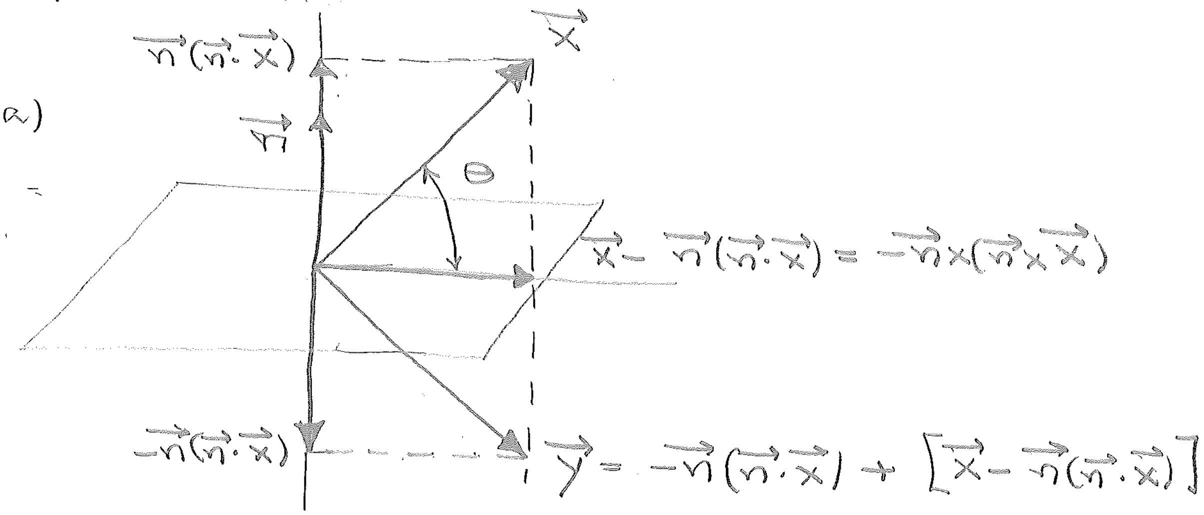
Phys 503

Homework # 3

Solution Set

3.1 Goldstein 4.18

(a)



$$\boxed{\vec{y} = \vec{x} - 2\vec{n}(\vec{n}\cdot\vec{x})}$$

(b) Active transformation: $\vec{y} = \tilde{A}\vec{x} \iff y_j = \tilde{A}_{jk}x_k$

$$y_j = x_j - 2n_j n_k x_k = \underbrace{(\delta_{jk} - 2n_j n_k)}_{\tilde{A}_{jk} = A_{kj}} x_k$$

$$\therefore \boxed{A_{jk} = \delta_{jk} - 2n_j n_k}$$

Notice, that $A = \tilde{A}$, i.e., $A_{jk} = \tilde{A}_{jk} = A_{kj}$, i.e., A is symmetric.

Orthogonality: $A_{jl}A_{kl} = (\delta_{jl}-2n_jn_l)(\delta_{kl}-2n_kn_l)$

$$= \underbrace{\delta_{jl}\delta_{kl}}_{\delta_{jk}} - 2\underbrace{\delta_{jl}n_kn_l}_{n_kn_j} - 2\underbrace{\delta_{kl}n_jn_l}_{n_jn_k} + 4n_jn_k\underbrace{n_ln_l}_{1}$$

$$= \delta_{jk} - 4n_jn_k + 4n_jn_k$$

$$\boxed{A_{jl}A_{kl} = \delta_{jk}}$$

Improper: $\det A = \epsilon_{jkl}A_{j1}A_{k2}A_{l3}$

$$= \epsilon_{jkl}(\delta_{j1}-2n_jn_1)(\delta_{k2}-2n_kn_2)(\delta_{l3}-2n_ln_3)$$

$$= \epsilon_{jkl}\delta_{j1}\delta_{k2}\delta_{l3}$$

+ (terms with one Kronecker delta; all these are zero because of the antisymmetry of ϵ_{jkl}; e.g., $\epsilon_{jkl}\delta_{j1}2n_kn_2 2n_ln_3 = 0$)

$$+ \epsilon_{jkl}\left(\delta_{j1}\delta_{k2}(-2n_ln_3) + \delta_{j1}\delta_{l3}(-2n_kn_2) + \delta_{k2}\delta_{l3}(-2n_jn_1)\right)$$

+ (terms with no Kronecker delta; all these are zero because of the antisymmetry of ϵ_{jkl})

$$= \epsilon_{123} - 2\left(\epsilon_{123}n_3^2 + \epsilon_{123}n_2^2 + \epsilon_{123}n_1^2\right)$$

$$= 1 - 2\vec{n}\cdot\vec{n}$$

$$\boxed{\det A = -1}$$

You can do this from the matrix

$$A = \begin{pmatrix} 1-2n_1^2 & -2n_1n_2 & -2n_1n_3 \\ -2n_1n_2 & 1-2n_2^2 & -2n_2n_3 \\ -2n_1n_3 & -2n_2n_3 & 1-2n_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} n_2^2+n_3^2 & -2n_1n_2 & -2n_1n_3 \\ -2n_1n_2 & n_1^2+n_3^2 & -2n_2n_3 \\ -2n_1n_3 & -2n_2n_3 & n_1^2+n_2^2 \end{pmatrix}$$

3.2. Goldstein 4.7

A describes a rotation by π about an axis $\vec{n}$

A^2 describes a rotation by 2π about $\vec{n}$; hence $A^2 = 1$

$$P_{\pm} = \frac{1}{2}(1 \pm A)$$

$$P_{\pm}^2 = \frac{1}{4}\underbrace{(1 \pm A)(1 \pm A)}_{1 \pm 2A + \underset{1}{\underset{\shortparallel}{A^2}} = 2(1 \pm A)} = \frac{1}{2}(1 \pm A) = \boxed{P_{\pm} = P_{\pm}^2}$$

From the rotation formula, the matrix elements of A are

$$A_{jk} = \delta_{jk} \underset{-1}{\underset{\shortparallel}{\cos\pi}} + n_j n_k \underbrace{(1 - \cos\pi)}_{2} + \epsilon_{jkl} n_l \underset{0}{\underset{\shortparallel}{\sin\pi}}$$

$$A_{jk} = -\delta_{jk} + 2 n_j n_k$$

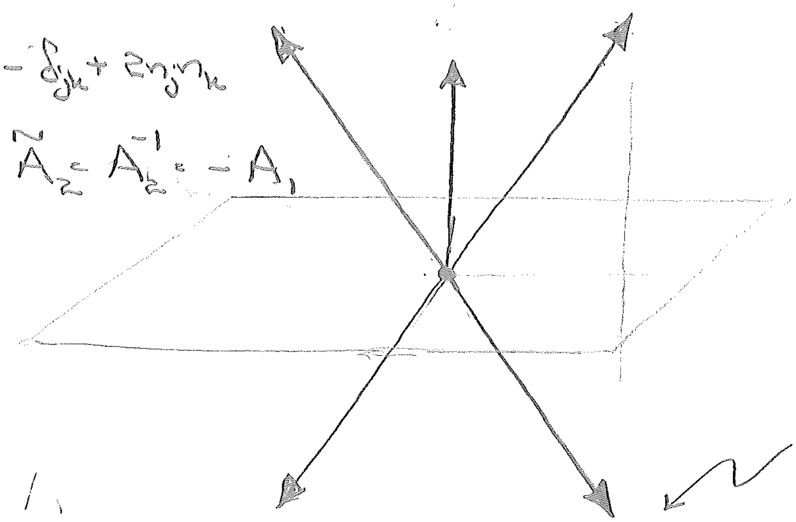
Digression: Notice that the matrix found here is the negative of the one in Goldstein 4.18, as shown in the following picture. Changing the sign of a 3×3 matrix changes the sign of $\det A$, so this goes back and forth between proper and improper orthogonal transformations. The sign change is an inversion through the origin.

rotation by π about $\vec{n}$

$$\vec{y}_2 = \tilde{A}_2\vec{x} = -\vec{x} + 2\vec{n}(\vec{n}\cdot\vec{x}) = -\vec{y}_1$$

$$(\tilde{A}_2)_{jk} = -\delta_{jk} + 2n_jn_k$$

$$A_2 = \tilde{A}_2 = A_2^{-1} = -A_1$$



reflection thru plane

$$\vec{y}_1 = \tilde{A}_1\vec{x} = \vec{x} - 2\vec{n}(\vec{n}\cdot\vec{x})$$

$$(\tilde{A}_1)_{jk} = \delta_{jk} - 2n_jn_k$$

$$A_1 = \tilde{A}_1 = A_1^{-1} = -A_2$$

$$\vec{y}_3 = \tilde{A}_2\tilde{A}_1\vec{x} = (\widetilde{A_1A_2})\vec{x}$$
$$= \tilde{A}_1\tilde{A}_2\vec{x} = (\widetilde{A_2A_1})\vec{x}$$
$$= -\tilde{A}_2^2\vec{x}$$
$$= -\vec{x}$$

parity inversion thru origin

$$P_\pm = \frac{1}{2}(1 \pm A) \Longleftrightarrow (P_\pm)_{jk} = \frac{1}{2}(\delta_{jk} \pm A_{jk})$$

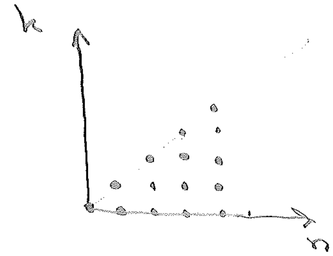
$$\boxed{\begin{aligned}(P_+)_{jk} &= n_jn_k \Longleftrightarrow P_+\vec{F} = \vec{n}(\vec{n}\cdot\vec{F}) = \begin{pmatrix}\text{projection of } \vec{F}\\ \text{onto } \vec{n}\end{pmatrix}\\ (P_-)_{jk} &= \delta_{jk} - n_jn_k \Longleftrightarrow P_-\vec{F} = \vec{F} - \vec{n}(\vec{n}\cdot\vec{F}) = \begin{pmatrix}\text{projection of } \vec{F}\\ \text{orthogonal to } \vec{n}\end{pmatrix}\end{aligned}}$$

3.3. Goldstein 4.10 (a)–(d) plus (e)

(a) $$e^{B+C} = \sum_{n=0}^{\infty} \frac{1}{n!}(B+C)^n$$

Since B & C commute, $(B+C)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B^k C^{n-k}$

$$\therefore\ e^{B+C} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B^k C^{n-k}}{k!(n-k)!}$$



We can write this sum as k running from 0 to ∞ and $l = n-k$ running from 0 to ∞

$$\therefore\ e^{B+C} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{B^k C^l}{k!\,l!} = \underbrace{\left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right)}_{e^B}\underbrace{\left(\sum_{l=0}^{\infty} \frac{C^l}{l!}\right)}_{e^C}$$

$$\boxed{e^{B+C} = e^B e^C}$$

In short, if B and C commute, they act like ordinary numbers.

(b) $$e^{-B}e^{B} = e^{B+(-B)} = e^0 = 1 \Rightarrow \boxed{e^{-B} = (e^B)^{-1}}$$

part (a) since B and $-B$ commute

(c) $$e^{CBC^{-1}} = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{(CBC^{-1})^n}_{} = C\left(\sum_{n=0}^{\infty}\frac{B^n}{n!}\right)C^{-1} = Ce^BC^{-1}$$

$$\underbrace{CBC^{-1}\,CBC^{-1}\ldots CBC^{-1}}_{n \text{ times}} = CB^nC^{-1}$$

Each "internal" C multiplies a C^{-1} to give 1 (d)'; Here we use $\widetilde{(CD)} = \tilde{D}\tilde{C}$, i.e.,

$$\left(\widetilde{(CD)}\right)_{jk} = (CD)_{kj} = C_{kl}D_{lj} = \tilde{D}_{jl}\tilde{C}_{lk} = (\tilde{D}\tilde{C})_{jk}$$

$$\therefore \tilde{A} = \widetilde{e^{B}} = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\widetilde{(B^n)}}_{=\tilde{B}^n} = \sum_{n=0}^{\infty}\frac{1}{n!}\tilde{B}^n = e^{\tilde{B}}$$

Assume B is antisymmetric, i.e., $\tilde{B} = -B$ $(B_{kj} = -B_{jk})$.
Then $\tilde{A} = e^{\tilde{B}} = e^{-B} = A^{-1}$, $\Rightarrow$ $\boxed{A \text{ is orthogonal}}$

(e) Here we use (1) $(aC)^{\dagger} = a^* C^{\dagger}$, i.e.,

$$((aC)^{\dagger})_{jk} = (aC)^*_{kj} = a^* C^*_{kj} = a^*(C^{\dagger})_{jk}$$

and (2) $(CD)^{\dagger} = D^{\dagger}C^{\dagger}$, i.e.,

$$((CD)^{\dagger})_{jk} = (CD)^*_{kj} = C^*_{kl}D^*_{lj} = (D^{\dagger})_{jl}(C^{\dagger})_{lk} = (D^{\dagger}C^{\dagger})_{jk}$$

$$\therefore (e^{iB})^{\dagger} = \left(\sum_{n=0}^{\infty}\frac{1}{n!}(iB)^n\right)^{\dagger} =$$

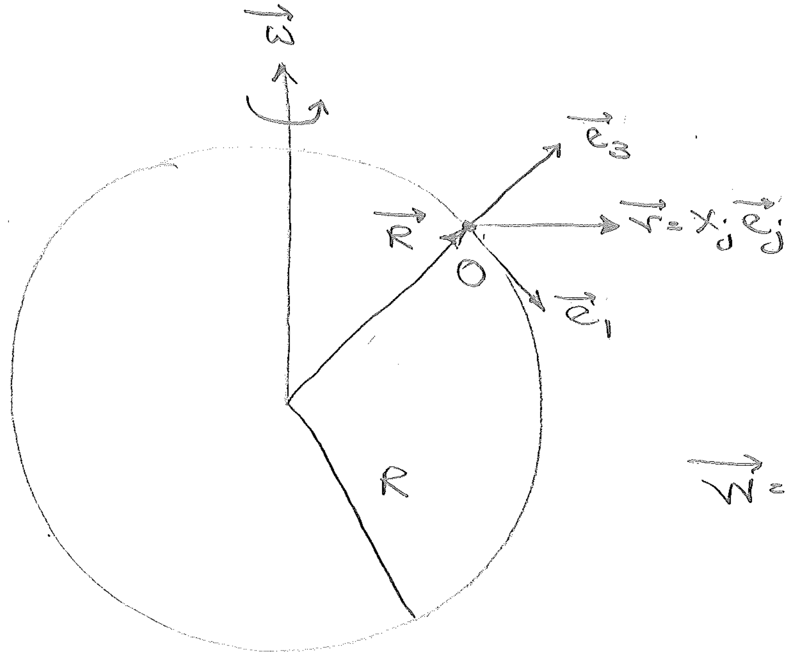
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{((i)^n B^n)^{\dagger}}_{(-i)^n (B^n)^{\dagger} = (-i)^n (B^{\dagger})^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}(B^{\dagger})^n = e^{-iB^{\dagger}}$$

Assume B is Hermitian, i.e. $B = B^{\dagger}$ $(B_{jk} = B_{kj}^{*})$.

Then $\left(e^{iB}\right)^{\dagger} = e^{-iB^{\dagger}} = e^{-iB} = \left(e^{iB}\right)^{-1} \Rightarrow$ $\boxed{e^{iB} \text{ is unitary.}}$

3.4.



$$\vec{W} = \left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{space}$$

(a) $\vec{R} = X_j \vec{e}_j$; $\boxed{X_j = R\,\delta_{j3}, \quad X_1 = X_2 = 0, \quad X_3 = R}$

(b) $$\vec{W} = W_j \vec{e}_j = \left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{space}$$

$$= \left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{rot} + \vec{\omega} \times (\vec{R} \times \vec{r})$$

$$= \cancelto{0}{\frac{dX_j}{dt}} \vec{e}_j + \frac{dx_j}{dt} \vec{e}_j + \vec{e}_j \epsilon_{jkl} \omega_k (X_l + x_l)$$

$$\boxed{W_j = \dot{x}_j + \epsilon_{jkl} \omega_k (X_l + x_l)}$$

(c) $T = \frac{1}{2} m \vec{W} \cdot \vec{W} = \frac{1}{2} m W_k W_k \, , \quad W_k = \dot{x}_k + \epsilon_{k\ell m} \omega_\ell (X_m + x_m)$

$$\boxed{\begin{aligned} L &= T - V \\ &= \frac{1}{2} m \left(\dot{x}_j + \epsilon_{jk\ell} \omega_k (X_\ell + x_\ell) \right) \left(\dot{x}_j + \epsilon_{jmn} \omega_m (X_n + x_n) \right) \\ &\quad - V(x_1, x_2, x_3) \end{aligned}}$$

(d) $$\frac{\partial L}{\partial \dot{x}_j} = m W_k \underbrace{\frac{\partial W_k}{\partial \dot{x}_j}}_{\delta_{jk}} = m W_j = m \dot{x}_j + m \epsilon_{jk\ell} \omega_k (X_\ell + x_\ell)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) = m \ddot{x}_j + m \epsilon_{jk\ell} \omega_k \dot{x}_\ell$$

$$\frac{\partial L}{\partial x_j} = m W_k \underbrace{\frac{\partial W_k}{\partial x_j}}_{\epsilon_{k\ell j} \omega_\ell} - \frac{\partial V}{\partial x_j}$$

$$= + m \epsilon_{jk\ell} W_k \omega_\ell - \frac{\partial V}{\partial x_j}$$

$$= - m \epsilon_{jk\ell} \omega_k W_\ell - \frac{\partial V}{\partial x_j}$$

$$= - m \epsilon_{jk\ell} \omega_k \dot{x}_\ell - m \epsilon_{jk\ell} \omega_k \epsilon_{\ell mn} \omega_m (X_n + x_n) - \frac{\partial V}{\partial x_j}$$

Notice that $0 = \left(\frac{d\vec{\omega}}{dt} \right)_{space} = \left(\frac{d\vec{\omega}}{dt} \right)_{rot} + \cancelto{0}{\vec{\omega} \times \vec{\omega}} \, , \quad \left(\frac{d\vec{\omega}}{dt} \right)_{rot} = \dot{\omega}_j$

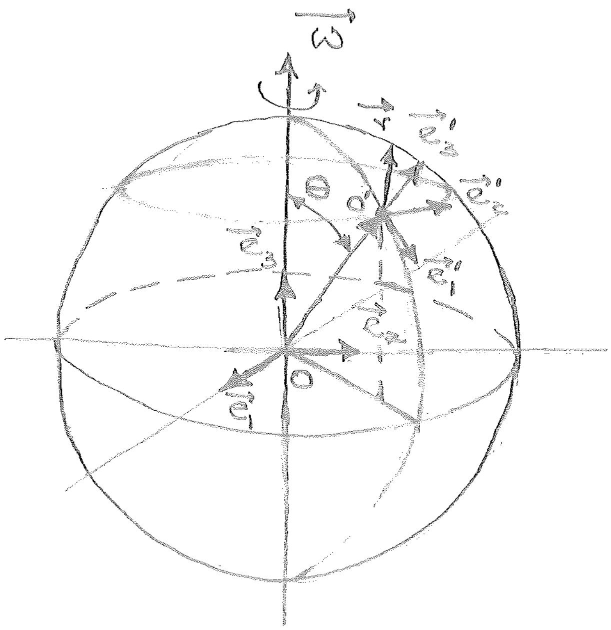
$\Rightarrow \dot{\omega}_j = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j}$$

$$m\ddot{x}_j + m\epsilon_{jkl}\omega_k\dot{x}_l = -m\epsilon_{jkl}\omega_k\dot{x}_l - m\epsilon_{jkl}\omega_k\epsilon_{lmn}\omega_m(X_n + x_n) - \frac{\partial V}{\partial x_j}$$

$$\Longrightarrow \boxed{\begin{gathered} m\ddot{x}_j = -2m\epsilon_{jkl}\omega_k\dot{x}_l - m\epsilon_{jkl}\omega_k\epsilon_{lmn}\omega_m(X_n + x_n) - \frac{\partial V}{\partial x_j} \\ \text{or} \\ m\vec{a} = -2m\vec{\omega}\times\vec{v} - m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r})) - \nabla V \end{gathered}}$$

3.5 Goldstein 4.22



In its discussion of centrifugal and Coriolis forces, the text uses a rotating frame whose origin is at the center of the Earth (point O). I prefer to use a rotating frame whose origin is at the place where the experiment is occurring (point O'). This means that I specify a particle's position by a vector $\vec{r}$ from O', whereas the text uses $\vec{R}+\vec{r}$. This makes no difference to velocities and accelerations in the rotating frame because $\vec{R} = R\vec{e}_3'$ is constant in the rotating frame, i.e.,

$$\left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{rot} = \left(\frac{d\vec{r}}{dt}\right)_{rot} \equiv \vec{v}_r$$

$$\left(\frac{d\vec{v}_r}{dt}\right)_{rot} = \left(\frac{d^2(\vec{R}+\vec{r})}{dt^2}\right)_{rot} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} \equiv \vec{a}_r$$

Thus the equation of motion in the rotating frame is [Eq. (4-128)]

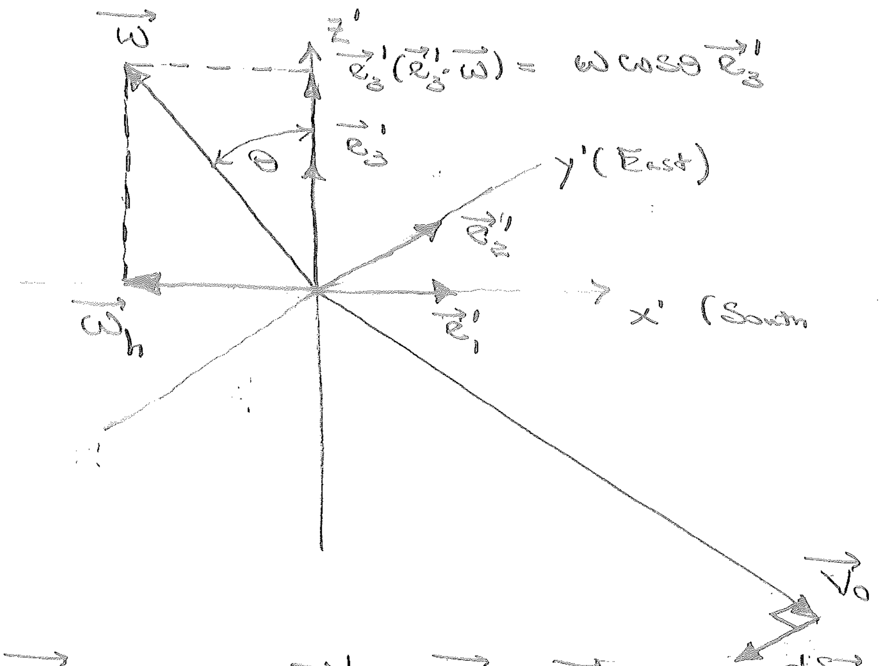
$$m\vec{a}_r = \underbrace{m\vec{g} - 2m\vec{\omega}\times\vec{v}_r - m\vec{\omega}\times(\vec{\omega}\times\vec{R}) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\equiv \vec{F}_r}$$

We are interested in the bullet's horizontal motion, i.e., motion in the plane defined by $\vec{e}_1'$ (South) and $\vec{e}_2'$. (East). Write the horizontal velocity as $\vec{v}_h = \vec{v}_0 + \delta\vec{v}$, where $\delta\vec{v}$ is the <u>horizontal</u> deviation from the initial horizontal velocity $\vec{v}_0$. The equation of motion for the horizontal velocity is

$$\frac{d\delta\vec{v}}{dt} = \frac{d\vec{v}_h}{dt} = \underbrace{\frac{\vec{F}_r}{m} - \vec{e}_3'\left(\vec{e}_3'\cdot\frac{\vec{F}_r}{m}\right)}_{\substack{\text{projection of } \vec{F}_r \\ \text{into horizontal plane}}}.$$

In $\vec{F}_r$, we neglect the constant term $m\vec{g} - m\vec{\omega}\times(\vec{\omega}\times\vec{R})$, because it defines the local vertical $\vec{e}_3'$ and thus does not appear in the horizontal projection. In addition, we neglect $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ relative to $-2m\vec{\omega}\times\vec{v}_r$ because it is small ($\omega|\vec{r}| \ll |\vec{v}_r|$). Thus we work with $\vec{F}_r/m = -2\vec{\omega}\times\vec{v}_r \simeq -2\vec{\omega}\times\vec{v}_0$.

$$\frac{d\delta\vec{v}}{dt} = -2\left[\vec{\omega}\times\vec{v}_0 - \vec{e}_3'(\vec{e}_3'\cdot\vec{\omega}\times\vec{v}_0)\right]$$



Write $\vec{\omega} = \omega\cos\theta\,\vec{e}_3{}' + \vec{\omega}_h$. Then: $\frac{d\delta\vec{v}}{dt} = -2\omega\cos\theta\,\vec{e}_3{}'\times\vec{V}_0$

$$\vec{\omega}\times\vec{V}_0 - \underbrace{\vec{e}_3{}'(\vec{e}_3{}'\cdot\vec{\omega}\times\vec{V}_0)} = (\vec{\omega}-\vec{\omega}_h)\times\vec{V}_0 = \omega\cos\theta\,\vec{e}_3{}'\times\vec{V}_0$$

$$\vec{e}_3{}'(\vec{e}_3{}'\cdot\underbrace{\vec{\omega}_h\times\vec{V}_0}) = \vec{\omega}_h\times\vec{V}_0$$

points in the $\vec{e}_3{}'$ direction

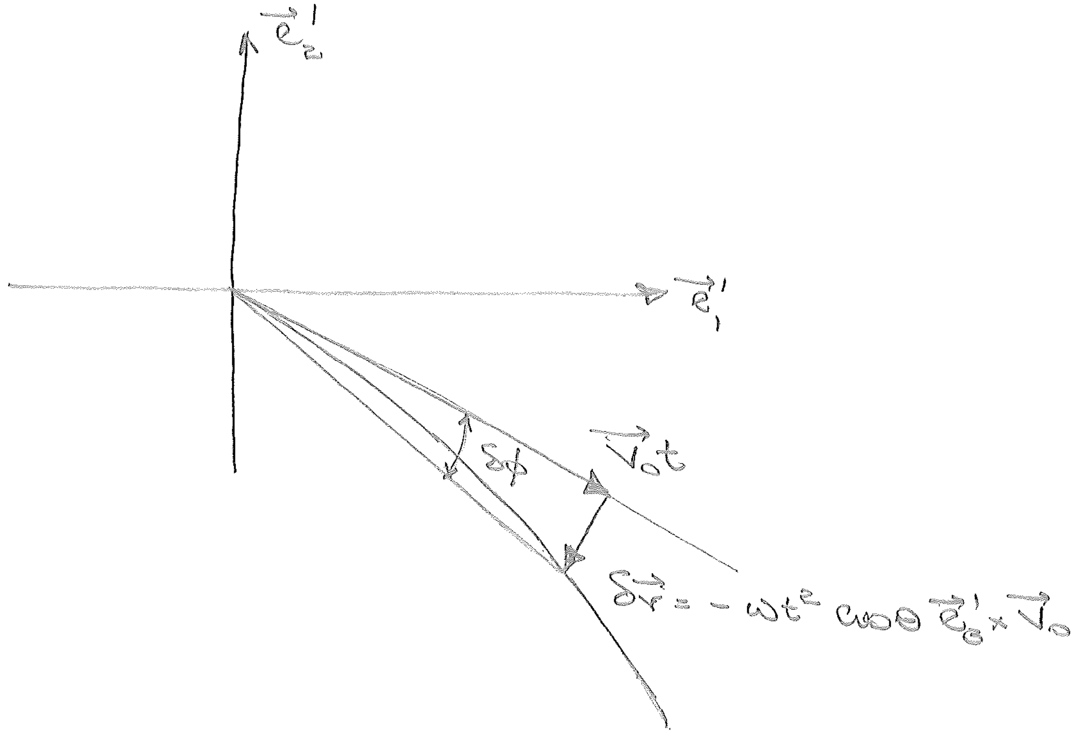
$$\therefore \boxed{\frac{d(\delta\vec{v})}{dt} = -2\omega\cos\theta\,\vec{e}_3{}'\times\vec{V}_0}$$

only the component of $\vec{\omega}$ along the local vertical counts

$$\boxed{\delta\vec{v} = -2\omega t\cos\theta\,\vec{e}_3{}'\times\vec{V}_0}$$

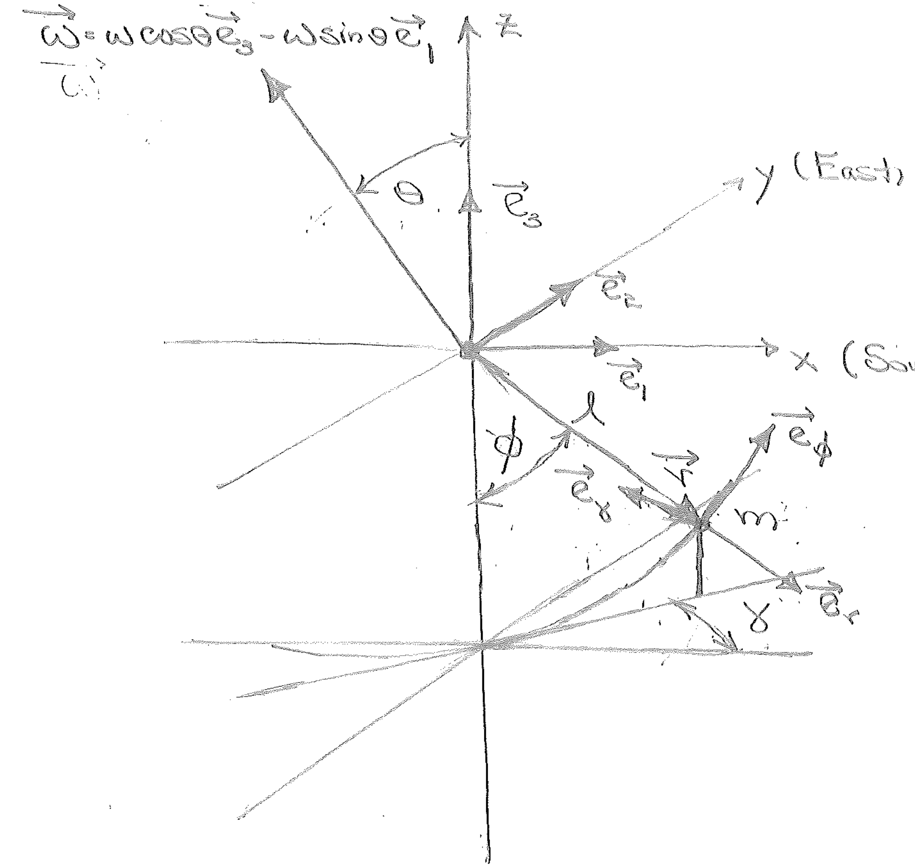
$$\frac{d\vec{r}_h}{dt} = \vec{v}_h = \vec{V}_0 + \delta\vec{v}$$

$$\boxed{\vec{r}_h = \vec{V}_0 t \underbrace{- \omega t^2\cos\theta\,\vec{e}_3{}'\times\vec{V}_0}_{\delta\vec{r}}}$$



$$\delta\phi = \begin{pmatrix}\text{angular}\\ \text{deviation}\end{pmatrix} = \frac{|\delta\vec{r}|}{|\vec{v}_0|t} = \frac{\omega t^{\cancel{2}} \cos\theta\, \cancel{|\vec{v}_0|}}{\cancel{|\vec{v}_0|}\cancel{t}} = \boxed{\omega t \cos\theta = \delta\phi}$$

3.6. Goldstein 4.23



$\vec{e}_r, \vec{e}_\gamma, \vec{e}_\phi$ are a right-handed system

Strategy: First use a Lagrangian to get the equations of motion for a spherical pendulum in the absence of rotation; then put in the Coriolis force by hand.

$$\vec{r} = \underbrace{l\sin\phi\cos\gamma}_{x}\,\vec{e}_1 + \underbrace{l\sin\phi\sin\gamma}_{y}\,\vec{e}_2 \underbrace{-l\cos\phi}_{z}\,\vec{e}_3$$

$$= l\,\vec{e}_r$$

$$\vec{v} = \dot{\vec{r}} = l\Big[\dot\phi\overbrace{(\cos\phi\cos\gamma\,\vec{e}_1 + \cos\phi\sin\gamma\,\vec{e}_2 + \sin\phi\,\vec{e}_3)}^{\vec{e}_\phi}$$

$$+ \dot\gamma\sin\phi\underbrace{(-\sin\gamma\,\vec{e}_1 + \cos\gamma\,\vec{e}_2)}_{\vec{e}_\gamma}\Big]$$

$$\vec{v} = l\dot{\phi}\vec{e}_\phi + l\dot{\gamma}\sin\phi\,\vec{e}_\gamma$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2(\dot{\phi}^2 + \dot{\gamma}^2\sin^2\phi)$$

$$V = mgz \longleftarrow$$ local vertical and g adjusted to include centrifugal force

$$= -mgl\cos\phi$$

$$L = T - V = \frac{1}{2}ml^2(\dot{\phi}^2 + \dot{\gamma}^2\sin^2\phi) + mgl\cos\phi$$

① $$\frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi}$$

$$\frac{\partial L}{\partial\phi} = ml^2\dot{\gamma}^2\sin\phi\cos\phi - mgl\sin\phi$$

$$\Longrightarrow \boxed{\underbrace{ml\ddot{\phi} - ml\dot{\gamma}^2\sin\phi\cos\phi}_{ma_\phi} = \underbrace{-mg\sin\phi}_{F_\phi}}$$

② $$\frac{\partial L}{\partial\dot{\gamma}} = ml^2\dot{\gamma}\sin^2\phi$$

$$ml^2\ddot{\gamma}\sin^2\phi + 2ml^2\dot{\gamma}\dot{\phi}\sin\phi\cos\phi = 0$$

$$\frac{\partial L}{\partial\gamma} = 0$$

$$\Longrightarrow \boxed{\underbrace{ml\ddot{\gamma}\sin\phi + 2ml\dot{\gamma}\dot{\phi}\cos\phi}_{ma_\gamma} = 0}$$

Now include the Coriolis force $\vec{F}_c = -2m\vec{\omega}\times\vec{v}$

$$\vec{\omega}\times\vec{v} = \omega l(\cos\theta\,\vec{e}_3 - \sin\theta\,\vec{e}_1)\times(\dot{\phi}\vec{e}_\phi + \dot{\gamma}\sin\phi\,\vec{e}_\gamma)$$

$$= \omega l\big(\dot{\phi}\cos\theta\,\vec{e}_3\times\vec{e}_\phi + \dot{\phi}\sin\theta\,\vec{e}_1\times\vec{e}_\phi$$

$$+ \dot{\gamma}\sin\phi\cos\theta\,\vec{e}_3\times\vec{e}_\gamma - \dot{\gamma}\sin\phi\sin\theta\,\vec{e}_1\times\vec{e}_\gamma\big)$$

$$\boxed{\begin{aligned} \vec{e}_r &= \sin\phi\,(\cos\gamma\,\vec{e}_1 + \sin\gamma\,\vec{e}_2) - \cos\phi\,\vec{e}_3 \\ \vec{e}_\phi &= \cos\phi\,(\cos\gamma\,\vec{e}_1 + \sin\gamma\,\vec{e}_2) + \sin\phi\,\vec{e}_3 \\ \vec{e}_\gamma &= -\sin\gamma\,\vec{e}_1 + \cos\gamma\,\vec{e}_2 \end{aligned}}$$

$$\left[\begin{aligned} \cos\gamma\,\vec{e}_1 + \sin\gamma\,\vec{e}_2 &= \cos\phi\,\vec{e}_\phi + \sin\phi\,\vec{e}_r \\ \vec{e}_3 &= \sin\phi\,\vec{e}_\phi - \cos\phi\,\vec{e}_r \end{aligned}\right.$$

$$\left[\begin{aligned} \vec{e}_1 &= \cos\gamma\,(\cos\phi\,\vec{e}_\phi + \sin\phi\,\vec{e}_r) - \sin\gamma\,\vec{e}_\gamma \\ \vec{e}_2 &= \sin\gamma\,(\cos\phi\,\vec{e}_\phi + \sin\phi\,\vec{e}_r) + \cos\gamma\,\vec{e}_\gamma \end{aligned}\right.$$

$$\boxed{\begin{aligned} \vec{e}_1 &= \cos\gamma\cos\phi\,\vec{e}_\phi + \cos\gamma\sin\phi\,\vec{e}_r - \sin\gamma\,\vec{e}_\gamma \\ \vec{e}_2 &= \sin\gamma\cos\phi\,\vec{e}_\phi + \sin\gamma\sin\phi\,\vec{e}_r + \cos\gamma\,\vec{e}_\gamma \\ \vec{e}_3 &= \sin\phi\,\vec{e}_\phi - \cos\phi\,\vec{e}_r \end{aligned}}$$

$$\vec{e}_3 \times \vec{e}_\phi = +\cos\phi\,\vec{e}_\gamma$$

$$\vec{e}_1 \times \vec{e}_\phi = -\cos\gamma\sin\phi\,\vec{e}_\gamma - \sin\gamma\,\vec{e}_r$$

$$\vec{e}_3 \times \vec{e}_\gamma = -\sin\phi\,\vec{e}_r - \cos\phi\,\vec{e}_\phi$$

$$\vec{e}_1 \times \vec{e}_\gamma = -\cos\gamma\cos\phi\,\vec{e}_r + \cos\gamma\sin\phi\,\vec{e}_\phi$$

$$\vec{F}_c = +2m\omega l \Big(-\dot{\phi}\cos\theta\cos\phi\,\vec{e}_\gamma$$
$$+\dot{\phi}\sin\theta\left(\cos\gamma\sin\phi\,\vec{e}_\gamma + \sin\gamma\,\vec{e}_r\right)$$
$$+\dot{\gamma}\sin\phi\cos\theta\left(\sin\phi\,\vec{e}_r + \cos\phi\,\vec{e}_\phi\right)$$
$$+\dot{\gamma}\sin\phi\sin\theta\left(\cos\gamma\cos\phi\,\vec{e}_r - \cos\gamma\sin\phi\,\vec{e}_\phi\right)\Big)$$

$$\boxed{\begin{aligned}\vec{F}_c = 2m\omega l \Big[&\vec{e}_r\left(-\dot{\phi}\sin\theta\sin\gamma + \dot{\gamma}\sin\phi\left(\cos\theta\sin\phi - \sin\theta\cos\gamma\cos\phi\right)\right) \\ &+\vec{e}_\phi\left(\dot{\gamma}\sin\phi\left(\cos\theta\cos\phi + \sin\theta\cos\gamma\sin\phi\right)\right) \\ &+\vec{e}_\gamma\left(-\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\cos\gamma\sin\phi\right)\right)\Big]\end{aligned}}$$

Including the Coriolis force, the equations of motion become

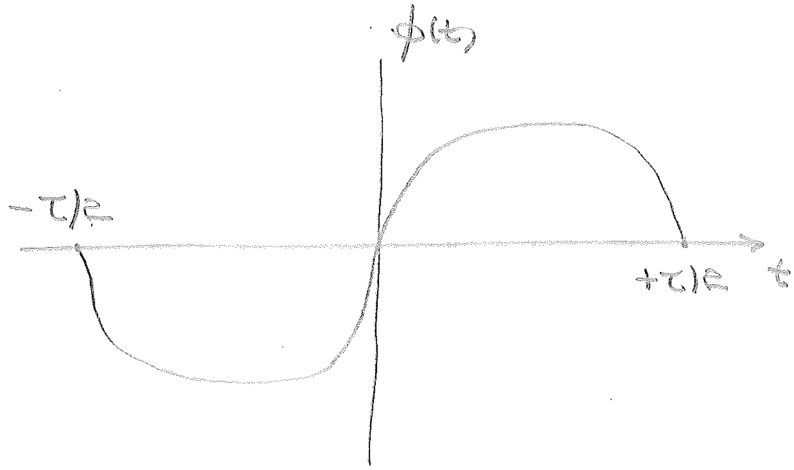
$$\boxed{\begin{aligned} &l\ddot{\phi} - l\dot{\gamma}^2\sin\phi\cos\phi \\ &\quad = -g\sin\phi + 2\omega l\dot{\gamma}\sin\phi\left(\cos\theta\cos\phi + \sin\theta\cos\gamma\sin\phi\right) \\ &l\ddot{\gamma}\sin\phi + 2l\dot{\gamma}\dot{\phi}\cos\phi = -2\omega l\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\cos\gamma\sin\phi\right)\end{aligned}}$$

Now <u>assume</u> that the pendulum makes <u>rapid</u> oscillations in ϕ ($\dot{\phi} \sim \sqrt{g/l} = \Omega$), while there is a slow rotation in γ ($\dot{\gamma} \sim \omega$). Dropping negligible terms leads to

$$l\ddot{\phi} = -g\sin\phi$$ ← standard equation for oscillations of spherical pendulum in a plane.

$$\dot{\gamma}\cos\phi = -\omega\left(\cos\theta\cos\phi + \sin\theta\cos\gamma\sin\phi\right)$$ ← rotation of plane

$$\boxed{\dot{\gamma} = -\omega\cos\theta - \omega\sin\theta\cos\gamma\tan\phi}$$



ϕ is an odd function of t

$\Longrightarrow \tan\phi$ is an odd function of t

$\Longrightarrow$ the second term in the $\dot{\gamma}$ equation averages to zero over a period.

$$\dot{\gamma}_{av} = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2}\dot{\gamma}\,dt = \boxed{-\omega\cos\theta = \dot{\gamma}_{av}}$$

Plane undergoes a negative rotation about the vertical axis, with rate $\omega\cos\theta$