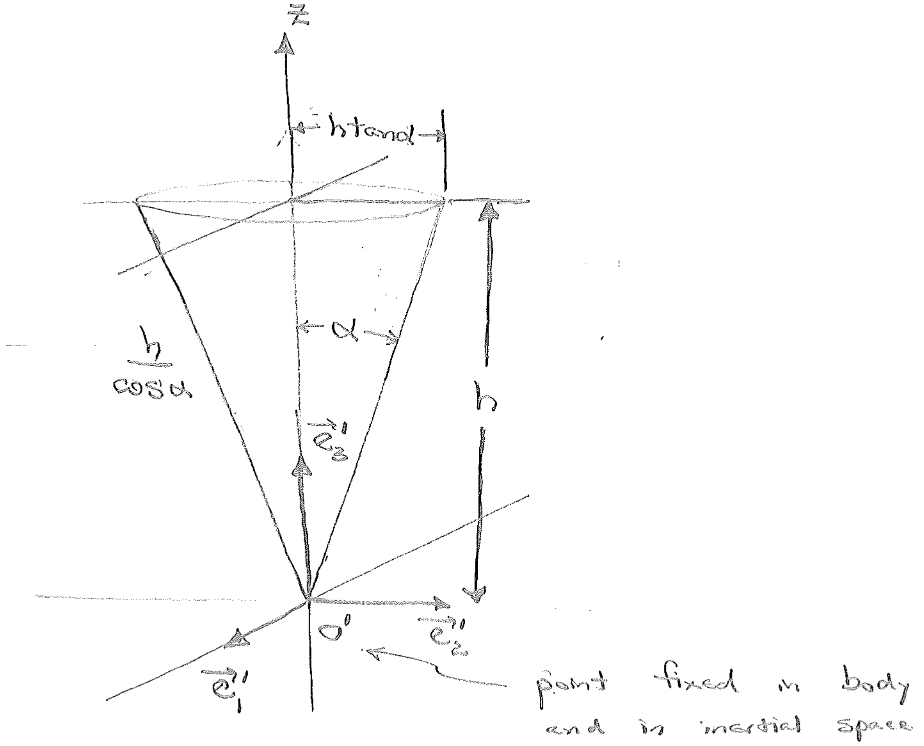
Phys 503

Homework # 4

Solution Set

4.1. Goldstein 5.17



$$\overleftrightarrow{I}_{O''} = \int dV \rho \left(\vec{r}\cdot\vec{r}\,\overleftrightarrow{1} - \vec{r}\otimes\vec{r}\right)$$

By cylindrical symmetry, axes shown are the (body) principal axes, with $I_{11} = I_{22} = I$. Use cylindrical coordinates r, ϕ, and z.

$$I_3 = \int dV \rho \, (\underbrace{\vec{r}\cdot\vec{r} - z^2}_{x^2+y^2 = r^2}) =$$

$$= \rho \int r\,dr\,d\phi\,dz\, r^2$$

$$= \underbrace{2\pi\rho \int_0^h dz \underbrace{\int_0^{z\tan\alpha} r^3\,dr}_{\frac{1}{4} z^4 \tan^4\alpha}}$$

$$\frac{1}{4}\tan^4\alpha \int_0^h dz\, z^4$$

$$= \frac{1}{20} h^5 \tan^4\alpha$$

$$\boxed{I_3 = \frac{\pi}{10}\rho h^5 \tan^4\alpha}$$

$$I_{11} = \int dV\, \rho\,(\underbrace{\vec{r}\cdot\vec{r} - x^2}_{y^2+z^2})$$

$$I_{22} = \int dV\, \rho\,(\underbrace{\vec{r}\cdot\vec{r} - y^2}_{x^2+z^2})$$

$$2I - I_3 = 2\int dV\, \rho\, z^2$$

$$= 2\rho\int r\, dr\, d\phi\, dz\, z^2$$

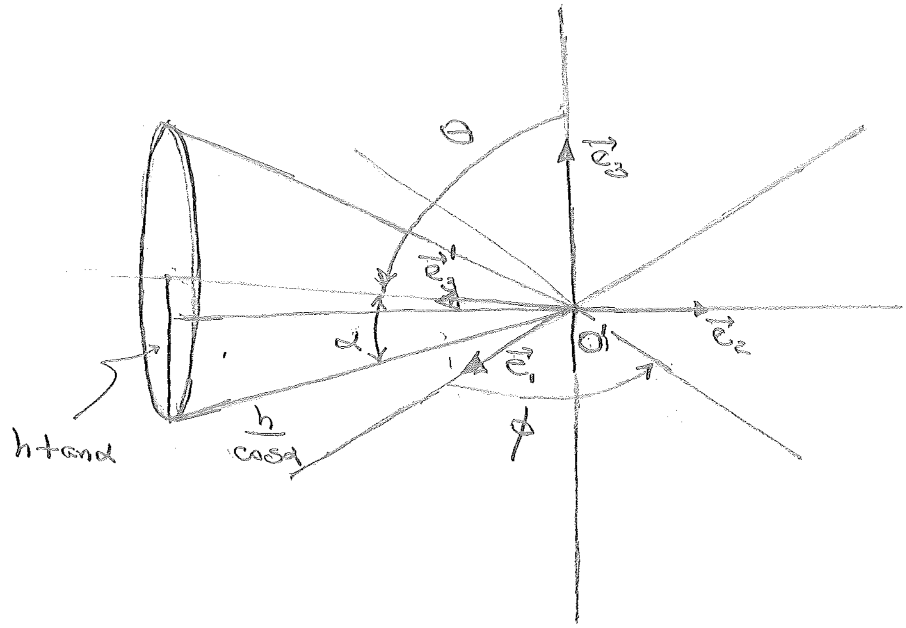
$$= 4\pi\rho \underbrace{\int_0^h dz\, z^2 \underbrace{\int_0^{z\tan\alpha} dr\, r}_{\frac{1}{2} z^2\tan^2\alpha}}_{\frac{1}{2}\tan^2\alpha\int_0^h dz\, z^4 = \frac{1}{10}h^5\tan^2\alpha}$$

$$\downarrow$$

$$= \frac{4\pi}{10}\rho h^5 \tan^2\alpha$$

$$I = \frac{1}{2}(\underbrace{I_3 + \frac{4\pi}{10}\rho h^5\tan^2\alpha}_{\frac{\pi}{10}\rho h^5\tan^2\alpha\left(\tan^2\alpha + 4\right)})$$

$$\boxed{I = \frac{\pi}{20}\rho h^5 \tan^2\alpha\,(4 + \tan^2\alpha)}$$



$\theta = \frac{\pi}{2} - \alpha$

ψ (not shown) describes body's rotation about its $\vec{e}_3'$ axis

Rolling constraint: $\dot\phi \frac{h}{\cos\alpha} = -\dot\psi\, h\tan\alpha \Rightarrow \boxed{\dot\psi = -\frac{\dot\phi}{\sin\alpha}}$

Components of $\vec{\omega}$ in body frame:

$$\omega_1' = \cancelto{0}{\dot\theta \cos\psi} + \dot\phi \sin\theta \sin\psi = \boxed{\dot\phi \cos\alpha \sin\psi = \omega_1'}$$

$$\omega_2' = -\cancelto{0}{\dot\theta \sin\psi} + \dot\phi \sin\theta \cos\psi = \boxed{\dot\phi \cos\alpha \cos\psi = \omega_2'}$$

$$\omega_3' = \dot\psi + \dot\phi\cos\theta = -\frac{\dot\phi}{\sin\alpha} + \dot\phi\sin\alpha = -\frac{\dot\phi}{\sin\alpha}\underbrace{\left(1 - \sin^2\alpha\right)}_{\cos^2\alpha} = -\dot\phi\frac{\cos^2\alpha}{\sin\alpha}$$

$$\boxed{\omega_3' = -\dot\phi \cos\alpha \cot\alpha}$$

Since the cone returns to it's original position in time τ, we have that $\boxed{\dot\phi = 2\pi/\tau.}$

Angular momentum in body frame:

$$\boxed{L_1' = I\omega_1', \quad L_2' = I\omega_2', \quad L_3' = I_3\omega_3'}$$

Kinetic energy:

$$T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}\left(I\left(\omega_1'^2 + \omega_2'^2\right) + I_3\,\omega_3'^2\right)$$

$$= \frac{1}{2}\left(I\,\dot{\phi}^2\cos^2\alpha + I_3\,\dot{\phi}^2\cos^2\alpha\,\cot^2\alpha\right)$$

$$= \frac{1}{2}\,\dot{\phi}^2\cos^2\alpha\underbrace{\left(I + I_3\cot^2\alpha\right)}_{\frac{\pi}{10}\rho h^5\tan^2\alpha\left(2+\frac{1}{2}\tan^2\alpha + 1\right)}$$

$$= \frac{\pi}{10}\rho h^5\tan^2\alpha\left(3 + \frac{1}{2}\tan^2\alpha\right)$$

$$\boxed{T = \frac{\pi}{20}\rho h^5\dot{\phi}^2\sin^2\alpha\left(3 + \frac{1}{2}\tan^2\alpha\right)}$$

4.2. Goldstein 5.7

$$L = T - V$$

⟶ assume this depends only on Euler angles ϕ, θ, ψ

$$T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \longrightarrow \text{use body principle axes}$$

Write components of $\vec{\omega}$ in <u>body</u> frame:

$$\omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$

$$\omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

Equation of motion for ψ:

$$\frac{\partial L}{\partial\dot{\psi}} = \frac{\partial T}{\partial\dot{\psi}} = I_3\omega_3\underset{\overset{\shortparallel}{1}}{\frac{\partial\omega_3}{\partial\dot{\psi}}} = I_3\omega_3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right) = I_3\dot{\omega}_3$$

$$\frac{\partial L}{\partial\psi} = \frac{\partial T}{\partial\psi} - \frac{\partial V}{\partial\psi}$$

Since ψ describes rotations about the $\vec{e}_3^{\,\prime}$ body axis,

$$-\frac{\partial V}{\partial\psi} = N_3 = (\vec{e}_3^{\,\prime}\text{- component of torque})$$

$$\frac{\partial L}{\partial \psi} = \frac{\partial T}{\partial \psi} + N_3 = (I_1 - I_2)\,\omega_1\omega_2 + N_3$$

$$\frac{\partial T}{\partial \psi} = I_1\omega_1 \underbrace{\frac{\partial \omega_1}{\partial \psi}}_{= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi = \omega_2} + I_2\omega_2 \underbrace{\frac{\partial \omega_2}{\partial \psi}}_{= -\dot{\theta}\cos\psi - \dot{\phi}\sin\theta\sin\psi = -\omega_1}$$

$$= (I_1 - I_2)\,\omega_1\omega_2$$

$$\Longrightarrow \quad 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = \boxed{I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 - N_3 = 0}$$

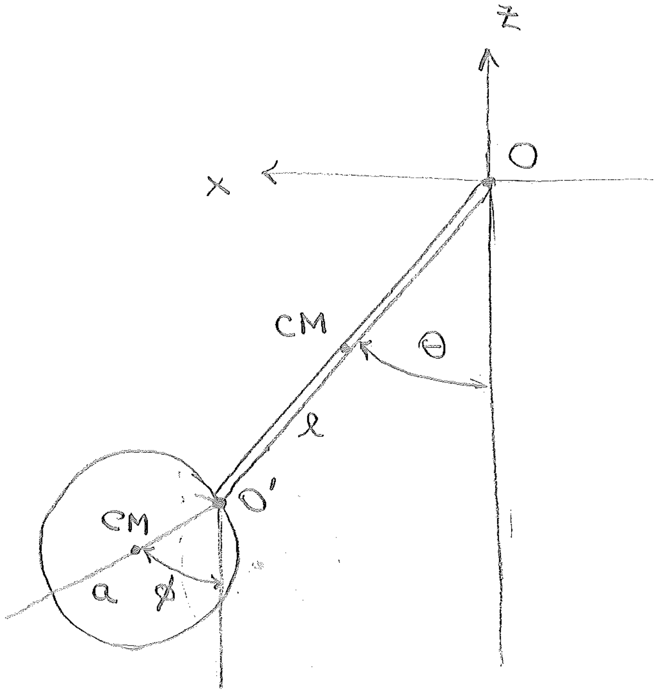
<u>But</u>, we could have chosen any of the body's principal axes to be the $\vec{e}_3^{\,\prime}$ axis, so we get the other 2 equations by cyclic permutation.

Why does this work for ψ and not for θ and ϕ? Notice that $\dot{\psi}$ appears only in ω_3, and in ω_3 $\dot{\psi}$ appears by itself, without being multiplied by anything. So

$$\frac{\partial L}{\partial \dot{\psi}} = I_3\omega_3 \underset{\overset{\shortparallel}{1}}{\frac{\partial \omega_3}{\partial \dot{\psi}}} = I_3\omega_3 = L_3 .$$

This doesn't work for θ or ϕ, so $\partial L/\partial\dot{\theta}$ and $\partial L/\partial\dot{\phi}$ are not components of $\vec{L}$.

4.3. Goldstein 5.20



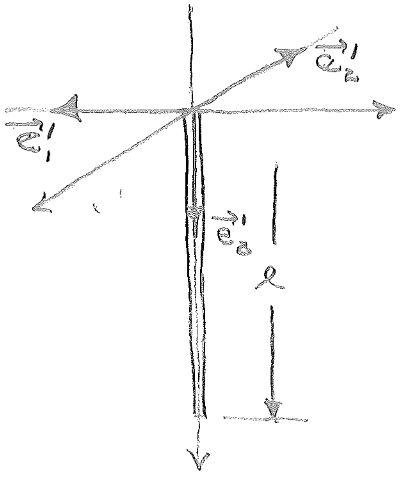
$$L = T - V$$

$$T = T_{bar} + T_{disk}$$

$$V = V_{bar} + V_{disk}$$

Bar: Point O is fixed in the body <u>and</u> in inertial space, so we can write

$$T_{bar} = T_{rot} = \frac{1}{2} I_{bar} \dot{\theta}^2 = \frac{1}{6} m\ell^2 \dot{\theta}^2$$



$$I_{bar} = I_{zz} = \int \lambda \, dz \, (\cancel{x^2}^{\,0} + z^2)$$

$$= \lambda \int dz \, z^2$$

$$= \frac{1}{3} \lambda \ell^3 = \frac{1}{3} m \ell^2$$

$$V_{bar} = mg \underset{\underset{-\frac{\ell}{2}\cos\theta}{\shortparallel}}{z_{cm}} = -\frac{1}{2} mg\ell \cos\theta$$

Disk:

Method 1: Use translation of CM and rotation about CM

$$T_{disk} = T_{cm} + T_{rot}$$

$$x_{cm} = l\sin\theta + a\sin\phi$$

$$z_{cm} = -l\cos\theta - a\cos\phi$$

$$\dot{x}_{cm} = l\dot{\theta}\cos\theta + a\dot{\phi}\cos\phi$$

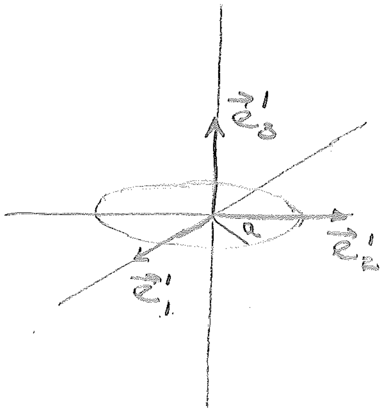
$$\dot{z}_{cm} = +l\dot{\theta}\sin\theta + a\dot{\phi}\sin\phi$$

$$T_{cm} = \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2)$$

$$= \frac{1}{2}M\Big[l^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2la\dot{\theta}\dot{\phi}\underbrace{(\cos\theta\cos\phi + \sin\theta\sin\phi)}_{\cos(\theta-\phi)}\Big]$$

$$T_{cm} = \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}Ma^2\dot{\phi}^2 + Mla\dot{\theta}\dot{\phi}\cos(\theta-\phi))$$

$$T_{rot} = \frac{1}{2}I_{disk}\dot{\phi}^2 = \frac{1}{4}Ma^2\dot{\phi}^2$$



$$I_{disk} = I_{33} = \int dA\,\sigma(x^2+y^2)$$

$$= \sigma\int r\,dr\,d\phi\, r^2$$

$$= 2\pi\sigma\underbrace{\int_0^a dr\, r^3}_{\frac{1}{4}a^4}$$

$$= \frac{\pi}{2}\sigma a^4$$

$$= \frac{1}{2}Ma^2$$

$$T_{disk} = T_{cm} + T_{rot} = \frac{1}{2}M\ell^2\dot{\theta}^2 + \frac{3}{4}Ma^2\dot{\phi}^2 + M\ell a\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$V_{disk} = Mg\,z_{cm} = -Mg(\ell\cos\theta + a\cos\phi)$$

Method 2: Calculate disk motion relative to O'

$$\vec{r}_i = \begin{pmatrix}\text{position vector for}\\ \text{a point in disk}\end{pmatrix} = \underbrace{\vec{r}_{O'}}_{\text{position vector of } O'} + \underbrace{\vec{r}_i\,'}_{\text{position vector of point relative to } O'}$$

$$T_{disk} = \sum_i \frac{1}{2}m_i\dot{\vec{r}}_i^{\,2}$$

$$= \frac{1}{2}M\dot{\vec{r}}_{O'}^{\,2} + \underbrace{\sum_i \frac{1}{2}m_i\dot{\vec{r}}_i^{\,\prime 2}}_{\text{rotational kinetic energy about } O'} + \dot{\vec{r}}_{O'}\cdot\sum_i m_i\dot{\vec{r}}_i\,'$$

$$\vec{r}_{O'} = \ell\sin\theta\,\vec{e}_x - \ell\cos\theta\,\vec{e}_z$$

$$\dot{\vec{r}}_{O'} = +\ell\dot{\theta}\cos\theta\,\vec{e}_x + \ell\dot{\theta}\sin\theta\,\vec{e}_z$$

$$\frac{1}{M}\sum_i m_i\vec{r}_i\,' = \begin{pmatrix}\text{position of CM}\\ \text{relative to } O'\end{pmatrix} = a\sin\phi\,\vec{e}_x - a\cos\phi\,\vec{e}_z$$

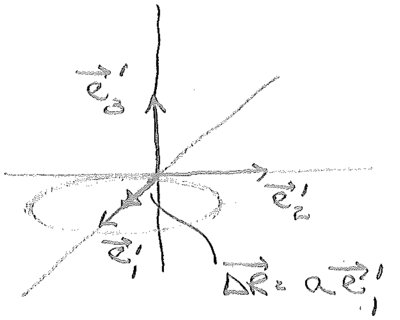
$$\frac{1}{M}\sum_i m_i\dot{\vec{r}}_i\,' = a\dot{\phi}\cos\phi\,\vec{e}_x + a\dot{\phi}\sin\phi\,\vec{e}_z$$

$$\frac{1}{2}M\dot{\vec{r}}_{O'}^{\,2} = \frac{1}{2}M\ell^2\dot{\theta}^2$$

$$\dot{\vec{r}}_{O'}\cdot\sum_i m_i\dot{\vec{r}}_i\,' = M\ell a\dot{\theta}\dot{\phi}\underbrace{(\cos\theta\cos\phi + \sin\theta\sin\phi)}_{\cos(\theta-\phi)}$$

$$= M\ell a\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$\sum_i \frac{1}{2} m_i \vec{v}_i^{\,\prime 2} = \begin{pmatrix}\text{rotational kinetic} \\ \text{energy about } O'\end{pmatrix} = \frac{1}{2} I'_{disk}\,\dot{\phi}^2 = \frac{3}{4} M a^2 \dot{\phi}^2$$



$$\overleftrightarrow{I}_{O'} = \overleftrightarrow{I}_{cm} + M\left(|\overrightarrow{\Delta R}|^2 \overleftrightarrow{1} - \overrightarrow{\Delta R} \otimes \overrightarrow{\Delta R}\right)$$

$$I'_{disk} = I'_{33}$$

$$= I_{33} + Ma^2$$

$$= I_{disk} + Ma^2$$

$$I'_{disk} = \frac{3}{2} M a^2$$

$$T_{disk} = \frac{1}{2} M \ell^2 \dot{\theta}^2 + \frac{3}{4} M a^2 \dot{\phi}^2 + M \ell a \dot{\theta} \dot{\phi} \cos(\theta - \phi)$$

Gather the results together

$$L = T_{bar} + T_{disk} - V_{bar} - V_{disk}$$

$$= \frac{1}{6} m \ell^2 \dot{\theta}^2 + \frac{1}{2} M \ell^2 \dot{\theta}^2 + \frac{3}{4} M a^2 \dot{\phi}^2 + M \ell a \dot{\theta} \dot{\phi} \cos(\theta - \phi)$$

$$+ \frac{1}{2} m g \ell \cos\theta + Mg(\ell \cos\theta + a \cos\phi)$$

$$\boxed{\begin{aligned} L = \frac{1}{2}\left(M + \frac{1}{3}m\right)\ell^2 \dot{\theta}^2 + \frac{3}{4} M a^2 \dot{\phi}^2 + M \ell a \dot{\theta}\dot{\phi}\cos(\theta - \phi) \\ + \left(M + \frac{1}{2}m\right) g \ell \cos\theta + M g a \cos\phi \end{aligned}}$$

4.4. Goldstein 5.14

$$|\vec{L}|^2 = I_1^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2$$

$$T = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3\omega_3^2$$

$$2TI_1 = I_1^2(\omega_1^2 + \omega_2^2) + I_1 I_3 \omega_3^2$$

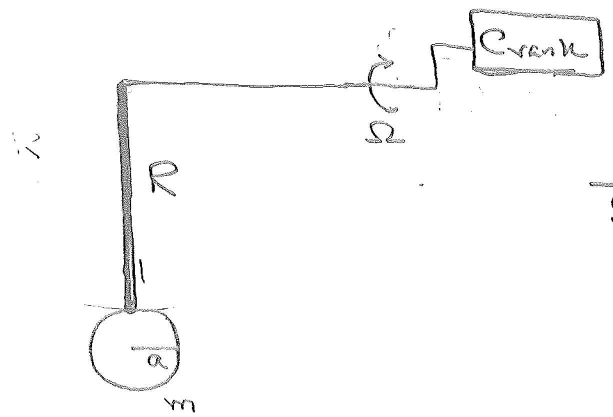
$$|\vec{L}|^2 = 2TI_1 - I_1 I_3\omega_3^2 + I_3^2\omega_3^2$$

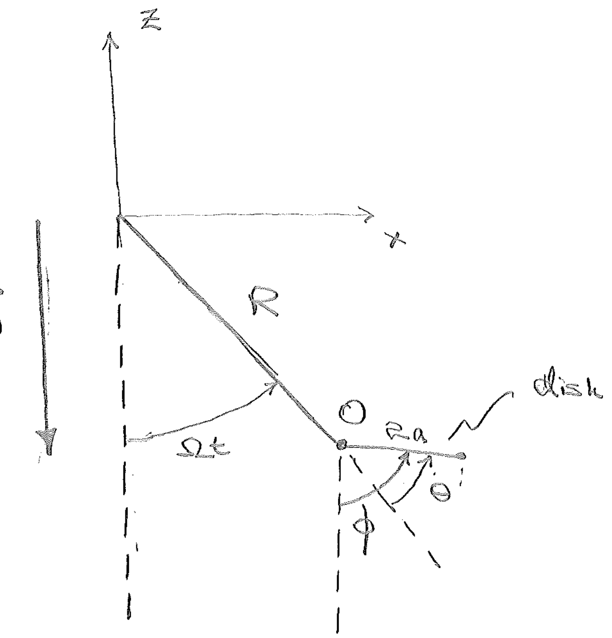
$$= 2(E - V) + \underbrace{I_3^2\omega_3^2}_{I_1 a}\left(1 - \frac{I_1}{I_3}\right)$$

$$\boxed{|\vec{L}|^2 = 2E + I_1 a\left(1 - \frac{I_1}{I_3}\right) - 2Mgl\cos\theta}$$

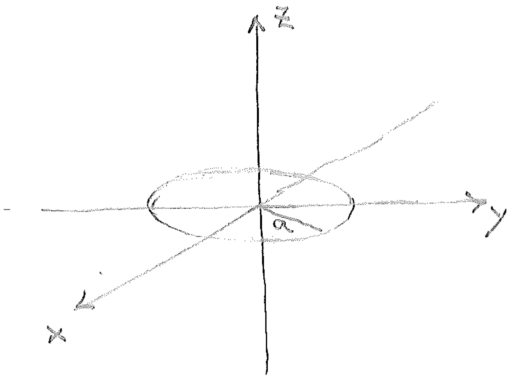
If $\vec{L}$ precesses uniformly, its magnitude is constant, which implies θ is constant, which implies uniform precession of ϕ.

4.5.





(a) Relevent moment of inertia about CM:



$m = \pi a^2 \sigma$

We want the moment of inertia about x-axis: (or y-axis)

$$I = I_{xx} = \int dA\,\sigma\,(y^2 + \cancel{z^2}^{\,0}) = \int dA\,\sigma\,y^2$$

$$I = I_{yy} = \int dA\,\sigma\,(x^2 + \cancel{z^2}^{\,0}) = \int dA\,\sigma\,x^2$$

$$2I = I_{xx} + I_{yy} = \int dA\,\sigma\,\underbrace{(x^2+y^2)}_{r^2}$$

$$= \int r\,dr\,d\phi\,\sigma r^2$$

$$= 2\pi\sigma \int_0^a dr\,r^3$$

$$= \frac{\pi}{2}\sigma a^4$$

$$= \frac{1}{2} m a^2$$

$$\boxed{I = \frac{1}{4} m a^2}$$

(b) The inertial rotation angle is $\phi = \theta + \Omega t$

Method 1: $T = T_{cm} + T_{rot}$

CM coordinates: $x = R\sin\Omega t + a\sin\phi$

$$z = -R\cos\Omega t - a\cos\phi$$

$$\dot{x} = R\Omega\cos\Omega t + a\dot{\phi}\cos\phi$$

$$\dot{z} = R\Omega\sin\Omega t + a\dot{\phi}\sin\phi$$

$$T_{cm} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\Big(R^2\Omega^2 + a^2\dot{\phi}^2 + 2Ra\Omega\dot{\phi}$$

$$+ 2aR\Omega\dot{\phi}\big(\underbrace{\cos\Omega t\cos\phi + \sin\Omega t\sin\phi}_{\cos(\phi - \Omega t) = \cos\theta}\big)\Big)$$

$$\boxed{T_{cm} = \frac{1}{2}m\left(R^2\Omega^2 + a^2\dot{\phi}^2 + 2aR\Omega\dot{\phi}\cos\theta\right)} \quad \boxed{\dot{\phi} = \dot{\theta} + \Omega}$$

$$\boxed{T_{rot} = \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}\frac{1}{4}ma^2\dot{\phi}^2}$$

$$\boxed{T = \frac{1}{2}m\left(R^2\Omega^2 + \frac{5}{4}a^2\dot{\phi}^2 + 2aR\Omega\dot{\phi}\cos\theta\right)}$$

$$\boxed{V = mgz = mg(-R\cos\Omega t - a\cos\phi)}$$

Discarding terms which do not depend on generalized coordinates or velocities, the Lagrangian becomes

$$\boxed{L = T - V = \frac{5}{8} m a^2 \dot{\phi}^2 + m a R \Omega \dot{\phi} \cos(\phi - \Omega t) + m g a \cos\phi}$$

Method 2: Let $\vec{r}_i = \underset{\substack{\uparrow \\ \text{position of attachment (point O)}}}{\vec{r}_0} + \vec{r}_i' \Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_0 + \dot{\vec{r}}_i'$

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 = \frac{1}{2} m \dot{\vec{r}}_0^2 + \underbrace{\frac{1}{2} \sum_i m_i \dot{\vec{r}}_i'^2}_{\frac{1}{2} \underset{\substack{\uparrow \\ \text{moment of} \\ \text{inertia about O}}}{I'} \dot{\phi}^2} + \underbrace{\left(\sum_i m_i \dot{\vec{r}}_i' \right)}_{\left(\substack{\text{momentum of CM} \\ \text{relative to O}}\right)} \cdot \dot{\vec{r}}_0$$

$$\vec{r}_0 = R \sin \Omega t \, \vec{e}_x - R \cos \Omega t \, \vec{e}_z$$

$$\sum_i m_i \vec{r}_i' = m (a \sin\phi \, \vec{e}_x - a \cos\phi \, \vec{e}_z)$$

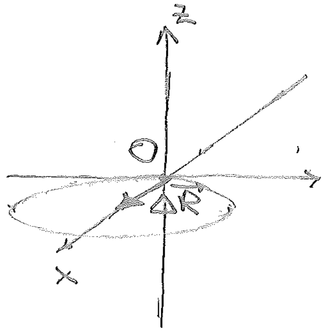
$$\dot{\vec{r}}_0 = R\Omega (\cos \Omega t \, \vec{e}_x + \sin \Omega t \, \vec{e}_z)$$

$$\sum_i m_i \dot{\vec{r}}_i' = m a \dot{\phi} (\cos\phi \, \vec{e}_x + \sin\phi \, \vec{e}_z)$$

$$\frac{1}{2} m \dot{\vec{r}}_0^2 = \frac{1}{2} m R^2 \Omega^2 = \text{constant}$$

$$\left(\sum_i m_i \dot{\vec{r}}_i' \right) \cdot \dot{\vec{r}}_0 = m a R \Omega \dot{\phi} \Big(\underbrace{\cos\phi \cos\Omega t + \sin\phi \sin \Omega t}_{\cos(\phi - \Omega t)} \Big)$$

$$= m a R \Omega \dot{\phi} \cos(\phi - \Omega t)$$



$$\overleftrightarrow{I}_0 = \overleftrightarrow{I}_{cm} + m(|\Delta\vec{R}|^2\overleftrightarrow{1} - \Delta\vec{R}\otimes\Delta\vec{R})$$

$$\Delta\vec{R} = a\vec{e}_x$$

$$I' = I'_{yy} = I_{yy} + ma^2 = I + ma^2 = \frac{5}{4}ma^2$$

$$\frac{1}{2}\sum_i m_i \dot{\vec{r}}_i'^2 = \frac{5}{8}ma^2\dot{\phi}^2$$

$$\boxed{T = \frac{1}{2}mR^2\Omega^2 + \frac{5}{8}ma^2\dot{\phi}^2 + maR\Omega\dot{\phi}\cos(\phi - \Omega t)}$$

(c) Equation of motion:

$$\frac{\partial L}{\partial\dot{\phi}} = \frac{5}{4}ma^2\dot{\phi} + maR\Omega\cos(\phi - \Omega t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{5}{4}ma^2\ddot{\phi} - maR\Omega\sin(\phi - \Omega t)(\dot{\phi} - \Omega)$$

$$\frac{\partial L}{\partial\phi} = -maR\Omega\dot{\phi}\sin(\phi - \Omega t) - mga\sin\phi$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \frac{5}{4}ma^2\ddot{\phi} + maR\Omega^2\sin(\phi - \Omega t) + mga\sin\phi$$

$$\frac{5}{4}\ddot{\phi} + \frac{g}{a}\sin\phi + \frac{R\Omega^2}{a}\sin(\phi - \Omega t) = 0$$

$$\phi = \theta + \Omega t\,,\quad \dot{\phi} = \dot{\theta} + \Omega,\quad \ddot{\phi} = \ddot{\theta}$$

$$\boxed{\frac{5}{4}\ddot{\theta} + \frac{g}{a}\sin(\theta + \Omega t) + \frac{R\Omega^2}{a}\sin\theta = 0}$$

(d) Assume ① $g \ll R\Omega^2$ and ② $a \ll R \Leftrightarrow \Omega \ll \underset{\uparrow}{\sqrt{\frac{R}{a}}\,\Omega}$

resonant frequency

Now linearize about $\theta = 0$

$\sin\theta \simeq \theta$

$$\sin(\theta + \Omega t) = \sin\theta\cos\Omega t + \cos\theta\sin\Omega t$$

$$\simeq \theta\cos\Omega t + \sin\Omega t$$

neglect: small amplitude, off resonance

$$\frac{5}{4}\ddot{\theta} + \frac{g}{a}\cancel{\theta\cos\Omega t} + \frac{g}{a}\sin\Omega t + \frac{R\Omega^2}{a}\theta = 0$$

$$\boxed{\frac{5}{4}\ddot{\theta} + \frac{R\Omega^2}{a}\theta = -\frac{g}{a}\sin\Omega t}$$

Steady-state Solution: $\theta(t) = A\sin\Omega t$

$$\underbrace{\left(-\frac{5}{4} + \frac{R}{a}\right)}_{\simeq R/a}\Omega^2 A\sin\Omega t = -\frac{g}{a}\sin\Omega t$$

$$\Longrightarrow A = -\frac{g}{R\Omega^2}$$

$$\boxed{\theta(t) = -\frac{g}{R\Omega^2}\sin\Omega t}$$

$|A| = g/R\Omega^2 \ll 1$, which justifies the approximation.