frac{r}{f}\frac{df}{dr} + 3 > 0$

$$f = -k\alpha^2\frac{e^{-r}}{r^2}(1+r)$$

$$\frac{df}{dr} = -k\alpha^2 e^{-r}\underbrace{\left(-\frac{1+r}{r^2} + \frac{1}{r^2} - \frac{2(1+r)}{r^3}\right)}_{=\frac{-r^2-2r-2}{r^3}}$$

$$\frac{df}{dr} = +k\alpha^2 e^{-r}\,\frac{r^2+2r+2}{r^3}$$

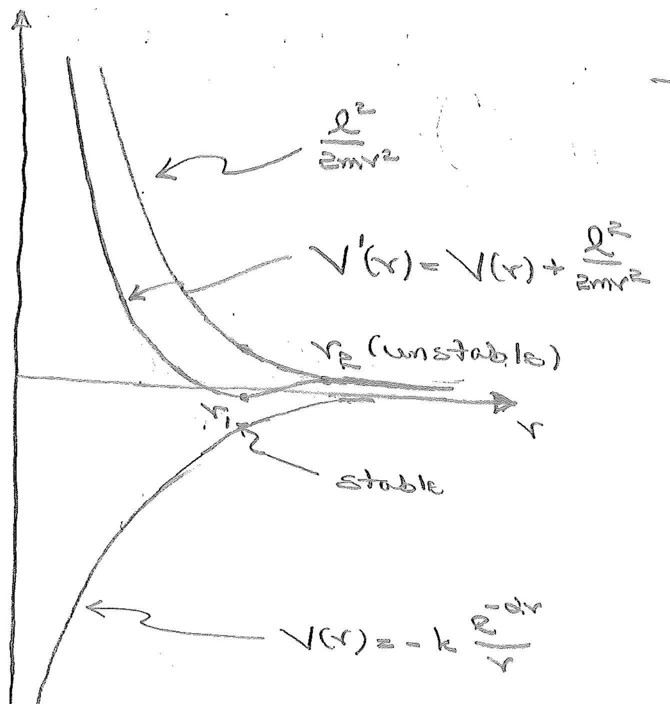
$$\frac{v}{f}\frac{df}{dv} = \frac{k\alpha^2 e^{-v}\,\dfrac{v^2+2v+2}{v^3}}{-k\alpha^2 e^{-v}\,\dfrac{1+v}{v^3}} = -\,\frac{v^2+2v+2}{1+v}$$

$$\frac{v}{f}\frac{df}{dv} + 3 = -\,\frac{v^2+2v+2-3(1+v)}{1+v} = -\,\frac{v^2-v-1}{1+v}$$

$v^2 - v - 1$ has roots at $\frac{1}{2}(-1 \pm \sqrt{5})$. Between the roots $v^2 - v - 1 < 0$; outside the roots, $v^2 - 2v - 1 > 0$.

$\therefore$ at the inner orbit r_1, where $v_1 = \alpha r_1 < v_c = \frac{1}{2}(-1+\sqrt{5})$, $\frac{v}{f}\frac{df}{dv} + 3 > 0$, and the orbit is stable; at the outer orbit r_2, where $v_2 = \alpha r_2 > v_c$, $\frac{v}{f}\frac{df}{dv} + 3 < 0$, and the orbit is unstable.

Method 3: Effective-potential diagram



The centrifugal potential dominates $V'(r)$ for small r and large r. The only way for $V'(r)$ to have two extrema is to have a minimum at a small $r = r_1$ and a maximum at larger $r = r_2$.

(d) $E = V'(r) = -k\,\dfrac{e^{-\alpha r_0}}{r_0} + \underbrace{\dfrac{\ell^2}{2mr_0^2}}$

$$\frac{r_0}{2}\frac{\ell^2}{mr_0^3} = +\frac{r_0}{2}\,k\,\frac{e^{-\alpha r_0}}{r_0^2}(1+\alpha r_0)$$

$$= +\frac{1}{2}k\,\frac{e^{-\alpha r_0}}{r_0}(1+\alpha r_0)$$

$$= -k\,\frac{e^{-\alpha r_0}}{r_0}\left(1 - \frac{1}{2}(1+\alpha r_0)\right)$$

$$= +k\,\frac{e^{-\alpha r_0}}{2r_0}(\alpha r_0 - 1)$$

$$E > 0 \iff \alpha r_0 > 1$$

$$\boxed{\text{There are stable circular orbits as long as } 1 < \alpha r_1 = \nu_1 < \nu_c\text{, i.e., if } 2mk/\alpha e < \ell^2 < \ell_c^2.}$$

No-math answer: r_2 always has positive energy; since r_1 and r_2 coalesce as $\ell = \ell_c$, there must be a range of values for which r_1 has positive energy.

R.5. Goldstein 3.28

Review the derivation of the Runge-Lenz vector:

$$\dot{\vec{p}} = f(r)\,\vec{e}_r\,, \qquad f(r) = -\frac{\partial V}{\partial r}$$

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \dot{\vec{p}}\times\vec{L} = f(r)\,\vec{e}_r\times\vec{L}$$

$$\vec{e}_r\times\vec{L} = \vec{e}_r\times(\vec{r}\times\vec{p}) = mr\,\underbrace{\vec{e}_r\times(\vec{e}_r\times\dot{\vec{r}})}_{\vec{e}_r\dot{r}-\dot{\vec{r}} = -r\frac{d\vec{e}_r}{dt}} = -mr^2\frac{d\vec{e}_r}{dt}$$

$$\therefore \boxed{\vec{e}_r\times\vec{L} = -mr^2\frac{d\vec{e}_r}{dt}} \longleftrightarrow$$ True whether or not $\vec{L}$ is conserved

$$\boxed{\frac{d}{dt}(\vec{p}\times\vec{L}) = -mr^2 f(r)\frac{d\vec{e}_r}{dt}} \longleftrightarrow$$ Assumes $\vec{L}$ is conserved.

(a) $$\dot{\vec{p}} = f(r)\,\vec{e}_r + \underbrace{\frac{g}{c}\,\dot{\vec{r}}\times\vec{B}}\,, \qquad \vec{B} = \frac{b}{r^3}\vec{r} = \frac{b}{r^2}\vec{e}_r$$

$$\downarrow$$

$$\frac{b}{r^3}\dot{\vec{r}}\times\vec{r} = -\frac{b}{mr^3}\vec{r}\times\vec{p} = -\frac{b}{mr^3}\vec{L}$$

$$\boxed{\dot{\vec{p}} = f(r)\,\vec{e}_r - \frac{gb}{mcr^3}\vec{L}}$$ Equation of motion

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \cancel{\dot{\vec{r}}\times\vec{p}} + \underbrace{\vec{r}\times\dot{\vec{p}}}_{\downarrow} = \frac{gb}{c}\frac{d\vec{e}_r}{dt}$$

$$-\frac{gb}{mcr^3}\vec{r}\times\vec{L} = -\frac{gb}{mcr^2}\vec{e}_r\times\vec{L}$$

$$= \frac{gb}{c}\frac{d\vec{e}_r}{dt}$$

$$\therefore \boxed{\frac{d}{dt}\underbrace{\left(\vec{L} - \frac{gb}{c}\vec{e}_r\right)}_{\equiv \vec{D}} = 0}$$

$\longleftarrow$ True for any $f(r)$

(b) $$\frac{d}{dt}(\vec{p}\times\vec{D}) = \dot{\vec{p}}\times\vec{D}$$

$$= \left(f(r)\vec{e}_r - \frac{gb}{mcr^3}\vec{L}\right)\times\left(\vec{L} - \frac{gb}{c}\vec{e}_r\right)$$

$$= \left(f(r) - \frac{1}{mr^3}\left(\frac{gb}{c}\right)^2\right)\vec{e}_r\times\vec{L}$$

$$= \left(-mr^2 f(r) + \frac{1}{r}\left(\frac{gb}{c}\right)^2\right)\frac{d\vec{e}_r}{dt}$$

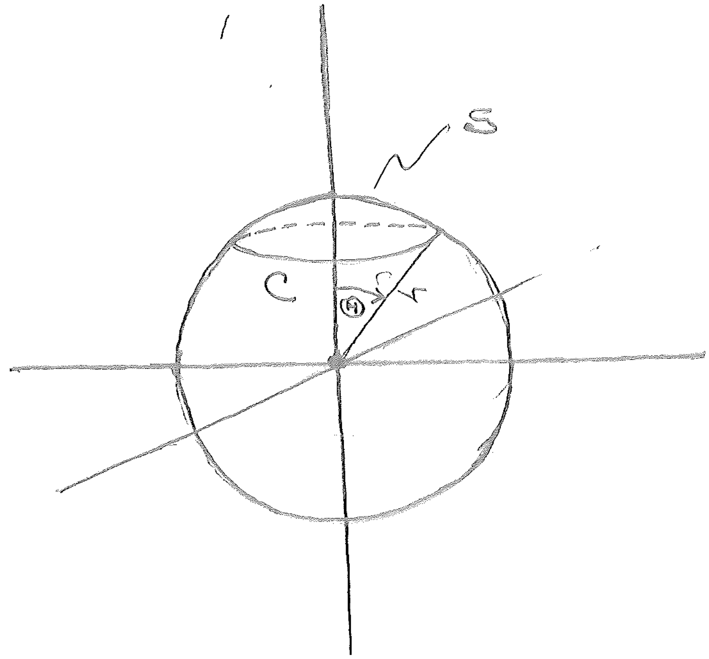
It looks to me that there will be a conserved quantity if the quantity in parentheses is a constant, i.e.,

$$-mr^2 f(r) + \frac{1}{r}\left(\frac{qb}{c}\right)^2 = km$$

$$\Rightarrow \boxed{f(r) = -\frac{k}{r^2} + \frac{(qb/c)^2}{mr^3}}$$

Then $\boxed{\frac{d}{dt}\left(\vec{p}\times\vec{D} - mk\vec{e}_r\right) = 0}$

You might have wondered why you're not asked to do part (a) by finding the conserved momentum associated with an angular coordinate. The reason is that a magnetic monopole is not a solution of the Maxwell equations ($\nabla\cdot B \neq 0$ at $r=0$), so technically there is no vector potential $\vec{A}$ such that $\nabla\times\vec{A} = \vec{B}$. Nonetheless, one can introduce a vector potential that works everywhere except along a line from $r=0$ to $r=\infty$. Pick an arbitrary z-direction, and let this line be the negative z-axis.



Apply Stokes's theorem to the "cap" on the sphere defined by $\theta \le \Theta$.

$$A_\varphi 2\pi r \sin\Theta = \int_C \vec{A}\cdot d\vec{s} = \int_S \vec{B}\cdot d\vec{s} = b\int \frac{1}{\cancel{r^2}}\cancel{r^2}\sin\theta \, d\theta \, d\varphi$$

$$= 2\pi b \int_0^\Theta d\theta \sin\theta$$

$$= 2\pi b\left(1 - \cos\Theta\right)$$

$$\Rightarrow A_\varphi = \frac{b}{r}\,\frac{1-\cos\theta}{\sin\theta}$$ ← singular on negative z-axis ($\theta = \pi$)

One easily verifies that

$$\vec{A} = \frac{b}{r}\,\frac{1-\cos\theta}{\sin\theta}\,\vec{e}_\varphi$$

satisfies $\nabla\times\vec{A} = \frac{b}{r^2}\vec{e}_r = \vec{B}$. Thus the Lagrangian of the particle is

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(r) + \frac{q}{c}\vec{A}\cdot\dot{\vec{r}}$$

In spherical coordinates

$$\vec{A}\cdot\vec{r} = A_\varphi v_\varphi = \frac{b}{r}\,\frac{1-\cos\theta}{\sin\theta}\, r\sin\theta\,\dot{\varphi} = b(1-\cos\theta)\dot{\varphi},$$

so the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2$$

$$- V(r) + \frac{gb}{c}(1-\cos\theta)\dot{\varphi}$$

The azimuthal coordinate φ is cyclic, so

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \underbrace{mr^2\sin^2\theta\,\dot{\varphi}}_{L_z = \vec{L}\cdot\vec{e}_z} + \frac{gb}{c}(1-\cos\theta)$$

is conserved. We can write p_φ as

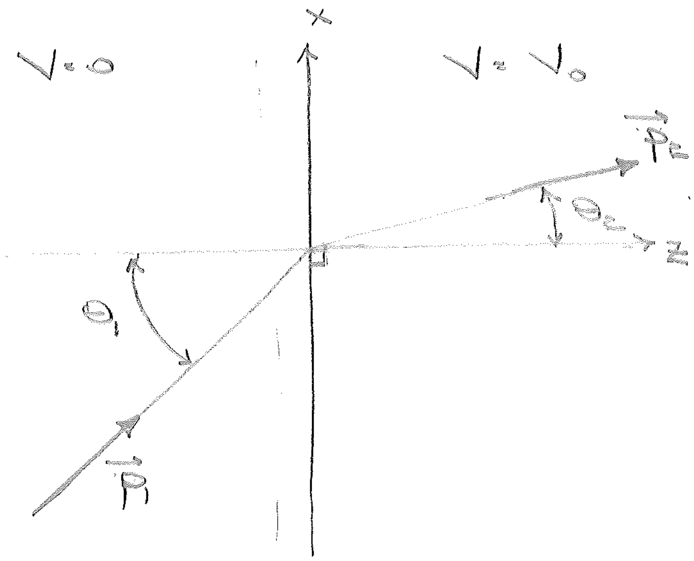
$$p_\varphi = \vec{L}\cdot\vec{e}_z + \frac{gb}{c} - \frac{gb}{c}\vec{e}_r\cdot\vec{e}_z$$

$$= \left(\vec{L} - \frac{gb}{c}\vec{e}_r\right)\cdot\vec{e}_z + \underbrace{\frac{gb}{c}}_{\text{constant}}$$

The z-direction was arbitrary, so $\vec{L} - \frac{gb}{c}\vec{e}_r$ is conserved.

2.6. Goldstein 3.32

What happens to a particle encountering a potential drop from $V=0$ to $V=-V_0$?



At the boundary the particle receives an impulsive force in the z-direction that increases its z-momentum. One would try to calculate the change in momentum from $\Delta p_z = \int F_z \, dt$, but one quickly runs into trouble from the δ-character of the force. It's better to use conservation laws:

Refraction problem

Energy Conservation : $E = \dfrac{p_1^2}{2m} = \dfrac{p_2^2}{2m} - V_0$

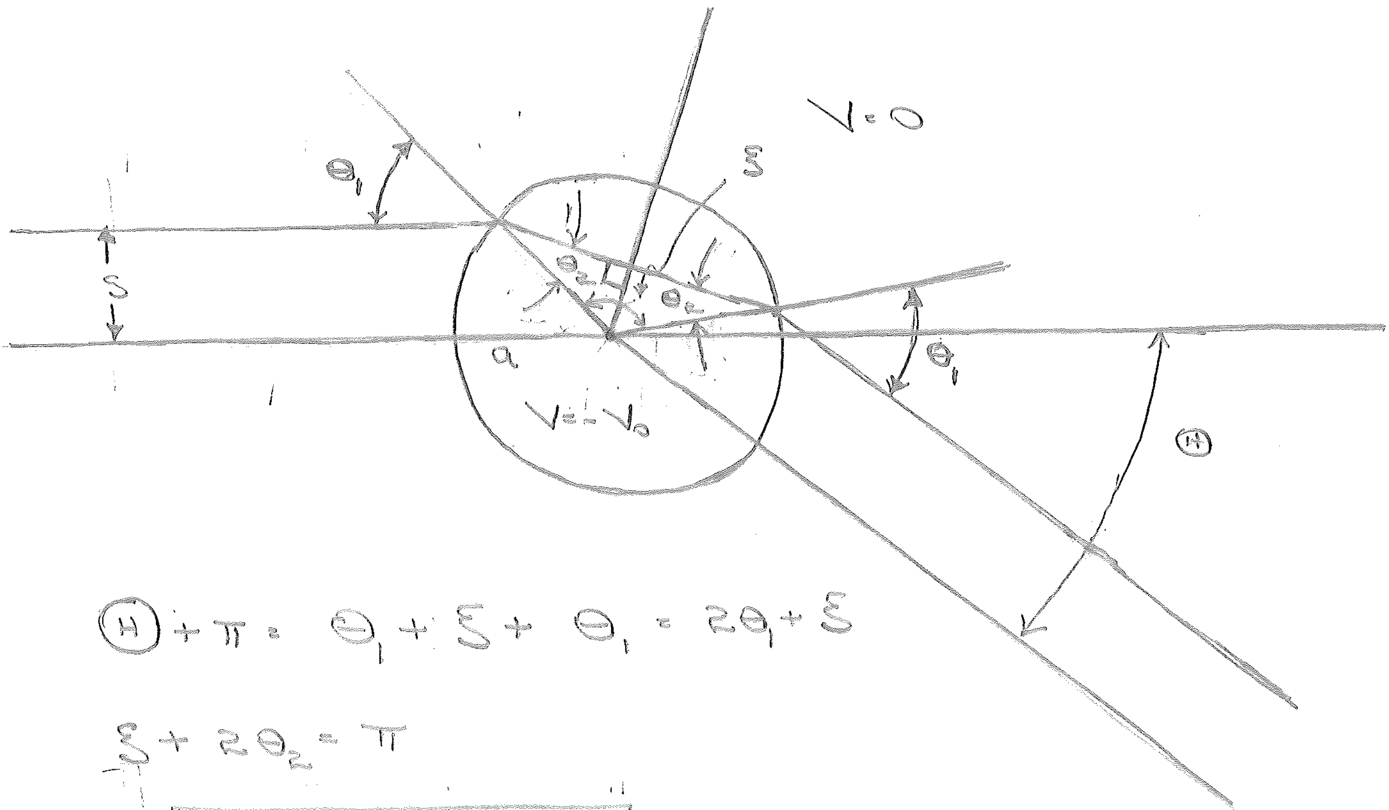
$E = \hbar\omega_1 = \hbar\omega_2$

x-momentum conservation : $\underset{\sqrt{2mE}}{\underset{\downarrow}{p_1}} \sin\theta_1 = \underset{\sqrt{2m(E+V_0)}}{\underset{\downarrow}{p_2}} \sin\theta_2$

$\hbar \underset{\frac{\omega_1}{c}}{\underset{\downarrow}{k_1}} \sin\theta_1 = \hbar \underset{\frac{\omega_2 n}{c}}{\underset{\downarrow}{k_2}} \sin\theta_2$

$$\Longrightarrow \boxed{\sin\theta_1 = \underbrace{\sqrt{\frac{E+V_0}{V_0}}}_{\equiv n} \sin\theta_2}$$

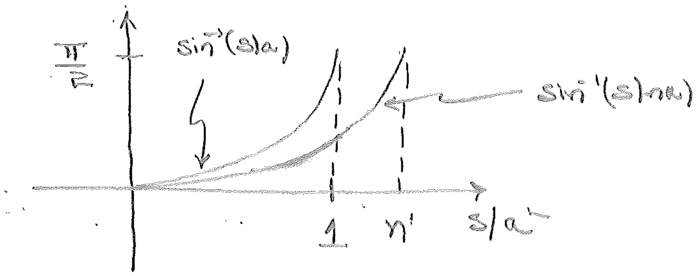
Snell's law



$$\Theta + \pi = \theta_1 + \xi + \theta_1 = 2\theta_1 + \xi$$

$$\xi + 2\theta_2 = \pi$$

$$\Longrightarrow \boxed{\begin{aligned} \Theta &= 2(\theta_1 - \theta_2) \\ a \sin\theta_1 &= s \\ \sin\theta_1 &= n \sin\theta_2 \end{aligned}} \Longrightarrow \boxed{\Theta = 2\left(\sin^{-1}(s/a) - \sin^{-1}(s/na)\right)}$$



$\Theta(s)$ runs monotonically from $\Theta = 0$ at $s = 0$ to

$\Theta = \Theta_{max} = 2\left(\frac{\pi}{2} - \sin^{-1}(n^{-1})\right)$ at $s = a$. $\left[\cos(\Theta_{max}/2) = \frac{1}{n}\right]$

$$\sigma(\Theta) = \frac{s}{\sin\Theta} \frac{ds}{d\Theta}$$

The hard part is to invert $\Theta(s)$ to get $s(\Theta)$

$$\sin \Theta \, d\Theta = 2 \sin(\Theta/2) \cos(\Theta/2) d\Theta = -4 \cos(\Theta/2) \, d \cos(\Theta/2)$$

$$s \, ds = a^2 \sin\theta_1 \, d \sin\theta_1 = \frac{1}{2} a^2 \, d \sin^2\theta_1$$

$$\sigma(\Theta) = \frac{\frac{1}{2} a^2 \, d \sin^2\theta_1}{-4 \cos(\Theta/2) \, d \cos(\Theta/2)} = -\frac{1}{8} a^2 \frac{1}{\cos(\Theta/2)} \frac{d \sin^2\theta_1}{d \cos(\Theta/2)}$$

$$\cos(\Theta/2) = \cos(\theta_1 - \theta_2)$$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

$$= \sqrt{1 - \sin^2\theta_1} \sqrt{1 - \frac{\sin^2\theta_1}{n^2}} + \frac{\sin^2\theta_1}{n}$$

$$x = \sin^2\theta_1 \quad ; \quad c = \cos(\Theta/2)$$

$$\left(c - \frac{x}{n}\right)^2 = (1 - x)\left(1 - \frac{x}{n^2}\right)$$

$$c^2 - \frac{2c}{n} x + \cancel{\frac{x^2}{n^2}} = 1 - x - \frac{x}{n^2} + \cancel{\frac{x^2}{n^2}}$$

$$x \left(1 + \frac{1}{n^2} - \frac{2c}{n}\right) = 1 - c^2 = \sin^2(\Theta/2)$$

$$x \left(1 + n^2 - 2n \cos(\Theta/2)\right) = n^2 \sin^2(\Theta/2)$$

$$\boxed{\left(\frac{s}{a}\right)^2 = \sin^2\theta_1 = x = \frac{n^2 \sin^2(\Theta/2)}{1 + n^2 - 2n \cos(\Theta/2)}}$$

$$\sigma(\Theta) = -\frac{a^2}{8}\,\frac{1}{\cos(\Theta/2)}\,\frac{dx}{dy}\,, \qquad y = \cos(\Theta/2)$$

$$x = \frac{n^2(1-y^2)}{1+n^2-2ny}$$

$$\frac{dx}{dy} = \frac{-2n^2y}{1+n^2-2ny} - \frac{n^2(1-y^2)(-2n)}{(1+n^2-2ny)^2}$$

$$= 2n^2\,\frac{-y(1+n^2-2ny) + n(1-y^2)}{(1+n^2-2ny)^2}$$

$$= 2n^2\,\frac{-y(1+n^2) + ny^2 + n}{(1+n^2-2ny)^2}$$

$$= 2n^2\,\frac{(ny-1)(y-n)}{(1+n^2-2ny)^2}$$

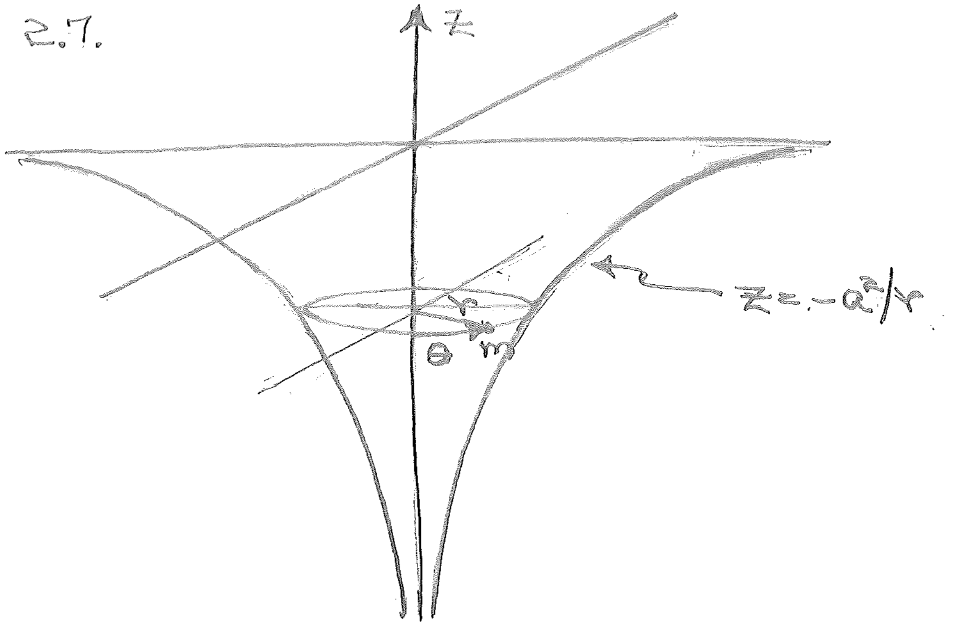
$$\boxed{\sigma(\Theta) = \frac{a^2n^2}{4}\,\frac{(ny-1)(n-y)}{y(1+n^2-2ny)^2}\,, \qquad y = \cos(\Theta/2)}$$

Note that $\sigma(\Theta_{max}) = \sigma(y = n^{-1}) = 0$

Total cross section:

$$\sigma_T = 2\pi\int_0^{\Theta_{max}} \underbrace{d\Theta\,\sin\Theta\,\sigma(\Theta)}_{s\,ds} = 2\pi\int_0^{a} s\,ds = \boxed{\pi a^2 = \sigma_T}$$

2.7.



A particle of mass m slides without friction on a cylindrically symmetric funnel defined by $z = -a^2/r$

(a) Give a Lagrangian in polar coördinates r and θ. In what limit does this system mimic an attractive $1/r^2$ central force.

(b) Derive the equation of motion for r.

(c) What is the radius r_0 of a circular orbit with angular momentum l?

(d) Derive an equation for the shape of an orbit in terms of $u = 1/r$.

(e) Apply the equation of part (d) to a nearly circular orbit to determine the precession of the point of minimum r.

(a) $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2\left(1+\frac{a^4}{r^4}\right) + r^2\dot{\theta}^2\right)$

$\dot{z} = \frac{a^2}{r^2}\dot{r}$

$V = mgz = -\frac{mga^2}{r}$

$$\boxed{L = T - V = \frac{1}{2}m\dot{r}^2\left(1+\frac{a^4}{r^4}\right) + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{mga^2}{r}}$$

Limit of $-1/r$ potential: $a \to 0$, $g \to \infty$, with $ga^2 =$ constant

(b) $\frac{\partial L}{\partial \dot{\theta}} = \boxed{mr^2\dot{\theta} = l}$ (conserved angular momentum)

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\left(1+\frac{a^4}{r^4}\right)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}\left(1+\frac{a^4}{r^4}\right) - 4m\dot{r}^2\frac{a^4}{r^5}$

$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{mga^2}{r^2} - \frac{4}{2}m\dot{r}^2\frac{a^4}{r^5}$

$m\ddot{r}\left(1+\frac{a^4}{r^4}\right) - 4m\dot{r}^2\frac{a^4}{r^5} = mr\dot{\theta}^2 - \frac{mga^2}{r^2} - 2m\dot{r}^2\frac{a^4}{r^5}$

$$\boxed{m\ddot{r}\left(1+\frac{a^4}{r^4}\right) - 2m\dot{r}^2\frac{a^4}{r^5} = +\frac{l^2}{mr^3} - \frac{mga^2}{r^2}}$$

(c) Circular orbit: $\dot{r} = 0 = \ddot{r} \implies \frac{l^2}{mr_0^3} = \frac{mga^2}{r_0^2}$

$$\implies \boxed{r_0 = \frac{l^2}{m^2ga^2}}$$

(a) $\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{l}{mr^2}\frac{d}{d\theta} = \frac{l}{m}u^2\frac{d}{d\theta}$

$\dot{r} = \frac{l}{mr^2}\frac{dr}{d\theta} = -\frac{l}{m}\frac{du}{d\theta}$; $\ddot{r} = \frac{l}{mr^2}\frac{d\dot{r}}{d\theta} = -\frac{l^2}{m^2r^2}\frac{d^2u}{d\theta^2} = -\frac{l^2}{m^2}u^2\frac{d^2u}{d\theta^2}$

$$+\frac{\cancel{l^2}}{\cancel{m}}\cancel{u^2}\frac{d^2u}{d\theta^2}\left(1+a^4u^4\right) + \frac{2\cancel{l^2}}{\cancel{m}}\left(\frac{du}{d\theta}\right)^2 a^4u^{\cancel{5}\,3}$$

$$= -\frac{\cancel{l^2}}{\cancel{m}}u^{\cancel{3}} + mga^2\cancel{r^2}\frac{m}{l^2}$$

$$\boxed{\frac{d^2u}{d\theta^2}\left(1+a^4u^4\right) + 2a^4u^3\left(\frac{du}{d\theta}\right)^2 + u = \frac{m^2ga^2}{l^2}}$$

Circular orbit: $u = u_0 = \frac{m^2ga^2}{l^2}$

(e) Nearly circular orbit:

$$u = u_0 + \delta u$$

$\uparrow$ work to linear order

$$\frac{d^2\delta u}{d\theta^2}\left(1+a^4u_0^4\right) + \cancel{u_0} + \delta u = \cancel{\frac{m^2ga^2}{l^2}}$$

$$\frac{d^2\delta u}{d\theta^2} + \underbrace{\boxed{\frac{1}{1+a^4u_0^4}}}_{\omega^2}\delta u = 0$$

$$\delta u = A\cos\omega\theta$$

Minimum $r \Longleftrightarrow$ Maximum $u \Longrightarrow \omega\theta_n = 2n\pi$

Each orbit the minimum r advances in θ by

$$\boxed{\begin{aligned} \Theta_{n+1} - (\Theta_n + 2\pi) &= \frac{2(n+1)\pi}{\omega} - \frac{2n\pi}{\omega} - 2\pi \\ &= 2\pi\left(\frac{1}{\omega} - 1\right) \\ &= 2\pi\left(\sqrt{1 + a^4 u_0^4} - 1\right) \end{aligned}}$$