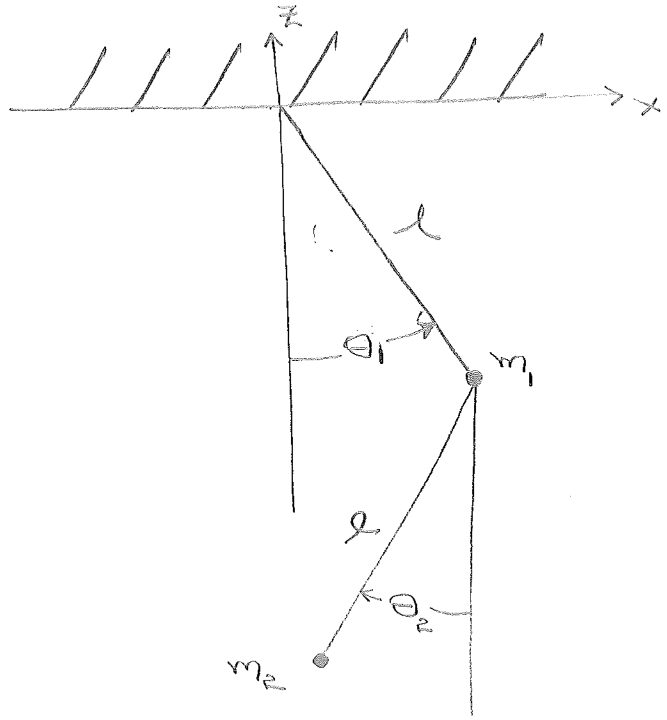
Phys 503

Homework # 5

Solution Set

5.1. Goldstein 6.4



$$x_1 = l\sin\theta_1 = l\theta_1$$

$$z_1 = -l\cos\theta_1 = l\left(-1 + \tfrac{1}{2}\theta_1^2\right)$$

$$x_2 = l\sin\theta_1 - l\sin\theta_2 = l(\theta_1 - \theta_2)$$

$$z_2 = -l(\cos\theta_1 + \cos\theta_2) = l\left(-2 + \tfrac{1}{2}(\theta_1^2 + \theta_2^2)\right)$$

$$T_1 = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{z}_1^2) = \tfrac{1}{2}m_1 l^2\dot{\theta}_1^2$$

$$T_2 = \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{z}_2^2) = \tfrac{1}{2}m_2 l^2(\dot{\theta}_1 - \dot{\theta}_2)^2$$

$$V_1 = m_1 g(z_1 + l) = \tfrac{1}{2}m_1 g l\,\theta_1^2$$

$$V_2 = m_2 g(z_2 + 2l) = \tfrac{1}{2}m_2 g l(\theta_1^2 + \theta_2^2)$$

$$\begin{aligned}
T = T_1 + T_2 &= \tfrac{1}{2}m_1 l^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2 l^2(\dot{\theta}_1 - \dot{\theta}_2)^2 \\
&= \tfrac{1}{2}l^2\left(m_1\dot{\theta}_1^2 + m_2(\dot{\theta}_1^2 - 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\right) \\
&= \tfrac{1}{2}l^2\left[(m_1+m_2)\dot{\theta}_1^2 - 2m_2\dot{\theta}_1\dot{\theta}_2 + m_2\dot{\theta}_2^2\right] \\
&= \tfrac{1}{2}(\dot{\theta}_1\ \ \dot{\theta}_2)\,\underbrace{l^2\begin{pmatrix} m_1+m_2 & -m_2 \\ -m_2 & m_2 \end{pmatrix}}_{T}\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix}
\end{aligned}$$

The kinetic energy matrix is

$$\boxed{T = \ell^2 \begin{pmatrix} m_1+m_2 & -m_2 \\ -m_2 & m_2 \end{pmatrix} = \ell^2 \begin{pmatrix} M & -m_2 \\ -m_2 & m_2 \end{pmatrix}}$$

$$\boxed{M \equiv m_1 + m_2}$$

$$V = V_1 + V_2 = \tfrac{1}{2} g\ell \left[(m_1+m_2)\,\theta_1^2 + m_2\,\theta_2^2 \right]$$

$$= \frac{1}{2} \begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix} \underbrace{g\ell \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}}_{V} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Equations of motion: $T_{jk}\ddot{\theta}_k + V_{jk}\theta_k = 0$

$$\ddot{\theta}_1 - \frac{m_2}{M}\ddot{\theta}_2 + \Omega^2\theta_1 = 0$$

$$-\ddot{\theta}_1 + \ddot{\theta}_2 + \Omega^2\theta_2 = 0$$

The potential energy matrix is

$$\boxed{V = g\ell \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix} = \ell^2\Omega^2 \begin{pmatrix} M & 0 \\ 0 & m_2 \end{pmatrix}}$$

$$\boxed{\Omega^2 \equiv g/\ell = \begin{pmatrix} \text{pendulum} \\ \text{frequency} \end{pmatrix}}$$

We want to solve for the eigenvalues ω^2 and eigenvectors a according to

$$0 = (V - \omega^2 T)\,a = \ell^2 \begin{pmatrix} (\Omega^2-\omega^2)M & +\omega^2 m_2 \\ +\omega^2 m_2 & (\Omega^2-\omega^2)m_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Secular equation:

$$0 = \det \begin{pmatrix} (\Omega^2 - \omega^2) M & \omega^2 m_2 \\ \omega^2 m_2 & (\Omega^2 - \omega^2) m_2 \end{pmatrix}$$

$$\boxed{0 = (\Omega^2 - \omega^2)^2 M m_2 - \omega^4 m_2^2}$$

$$\Rightarrow \pm (\Omega^2 - \omega_\pm^2) \sqrt{M m_2} = \pm \omega_\pm^2 m_2$$

$$\boxed{\Omega^2 - \omega_\pm^2 = \pm \omega^2 \sqrt{\frac{m_2}{M}}}$$

$$\boxed{\omega_\pm^2 \left(1 \pm \sqrt{\frac{m_2}{M}}\right) = \Omega^2}$$

$$\boxed{\omega_\pm = \frac{\Omega}{\sqrt{1 \pm \sqrt{m_2/M}}}}$$

Normal-mode frequencies

$$\boxed{\text{If } m_2 \ll M, \quad \omega_\pm = \Omega\left(1 \mp \frac{1}{2}\sqrt{\frac{m_2}{M}}\right)}$$

Normal modes: $\underbrace{(\Omega^2 - \omega_\pm^2)}_{\pm \omega_\pm^2 \sqrt{m_2 M}} M a_1 + \omega_\pm^2 m_2 a_2 = 0$

$$\Longrightarrow \quad a_1 \pm \sqrt{\frac{m_2}{M}}\, a_2 = 0$$

$$a_1 = \mp \sqrt{\frac{m_2}{M}}\, a_2$$

$$a = a_2 \begin{pmatrix} \mp \sqrt{m_2/M} \\ 1 \end{pmatrix}$$

Normalization:

$$\underline{1} = \tilde{a} T a = l^2 a_2^2 \left(\mp\sqrt{m_2/M} \quad 1 \right) \begin{pmatrix} M & -m_2 \\ -m_2 & m_2 \end{pmatrix} \begin{pmatrix} \mp\sqrt{m_2/M} \\ 1 \end{pmatrix}$$

$$= l^2 a_2^2 \left(m_2 + m_2 \mp 2\sqrt{\frac{m_2}{M}} (-m_2) \right)$$

$$= 2 m_2 l^2 a_2^2 \underbrace{\left(1 \pm \sqrt{m_2/M} \right)}_{\Omega^2/\omega_\pm^2}$$

$$a_2^2 = \frac{\omega_\pm^2}{2 m_2 l^2 \Omega^2} = \frac{\omega_\pm^2}{2 m_2 g l} \Longrightarrow a_2 = \frac{\omega_\pm}{\sqrt{2 m_2}\, \Omega l}$$

$$\boxed{a_\pm = \frac{1}{\sqrt{2}} \frac{\omega_\pm}{\Omega l} \begin{pmatrix} \mp 1/\sqrt{M} \\ 1/\sqrt{m_2} \end{pmatrix}}$$

Normal modes

Normal coordinates:

$$\boxed{A = \begin{pmatrix} a_{1+} & a_{1-} \\ a_{2+} & a_{2-} \end{pmatrix} = \frac{1}{\sqrt{2}} \frac{1}{\Omega l} \begin{pmatrix} -\omega_+/\sqrt{M} & \omega_-/\sqrt{M} \\ \omega_+/\sqrt{m_2} & \omega_-/\sqrt{m_2} \end{pmatrix}}$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = A \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = A^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \tilde{A} T \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\boxed{\begin{aligned}\Theta_1 &= \frac{1}{\sqrt{2M}}\frac{1}{\Omega\ell}\left(-\omega_+ J_+ + \omega_- J_-\right)\\ \Theta_2 &= \frac{1}{\sqrt{2m_2}}\frac{1}{\Omega\ell}\left(\omega_+ J_+ + \omega_- J_-\right)\end{aligned}}$$

$$A' = \tilde{A}T = \frac{1}{\sqrt{2}\,\Omega\ell}\begin{pmatrix} -\omega_+/\sqrt{M} & \omega_+/\sqrt{m_2} \\ \omega_-/\sqrt{M} & \omega_-/\sqrt{m_2}\end{pmatrix}\ell^2\begin{pmatrix} M & -m_2 \\ -m_2 & m_2\end{pmatrix}$$

$$= \frac{\ell}{\sqrt{2}\,\Omega}\begin{pmatrix} -\omega_+\sqrt{M} - \omega_+\sqrt{m_2} & \omega_+ m_2/\sqrt{M} + \omega_+\sqrt{m_2} \\ +\omega_-\sqrt{M} - \omega_-\sqrt{m_2} & -\omega_- m_2/\sqrt{M} + \omega_-\sqrt{m_2}\end{pmatrix}$$

$$-\omega_+\sqrt{M}\underbrace{\left(1+\sqrt{m_2/M}\right)}_{\Omega^2/\omega_+^2} = -\Omega^2\frac{\sqrt{M}}{\omega_+}$$

$$\omega_+\sqrt{m_2}\underbrace{\left(1+\sqrt{m_2/M}\right)}_{\Omega^2/\omega_+^2} = \Omega^2\frac{\sqrt{m_2}}{\omega_+}$$

$$\omega_-\sqrt{M}\underbrace{\left(1-\sqrt{m_2/M}\right)}_{\Omega^2/\omega_-^2} = \Omega^2\frac{\sqrt{M}}{\omega_-}$$

$$\omega_-\sqrt{m_2}\underbrace{\left(1-\sqrt{m_2/M}\right)}_{\Omega^2/\omega_-^2} = \Omega^2\frac{\sqrt{m_2}}{\omega_-}$$

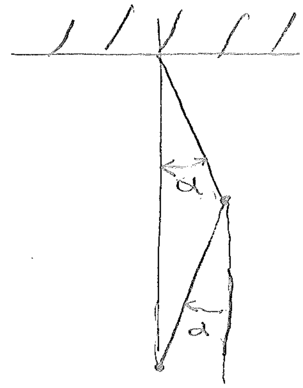
$$\boxed{A' = \frac{1}{\sqrt{2}}\Omega\ell\begin{pmatrix} -\sqrt{M}/\omega_+ & \sqrt{m_2}/\omega_+ \\ \sqrt{M}/\omega_- & \sqrt{m_2}/\omega_-\end{pmatrix}}$$

$$\begin{pmatrix} \zeta_+ \\ \zeta_- \end{pmatrix} = A^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} \zeta_+ &= \frac{1}{\sqrt{2}} \frac{\Omega l}{\omega_+} \left(-\sqrt{M}\,\theta_1 + \sqrt{m_2}\,\theta_2 \right) \\ \zeta_- &= \frac{1}{\sqrt{2}} \frac{\Omega l}{\omega_-} \left(\sqrt{M}\,\theta_1 + \sqrt{m_2}\,\theta_2 \right) \end{aligned}}$$

unnormalized normal coördinates (one can easily verify that these are the normal coördinates, with frequencies $\omega_\pm$, by substituting into the equations for θ_1 and θ_2)

Initial condition:



$$\theta_1(0) = \theta_2(0) = \alpha$$
$$\dot\theta_1(0) = \dot\theta_2(0) = 0$$

$$\zeta_\pm(0) = \frac{1}{\sqrt{2}} \frac{\Omega l}{\omega_\pm} \left(\mp\sqrt{M} + \sqrt{m_2} \right) \alpha$$
$$= \mp\sqrt{\frac{M}{2}} \frac{\Omega l}{\omega_\pm} \underbrace{\left(1 \mp \sqrt{\frac{m_2}{M}} \right)}_{\Omega^2/\omega_\mp^2} \alpha$$

$$\boxed{\zeta_\pm(0) = \mp\sqrt{\frac{M}{2}} \frac{\Omega l}{\omega_\pm} \frac{\Omega^2}{\omega_\mp^2} \alpha}$$

$$\dot\zeta_\pm(0) = 0$$

$$\Longrightarrow \boxed{\zeta_\pm(t) = \zeta_\pm(0) e^{-i\omega_\pm t}}$$

$$\boxed{\begin{aligned}\Theta_1 &= \frac{1}{2}\,\alpha\,\Omega^2\left(\frac{e^{-i\omega_+ t}}{\omega_-^2} + \frac{e^{-i\omega_- t}}{\omega_+^2}\right)\\ \Theta_2 &= \frac{1}{2}\sqrt{\frac{M}{m_2}}\,\alpha\,\Omega^2\left(-\frac{e^{-i\omega_+ t}}{\omega_-^2} + \frac{e^{-i\omega_- t}}{\omega_+^2}\right)\end{aligned}}$$

$$\Theta_1 = \underbrace{\frac{1}{2}}_{2 \longrightarrow \sqrt{\frac{m_1}{M}}}\alpha\, e^{-i(\omega_+ + \omega_-)t/2}\underbrace{\left(\frac{\Omega^2}{\omega_-^2}\, e^{-i(\omega_+ - \omega_-)t/2} + \frac{\Omega^2}{\omega_+^2}\, e^{+i(\omega_+ - \omega_-)t/2}\right)}$$

$$= \cos(\omega_+ - \omega_-)t/2\;\underbrace{\left(\frac{\Omega^2}{\omega_-^2} + \frac{\Omega^2}{\omega_+^2}\right)} \longrightarrow \left(1 - \sqrt{\frac{m_2}{M}} + 1 + \sqrt{\frac{m_2}{M}}\right) = 2$$

$$+\; i\sin(\omega_+ - \omega_-)t/2\;\underbrace{\left(\frac{\Omega^2}{\omega_+^2} - \frac{\Omega^2}{\omega_-^2}\right)} \longrightarrow \left(1 + \sqrt{\frac{m_2}{M}} - 1 + \sqrt{\frac{m_2}{M}}\right)$$

$$= 2\sqrt{\frac{m_2}{M}}$$

$$= 2\left(\underbrace{\cos(\omega_+ - \omega_-)t/2}_{\sqrt{\frac{m_1}{M}}} + i\underbrace{\sqrt{\frac{m_2}{M}}}_{\sqrt{\frac{m_1}{M}}}\sin(\omega_+ - \omega_-)t/2\right)$$

$$\boxed{\Theta_1 = \alpha\, e^{-i(\omega_+ + \omega_-)t/2}\left(\cos(\omega_+ - \omega_-)t/2 + i\sqrt{\frac{m_2}{M}}\sin(\omega_+ - \omega_-)t/2\right)}$$

$$\boxed{\Theta_2 = \alpha\sqrt{\frac{M}{m_2}}\, e^{-i(\omega_+ + \omega_-)t/2}\left(i\sin(\omega_+ - \omega_-)t/2 + \sqrt{\frac{m_2}{M}}\cos(\omega_+ - \omega_-)t/2\right)}$$

out-of-phase beats

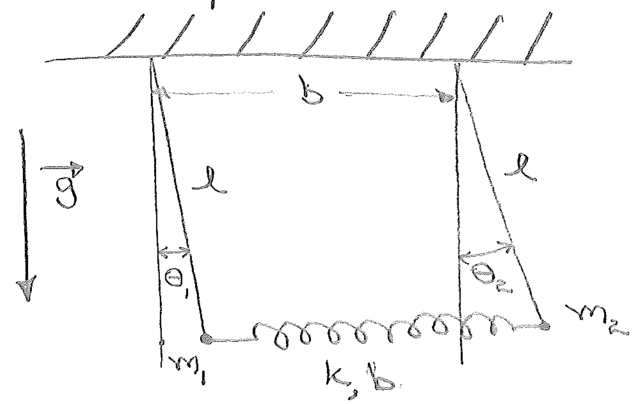
neglect these if $m_2 \ll m_1$

The beat frequency is

$$\frac{\omega_+ - \omega_-}{2} = \frac{1}{2}\Omega\left(\frac{1}{\sqrt{1+\sqrt{m_2/M}}} - \frac{1}{\sqrt{1-\sqrt{m_2/M}}}\right)$$

$$m_2 \ll m_1 \rightarrow \simeq \frac{1}{2}\Omega\left(\cancel{1} - \frac{1}{2}\sqrt{\frac{m_2}{M}} - \cancel{1} - \frac{1}{2}\sqrt{\frac{m_2}{M}}\right) = -\frac{1}{2}\Omega\sqrt{\frac{m_2}{M}}$$

5.2. Coupled pendulum



Small oscillations

$$x_1 = l\sin\theta_1 = l\theta_1$$

$$z_1 = l(1-\cos\theta_1) = \tfrac{1}{2}l\theta_1^2$$

$$x_2 = b + l\sin\theta_2 = b + l\theta_2$$

$$z_2 = l(1-\cos\theta_2) = \tfrac{1}{2}l\theta_2^2$$

(a) $$T = \frac{1}{2}l^2\left(m_1\dot\theta_1^2 + m_2\dot\theta_2^2\right) = \frac{1}{2}\dot{\tilde\theta}\cdot l^2\begin{pmatrix} m_1 & 0 \\ 0 & m_2\end{pmatrix}\dot\theta$$

Kinetic-energy matrix: $$\boxed{T = l^2\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2\end{pmatrix}}_{\equiv t}}$$

$$V = \underbrace{g(m_1z_1 + m_2z_2)}_{\frac{1}{2}gl(m_1\theta_1^2 + m_2\theta_2^2)} + \frac{1}{2}k\Big(\underbrace{\left[(x_2-x_1)^2 + (z_2-z_1)^2\right]^{1/2} - b}_{x_2 - x_1 - b \,=\, l(\theta_2-\theta_1)}\Big)^2$$

$$= \frac{1}{2}gl(m_1\theta_1^2 + m_2\theta_2^2) + \frac{1}{2}kl^2(\theta_2-\theta_1)^2$$

$$= \frac{1}{2}l^2\left[\frac{g}{l}(m_1\theta_1^2 + m_2\theta_2^2) + k(\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2)\right]$$

$$= \frac{1}{2}\tilde\theta\cdot l^2\begin{pmatrix} m_1g/l + k & -k \\ -k & m_2g/l + k\end{pmatrix}\tilde\theta$$

Potential-energy matrix: $$\boxed{V = l^2\underbrace{\begin{pmatrix} m_1g/l + k & -k \\ -k & m_2g/l + k\end{pmatrix}}_{\equiv v}}$$

(b) Eigenvalue problem: $0 = (V - \omega^2 T)a = l^2(v - \omega^2 t)a$

$$v - \omega^2 t = \begin{pmatrix} m_1 g/l + k - \omega^2 m_1 & -k \\ -k & m_2 g/l + k - \omega^2 m_2 \end{pmatrix}$$

Official method:

$$0 = \det(v - \omega^2 t)$$

$$= [m_1(g/l - \omega^2) + k][m_2(g/l - \omega^2) + k] - k^2$$

Let $\alpha \equiv \omega^2 - g/l$

$$0 = (m_1\alpha - k)(m_2\alpha - k) - k^2$$

$$= m_1 m_2 \alpha^2 - k(m_1 + m_2)\alpha$$

$$\Longrightarrow \alpha_1 = 0 \quad \text{and} \quad \alpha_1 = k\frac{m_1 + m_2}{m_1 m_2} = k/\mu$$

reduced mass

$$\boxed{\omega_1^2 = g/l \quad \text{and} \quad \omega_2^2 = g/l + k/\mu}$$

Guess method: It should be clear that one normal mode corresponds to the two masses moving together ($\theta_1 = \theta_2$), i.e., to vector $\begin{pmatrix}1\\1\end{pmatrix}$.

$$v\begin{pmatrix}1\\1\end{pmatrix} = \frac{g}{l}\begin{pmatrix}m_1\\m_2\end{pmatrix} \; ; \; t\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}m_1\\m_2\end{pmatrix}$$

$$(v - \omega^2 t)\begin{pmatrix}1\\1\end{pmatrix} = \left(\frac{g}{l} - \omega^2\right)\begin{pmatrix}m_1\\m_2\end{pmatrix} = 0 \quad \text{if} \quad \boxed{\omega_1^2 = g/l}$$

The second normal mode must be orthogonal to $\begin{pmatrix}1\\1\end{pmatrix}$, relative to T:

$$0 = \tilde{a}_1 T a_2 \Rightarrow 0 = (1 \ \ 1)\begin{pmatrix}m_1 & 0\\0 & m_2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha m_1 + \beta m_2$$

$$\Rightarrow a_2 \propto \begin{pmatrix}-m_2\\m_1\end{pmatrix}$$

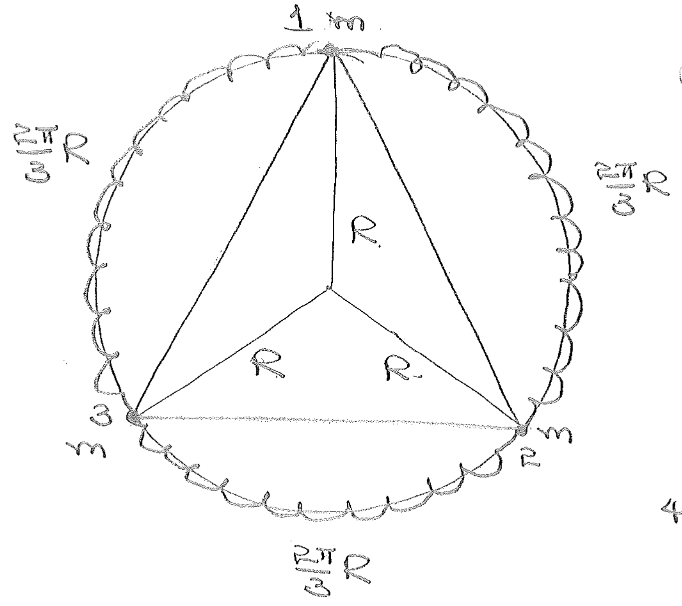
$$v\begin{pmatrix}-m_2\\m_1\end{pmatrix} = \begin{pmatrix}-m_1 m_2 g/l - (m_2+m_1)k\\+m_1 m_2 g/l + (m_2+m_1)k\end{pmatrix} = \left(\frac{m_1 m_2 g}{l} + (m_1+m_2)k\right)\begin{pmatrix}-1\\+1\end{pmatrix}$$

$$t\begin{pmatrix}-m_2\\m_1\end{pmatrix} = \begin{pmatrix}-m_1 m_2\\m_1 m_2\end{pmatrix} = m_1 m_2\begin{pmatrix}-1\\+1\end{pmatrix}$$

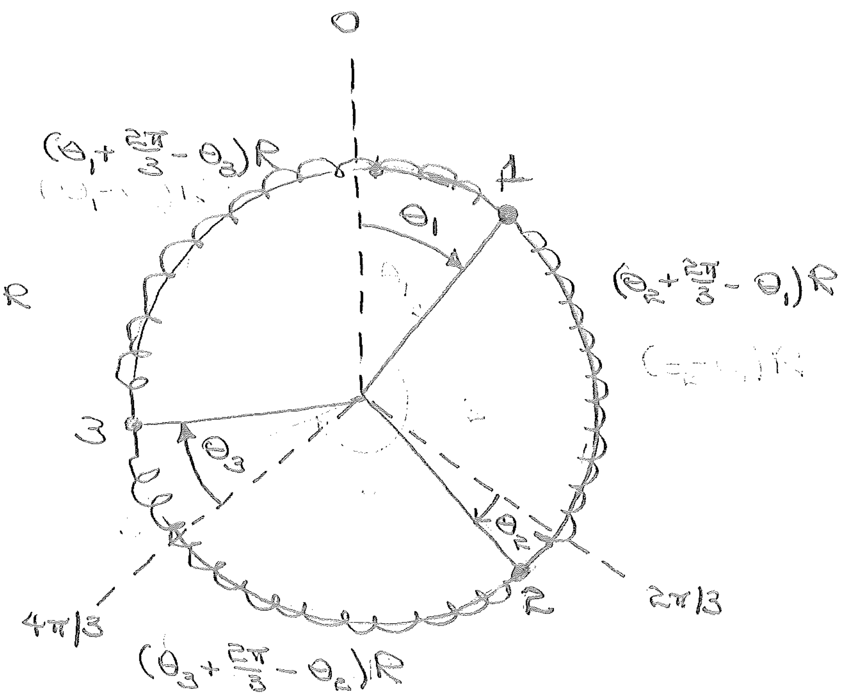
$$(v - \omega^2 t)\begin{pmatrix}-m_2\\m_1\end{pmatrix} = \left(\frac{m_1 m_2 g}{l} + (m_1+m_2)k - m_1 m_2 \omega^2\right)\begin{pmatrix}-1\\+1\end{pmatrix} = 0$$

if $\boxed{\omega_2^2 = \frac{g}{l} + k\,\frac{m_1+m_2}{m_1 m_2} = \frac{g}{l} + \frac{k}{\mu}}$

5.3. Goldstein 6.10



Equilibrium



(a)
$$T=\frac{1}{2}mR^2\left(\dot\theta_1^2+\dot\theta_2^2+\dot\theta_3^2\right)=\frac{1}{2}mR^2\,\tilde{\dot\theta}\,\overset{\text{unit matrix}}{\overset{\downarrow}{\mathbb{1}}}\,\dot\theta$$

$$V=\frac{1}{2}k\left(\left[\left(\theta_2+\frac{2\pi}{3}-\theta_1\right)R-\alpha R\right]^2+\left[\left(\theta_3+\frac{2\pi}{3}-\theta_2\right)R-\alpha R\right]^2+\left[\left(\theta_1+\frac{2\pi}{3}-\theta_3\right)R-\alpha R\right]^2\right)$$

αR is the unstretched length of the springs

$$=\frac{1}{2}kR^2\Big(\theta_2^2-2\theta_1\theta_2+\theta_1^2+\overset{0}{\overset{\uparrow}{2\left(\frac{2\pi}{3}-\alpha\right)(\theta_2-\theta_1)}}+\left(\frac{2\pi}{3}-\alpha\right)^2$$
$$+\theta_3^2-2\theta_3\theta_2+\theta_2^2+2\left(\frac{2\pi}{3}-\alpha\right)(\theta_3-\theta_2)+\left(\frac{2\pi}{3}-\alpha\right)^2$$
$$+\theta_1^2-2\theta_1\theta_3+\theta_3^2+2\left(\frac{2\pi}{3}-\alpha\right)(\theta_1-\theta_3)+\left(\frac{2\pi}{3}-\alpha\right)^2\Big)$$

Discard the constant term $\underbrace{\frac{1}{2}kR^2\,3\left(\frac{2\pi}{3}-\alpha\right)^2}_{\text{energy stored in springs at equilibrium}}$ by re-zeroing the potential energy.

$$V = \frac{1}{2} k R^2 \Theta \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \Theta$$

The kinetic and potential matrices are

$$T = \frac{1}{\,} m R^2 \mathbb{1} \; ; \quad V = \frac{1}{\,} k R^2 \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}}_{\equiv \mathcal{V}}$$

Eigenvalue problem:

$$0 = (V - \omega^2 T)\, a = m R^2 \left(\Omega^2 \mathcal{V} - \omega^2 \mathbb{1} \right) a$$

$$\Omega^2 = k/m = \begin{pmatrix} \text{natural frequency} \\ \text{of springs} \end{pmatrix}$$

$$0 = \frac{1}{\,} m R^2 \Omega^2 \left(\mathcal{V} - \alpha^2 \mathbb{1} \right) a \, , \quad \boxed{\alpha^2 = \frac{\omega^2}{\Omega^2}}$$

Work with the eigenvalue problem

$$\boxed{\mathcal{V} a = \alpha^2 a}$$

Instead of using the official method, we can guess the solutions.

$$\mathcal{V}\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \longrightarrow \alpha^2 = \omega^2/\Omega^2 = 0 \quad \left(\text{zero-frequency mode}\right)$$

$$\mathcal{V}\begin{pmatrix}0\\1\\-1\end{pmatrix} = 3\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\mathcal{V}\begin{pmatrix}-1\\0\\1\end{pmatrix} = 3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\mathcal{V}\begin{pmatrix}1\\-1\\0\end{pmatrix} = 3\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$\longrightarrow \alpha^2 = \omega^2/\Omega^2 = 3$

Only two of these are linearly independent.

Using the official normalization, i.e., $1 = \tilde{a}\, T\, a = mR^2\, \tilde{a}\, a$, we have 4 normalized eigenvectors

$$a_0 = \frac{1}{\sqrt{m}\,R}\,\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

all 3 masses move in common, without stretching springs

$\omega_0 = 0$

$$a_1 = \frac{1}{\sqrt{m}\,R}\,\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

1 is at rest; 2 & 3 move in opposite directions

$$a_2 = \frac{1}{\sqrt{m}\,R}\,\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

2 is at rest; 1 and 3 move in opposite directions

$$a_3 = \frac{1}{\sqrt{m}\,R}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

3 is at rest; 1 and 2 move in opposite directions

$\longrightarrow \omega = \sqrt{3}\,\Omega$

a_1, a_2, and a_3 define an equilateral triangle in the plane orthogonal to a_0. We should choose two orthogonal vectors in that plane. For example, we could choose a_1 and

$$\boxed{a_{1\perp} = \frac{1}{\sqrt{m}\,R}\,\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \frac{1}{\sqrt{3}}\left(a_3 - a_2\right);\quad \omega = \sqrt{3}\,\Omega}$$

(b) Suppose that it is the spring between 2 and 3 whose force constant changes to $k' = k + \delta k$. It should be clear that a_0 and a_1 remain normal modes; hence, by orthogonality, so does $a_{1\perp}$. Let's figure out the frequencies. The kinetic energy matrix is unchanged, but the potential energy becomes

Since k' doesn't compress in $a_{1\perp}$, it is also clear that it remains a normal mode w/ unchanged frequency.

$$V = \frac{1}{2}\Big(k\big[(\theta_2 + \underbrace{\beta}_{\text{new equilibrium length}} - \theta_1)R - \alpha R\big]^2 + k\big[(\theta_1 + \underbrace{\beta}_{\text{new equilibrium length}} - \theta_3)R - \alpha R\big]^2$$

$$+\, k'\big[(\theta_3 + \underbrace{\beta'}_{\text{new equilibrium length}} - \theta_2)R - \underbrace{\alpha' R}_{\text{new unstretched length}}\big]^2 \Big)$$

$$= \frac{1}{2}R^2\Big(k(\theta_2 - \theta_1)^2 + 2k(\beta - \alpha)(\theta_2 - \theta_1) + k(\beta - \alpha)^2$$

$$+\, k(\theta_1 - \theta_3)^2 + 2k(\beta - \alpha)(\theta_1 - \theta_3) + k(\beta - \alpha)^2$$

$$+\, k'(\theta_3 - \theta_2)^2 + 2k'(\beta' - \alpha')(\theta_3 - \theta_2) + k'(\beta' - \alpha')^2 \Big)$$

omit

$$2\big(\underbrace{k(\beta - \alpha) - k'(\beta' - \alpha')}_{=0 \text{ (defines equilibrium lengths)}}\big)(\theta_2 - \theta_3)$$

Discarding the constant in V, we get

$$V = \frac{1}{2} m \Omega^2 R^2 \left((\theta_2 - \theta_1)^2 + (\theta_1 - \theta_3)^2 + \frac{k'}{k} (\theta_3 - \theta_2)^2 \right)$$

$$= \frac{1}{2} m \Omega^2 R^2 \tilde{\theta} \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1+k'/k & -k'/k \\ -1 & -k'/k & 1+k'/k \end{pmatrix}}_{v'} \theta$$

The potential energy matrix is

$$\underline{V' = m \Omega^2 R^2 v'.}$$

Eigenvalue problem:

$$0 = (V' - \omega^2 T) a = m R^2 (\Omega^2 v' - \omega^2 1) a$$

$$\Longleftrightarrow \boxed{v' a = \alpha^2 a}$$

Try a_0, a_1, and $a_{1\perp}$:

$$\boxed{\begin{array}{ll} v' a_0 = 0 , & \omega_0 = 0 \\ v' a_1 = (1 + 2k'/k) a_1 , & \omega_1 = \Omega \sqrt{1 + 2k'/k} \\ v' a_{1\perp} = 3 a_{1\perp} , & \omega_{1\perp} = \sqrt{3}\, \Omega \end{array}}$$

Frequency changes because k' is compressed

Frequency doesn't change because k' is not compressed.

The perturbation lifts the degeneracy in the plane spanned by a_1, a_2, and a_3, and picks out particular vectors a_1 and $a_{1\perp}$.

(c) Let 1 have mass $m' = m + \delta m$. Again a_0 and a_1 remain normal modes; the 3rd normal mode must be orthogonal to a_0 and a_1 with respect to T'.

$$T = \frac{1}{2}R^2\left(m'\dot{\theta}_1^2 + m\dot{\theta}_2^2 + m\dot{\theta}_3^2\right)$$

$$= \frac{1}{2}mR^2\,\dot{\tilde{\theta}}\underbrace{\begin{pmatrix} m'/m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{t'}\dot{\theta}$$

The kinetic energy matrix is

$$T' = mR^2 t'$$

The potential energy is unchanged.

Eigenvalue problem:

$$0 = (V - \omega^2 T')a = mR^2\left(\Omega^2 v - \omega^2 t'\right)a$$

$$\Longleftrightarrow \quad va = \alpha^2 t' a$$

Try a_0, a_1, and $a'_{1\perp} = \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}$, where

$$0 = \tilde{a}_0 T' a'_{1\perp} \Longrightarrow 0 = \frac{m'}{m}\beta + \gamma + \delta$$

$$0 = \tilde{a}_1 T' a'_{1\perp} \Longrightarrow 0 = \gamma - \delta$$

$$\Longrightarrow \quad a'_{1\perp} = \underset{\substack{\uparrow \\ \text{normalization}}}{\mu}\begin{pmatrix} 2\frac{m}{m'} \\ -1 \\ -1 \end{pmatrix}$$

mass 1 moves less as its mass increases

$\mathcal{V} a_0 = 0, \qquad \omega_0 = 0$

$\mathcal{V} a_1 = 3 a_1 = 3 t' a_1, \qquad \omega_1 = \sqrt{3}\,\Omega \longleftarrow$ This frequency is unchanged because $\underline{1}$ is at rest

$t' a_1 = a_1$

$\mathcal{V} a'_{1\perp} = (\underline{1} + 2m/m') a'_{1\perp} = (\underline{1} + 2m/m') t a'_{1\perp},$

$t' a'_{1\perp} = a'_{1\perp}$

This frequency changes because $\underline{1}$ is moving

$\omega_{1\perp} = \Omega \sqrt{1 + 2m/m'}$

The normal modes would have to be re-normalized in terms of the new T', but that only changes them by a multiplicative constant.