Phys 503

Homework #6

Solution Set

6.1. Goldstein 8.16

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{1}{2}abq^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{1}{2}kq^2$$

(a) $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{a} - bqe^{-\alpha t} \Rightarrow p = a(\dot{q} + bqe^{-\alpha t})$

$$L = p\dot{q} - H$$

$$= \frac{p^2}{a} - \cancel{bqpe^{-\alpha t}} - \frac{p^2}{2a} + \cancel{bqpe^{-\alpha t}}$$

$$- \frac{1}{2}ab\,q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{1}{2}kq^2$$

$$= \frac{p^2}{2a} - \frac{1}{2}abq^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{1}{2}kq^2$$

$$\hookrightarrow = \frac{1}{2}a(\dot{q}^2 + 2bq\dot{q}e^{-\alpha t} + b^2q^2e^{-2\alpha t})$$

$$= \frac{1}{2}a\dot{q}^2 + abq\dot{q}e^{-\alpha t} + \frac{1}{2}abq^2e^{-\alpha t}(-\alpha - \cancel{be^{-\alpha t}} + \cancel{be^{-\alpha t}})$$

$$- \frac{kq^2}{2}$$

$$\boxed{L = \frac{1}{2}a\dot{q}^2 + abq\dot{q}e^{-\alpha t} - \frac{1}{2}\alpha abq^2e^{-\alpha t} - \frac{1}{2}kq^2}$$

(b) We can write L as

$$L = \frac{1}{2}a\dot{q}^2 - \frac{1}{2}kq^2 + \frac{d}{dt}\left(\frac{1}{2}abq^2e^{-\alpha t}\right)$$

Since total time derivatives of functions of q and t don't change the equations of motion, we can define a new, equivalent Lagrangian

$$\boxed{L' = \frac{1}{2} a \dot{q}^2 - \frac{1}{2} k q^2}$$

(c) $p' = \dfrac{\partial L'}{\partial \dot{q}} = a\dot{q} = p - abq e^{-\alpha t}$

$$\boxed{H' = p\dot{q} - L' = \frac{p'^2}{2a} + \frac{1}{2} k q^2}$$

Write H' in terms of q and p to find out its relation to H:

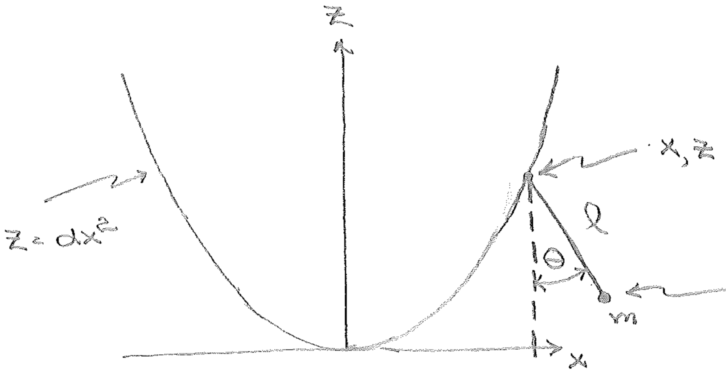
$$H' = \frac{p^2}{2a} - bpq e^{-\alpha t} + \frac{1}{2} a b^2 q^2 e^{-2\alpha t} + \frac{1}{2} k q^2$$

$$\downarrow$$

$$H - \frac{1}{2} a b q^2 e^{-\alpha t} (\alpha + b e^{-\alpha t})$$

$$\boxed{H' = H - \frac{1}{2} \alpha a b q^2 e^{-\alpha t}}$$

6.2. Goldstein 8.19



$x' = x + \ell \sin\theta$

$z' = z - \ell\cos\theta = \alpha x^2 - \ell\cos\theta$

$\dot{x}' = \dot{x} + \ell\dot{\theta}\cos\theta$

$\dot{z}' = 2\alpha x\dot{x} + \ell\dot{\theta}\sin\theta$

$$T = \frac{1}{2}m\left(\dot{x}'^2 + \dot{z}'^2\right)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\cos^2\theta + 4\alpha^2x^2\dot{x}^2 + 4\alpha\ell x\dot{x}\dot{\theta}\sin\theta + \ell^2\dot{\theta}^2\sin^2\theta\right)$$

$$= \frac{1}{2}m\left(\ell^2\dot{\theta}^2 + (1+4\alpha^2x^2)\dot{x}^2 + 2\ell\dot{x}\dot{\theta}(\cos\theta + 2\alpha x\sin\theta)\right)$$

$$V = mgz' = mg(\alpha x^2 - \ell\cos\theta)$$

$$\boxed{\begin{aligned} L = T - V = {} & \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2(1+4\alpha^2x^2) \\ & + m\ell\dot{x}\dot{\theta}(\cos\theta + 2\alpha x\sin\theta) \\ & - mg\alpha x^2 + mg\ell\cos\theta \end{aligned}}$$

Derive the Hamiltonian: Notice that the Lagrangian is quadratic in velocities, i.e.,

$$L = \underbrace{\tfrac{1}{2} M_{jk}\, \dot{q}_j \dot{q}_k}_{T} - V \qquad \begin{pmatrix} q_1 = \theta \\ q_2 = x \end{pmatrix}$$

Thus $p_j = \dfrac{\partial L}{\partial \dot{q}_j} = M_{jk}\dot{q}_k \iff \dot{q}_j = (M^{-1})_{jk}\, p_k$

$$H = p_j \dot{q}_j - L = M_{jk}\dot{q}_j\dot{q}_k - \tfrac{1}{2} M_{jk}\dot{q}_j\dot{q}_k + V$$

$$\Rightarrow \quad H = T + V = \tfrac{1}{2} M_{jk}\dot{q}_j\dot{q}_k + V = \tfrac{1}{2}(M^{-1})_{jk}\, p_j p_k + V$$

In our case,

$$T = \tfrac{1}{2} m l^2 \dot{\theta}^2 + \tfrac{1}{2} m \dot{x}^2 (1 + 4d^2x^2) + m l \dot{x}\dot{\theta}(\cos\theta + 2dx\sin\theta)$$

$$= \tfrac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{x}\end{pmatrix} \underbrace{\boxed{m\begin{pmatrix} l^2 & l\underbrace{(\cos\theta + 2dx\sin\theta)}_{\equiv \mu} \\ l\underbrace{(\cos\theta + 2dx\sin\theta)}_{\equiv \mu} & \underbrace{(1+4d^2x^2)}_{\nu} \end{pmatrix}}}_{M} \begin{pmatrix}\dot{\theta} \\ \dot{x}\end{pmatrix}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m l \dot{x}(\cos\theta + 2dx\sin\theta)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}(1 + 4d^2x^2) + m l \dot{\theta}(\cos\theta + 2dx\sin\theta)$$

$$\begin{pmatrix} p_\theta \\ p_x \end{pmatrix} = M \begin{pmatrix} \dot\theta \\ \dot x \end{pmatrix}$$

$$\frac{p_\theta}{m\ell} = \ell\dot\theta + \dot x \underbrace{(\cos\theta + 2\alpha x \sin\theta)}_{\equiv \mu} = \ell\dot\theta + \mu \dot x$$

$$\frac{p_x}{m} = \dot x \overbrace{(1+4\alpha^2 x^2)}^{\equiv \nu} + \ell\dot\theta\mu = \nu \dot x + \mu \ell \dot\theta$$

$$= \nu\dot x + \mu\left(\frac{p_\theta}{m\ell} - \mu\dot x\right)$$

$$= \underbrace{(\nu - \mu^2)}\dot x + \mu\frac{p_\theta}{m\ell}$$

$$= 1 + 4\alpha^2 x^2 - (\cos^2\theta + 4\alpha x \cos\theta\sin\theta + 4\alpha^2 x^2 \sin^2\theta)$$

$$= \sin^2\theta + 4\alpha^2 x^2 \cos^2\theta - 4\alpha x \cos\theta \sin\theta$$

$$= (\sin\theta - 2\alpha x \cos\theta)^2 \equiv \beta^2 = \nu - \mu^2$$

$$\dot x = \frac{1}{m\beta^2}\left(p_x - \frac{\mu}{\ell} p_\theta\right)$$

$$\ell\dot\theta = \frac{p_\theta}{m\ell} - \mu\dot x = \frac{p_\theta}{m\ell} - \frac{\mu}{m\beta^2} p_x + \frac{\mu^2}{m\ell\beta^2} p_\theta$$

$$= -\frac{\mu}{m\beta^2} p_x + \frac{p_\theta}{m\ell}\underbrace{\left(1 + \frac{\mu^2}{\beta^2}\right)}$$

$$\frac{\beta^2 + \mu^2}{\beta^2} = \frac{\nu}{\beta^2}$$

$$= \frac{1}{m\beta^2}\left(-\mu p_x + \frac{\nu}{\ell} p_\theta\right)$$

$$\boxed{\begin{pmatrix}\dot\theta\\ \dot x\end{pmatrix} = \underbrace{\frac{1}{m\beta^2}\begin{pmatrix}\nu/\ell^2 & -\mu/\ell\\ -\mu/\ell & 1\end{pmatrix}}_{M^{-1}}\begin{pmatrix}p_\theta\\ p_x\end{pmatrix}}$$

$$H = \frac{1}{2}(M^{-1})_{jk}\, p_j p_k + V$$

$$\boxed{H = \frac{1}{2m\beta^2}\left(\frac{\nu}{\ell^2}p_\theta^2 - \frac{2\mu}{\ell}p_\theta p_x + p_x^2\right) + mg(\alpha x^2 - \ell\cos\theta)}$$

$$\mu \equiv \cos\theta + 2\alpha x\sin\theta$$

$$\nu \equiv 1 + 4\alpha^2 x^2$$

$$\beta^2 \equiv \nu - \mu^2 = (\sin\theta - 2\alpha x\cos\theta)^2 \;;\quad \beta \equiv 2\alpha x\cos\theta - \sin\theta$$

Equations of motion: We will show formally that Hamilton's equations are identical to the Lagrange equations and thus effectively derive the equations of motion from the Lagrangian

$$H = \frac{1}{2}(M^{-1})_{jk}\, p_j p_k + V$$

$$\boxed{\dot q_j = \frac{\partial H}{\partial p_j} = (M^{-1})_{jk}\, p_k}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = -\frac{1}{2}\underbrace{\frac{\partial (M^{-1})_{kl}}{\partial q_j} p_j p_k} - \frac{\partial V}{\partial q_j}$$

$$= \tilde{p}\,\frac{\partial M^{-1}}{\partial q_j}\,p$$

Use $M M^{-1} = 1$

$$\Rightarrow \frac{\partial M}{\partial q_j} M^{-1} + M \frac{\partial M^{-1}}{\partial q_j} = 0$$

$$\Rightarrow \frac{\partial M^{-1}}{\partial q_j} = -M^{-1}\frac{\partial M}{\partial q_j} M^{-1}$$

$$= -\tilde{p}\, M^{-1}\frac{\partial M}{\partial q_j} M^{-1} p$$

$$= -\tilde{p}\,\tilde{M}^{-1}\frac{\partial M}{\partial q_j} M^{-1} p \qquad (\text{since } \tilde{M} = M)$$

$$= -\tilde{\dot{q}}\,\frac{\partial M}{\partial q_j}\,\dot{q} \qquad (\text{by other equation of motion})$$

$$= -\dot{q}_k \frac{\partial M_{kl}}{\partial q_j}\dot{q}_l$$

$$\boxed{\dot{p}_j = +\frac{1}{2}\frac{\partial M_{kl}}{\partial q_j}\dot{q}_k \dot{q}_l - \frac{\partial V}{\partial q_j} = \frac{\partial L}{\partial q_j}} \longleftarrow \text{Lagrange equations}$$

$$\dot{p}_j = \dot{M}_{jk}\dot{q}_k + M_{jk}\ddot{q}_k$$

$$= \frac{\partial M_{jk}}{\partial q_l}\dot{q}_k \dot{q}_l + M_{jk}\ddot{q}_k$$

$$\Rightarrow \boxed{M_{jk}\ddot{q}_k = \left(\frac{1}{2}\frac{\partial M_{kl}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_l}\right)\dot{q}_k \dot{q}_l - \frac{\partial V}{\partial q_j}}$$

Now we actually have to do some specific calculations:

$$\frac{\partial M_{11}}{\partial \theta} = \frac{\partial M_{11}}{\partial x} = 0$$

$$\frac{\partial M_{21}}{\partial \theta} = \frac{\partial M_{12}}{\partial \theta} = ml\frac{\partial u}{\partial \theta} = ml\left(-\sin\theta + 2\alpha x\cos\theta\right) = ml\beta$$

$$\frac{\partial M_{21}}{\partial x} = \frac{\partial M_{12}}{\partial x} = ml\frac{\partial u}{\partial x} = 2ml\alpha\sin\theta$$

$$\frac{\partial M_{22}}{\partial \theta} = m\frac{\partial v}{\partial \theta} = 0$$

$$\frac{\partial M_{22}}{\partial x} = m\frac{\partial v}{\partial x} = 8m\alpha^2 x$$

$$j=1: \quad M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 = \cancel{\frac{\partial M_{12}}{\partial \theta}}\dot{q}_1\dot{q}_2 - \cancel{\frac{\partial M_{12}}{\partial \theta}}\dot{q}_2\dot{q}_1 - \frac{\partial M_{12}}{\partial x}\dot{q}_2^2 - \frac{\partial V}{\partial \theta}$$

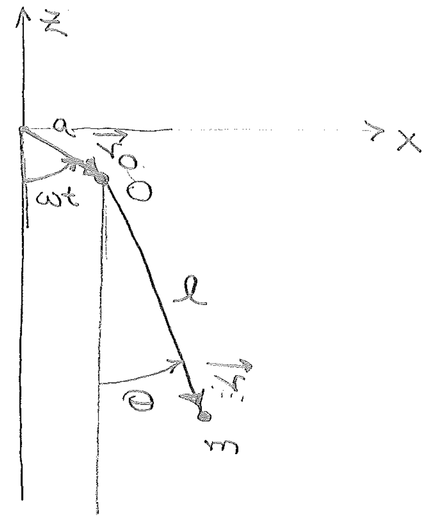
$$\Longrightarrow \boxed{ml^2\ddot{\theta} + ml(\cos\theta + 2\alpha x\sin\theta)\ddot{x} = -2ml\alpha\dot{x}^2\sin\theta - mgl\sin\theta}$$

$$j=2: \quad M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 = \cancel{\frac{\partial M_{12}}{\partial x}}\dot{q}_1\dot{q}_2 + \frac{1}{2}\frac{\partial M_{22}}{\partial x}\dot{q}_2^2 - \frac{\partial M_{21}}{\partial \theta}\dot{q}_1^2 - \cancel{\frac{\partial M_{21}}{\partial x}}\dot{q}_1\dot{q}_2 - \frac{\partial M_{22}}{\partial x}\dot{q}_2^2 - \frac{\partial V}{\partial x}$$

$$\boxed{\begin{aligned} &ml(\cos\theta + 2\alpha x\sin\theta)\ddot{\theta} + m(1+4\alpha^2x^2)\ddot{x} \\ &\quad = -4m\alpha^2 x\dot{x}^2 + ml(\sin\theta - 2\alpha x\cos\theta)\dot{\theta}^2 - 2mg\alpha x \end{aligned}}$$

These equations have been checked directly from the Lagrangian.

6.3 Goldstein 8.20



$x = a\sin\omega t + l\sin\theta$

$z = -a\cos\omega t - l\cos\theta$

$\dot{x} = a\omega\cos\omega t + l\dot{\theta}\cos\theta$

$\dot{z} = a\omega\sin\omega t + l\dot{\theta}\sin\theta$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\Big(a^2\omega^2\cos^2\omega t + 2al\omega\dot{\theta}\cos\omega t\cos\theta + l^2\dot{\theta}^2\cos^2\theta$$

$$+ a^2\omega^2\sin^2\omega t + 2al\omega\dot{\theta}\sin\omega t\sin\theta + l^2\dot{\theta}^2\sin^2\theta\Big)$$

$$= \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2al\omega\dot{\theta}\cos(\theta - \omega t) + a^2\omega^2\right)$$

$$V = mgz = -mg\left(l\cos\theta + a\cos\omega t\right)$$

Leaving out terms that don't depend on θ or $\dot{\theta}$, the Lagrangian is

$$\boxed{L = \frac{1}{2}ml^2\dot{\theta}^2 + mal\omega\dot{\theta}\cos(\theta - \omega t) + mgl\cos\theta}$$

$$\boxed{p_\theta = \frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta + mal\omega\cos(\theta-\omega t)}$$

$$H = p_\theta\dot\theta - L$$

$$= ml^2\dot\theta^2 + mal\omega\dot\theta\cos(\theta-\omega t) - L$$

$$\boxed{H = \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta = \frac{1}{2ml^2}\left(p_\theta - mal\omega\cos(\theta-\omega t)\right)^2 - mgl\cos\theta}$$

Equations of motion:

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{1}{ml^2}\left(p_\theta - mal\omega\cos(\theta-\omega t)\right)$$

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = -\frac{1}{ml^2}\left(p_\theta - mal\omega\cos(\theta-\omega t)\right)\left(+mal\omega\sin(\theta-\omega t)\right)$$

$$-mgl\sin\theta$$

$$ml^2\ddot\theta = mal\omega\sin(\theta-\omega t)(\cancel{\dot\theta}-\omega)$$

$$= \cancel{-\dot\theta\, mal\omega\sin(\theta-\omega t)} - mgl\sin\theta$$

$$\boxed{ml^2\ddot\theta = -mal\omega^2\sin(\theta-\omega t) - mgl\sin\theta}$$

Interpretation: $\vec{r}_0 = a(\sin\omega t\,\vec{e}_x - \cos\omega t\,\vec{e}_z)$

$$\vec{r} = l(\sin\theta\,\vec{e}_x - \cos\theta\,\vec{e}_z)$$

$$\dot{\vec{r}}_0 = a\omega(\cos\omega t\,\vec{e}_x + \sin\omega t\,\vec{e}_z)$$

$$\dot{\vec{r}} = l\dot{\theta}(\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_z)$$

$$\boxed{\begin{aligned} H &= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \\ &= \frac{1}{2}m\dot{\vec{r}}^2 + mg\vec{e}_z\cdot\vec{r} \\ &= \left(\begin{array}{l}\text{energy measured by observer who moves} \\ \text{with the suspension point O}\end{array}\right) \end{aligned}}$$

$$\vec{r}\times\dot{\vec{r}} = -l^2\dot{\theta}\,\vec{e}_y$$

$$\vec{r}\times\dot{\vec{r}}_0 = al\omega\,\underbrace{(\sin\theta\,\vec{e}_x - \cos\theta\,\vec{e}_z)\times(\cos\omega t\,\vec{e}_x + \sin\omega t\,\vec{e}_z)}$$

$$= -\sin\theta\sin\omega t\,\vec{e}_y - \cos\theta\cos\omega t\,\vec{e}_y$$

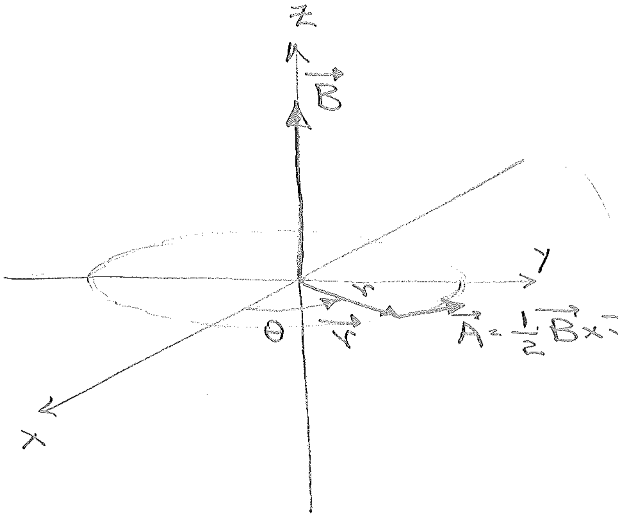
$$= -\cos(\theta-\omega t)\,\vec{e}_y$$

$$= -al\omega\cos(\theta-\omega t)\,\vec{e}_y$$

$$\boxed{m\vec{r}\times\dot{\vec{r}} + m\vec{r}\times\dot{\vec{r}}_0 = -\vec{e}_y\left(ml^2\dot{\theta} + mal\omega\cos(\theta-\omega t)\right) = -p_\theta\vec{e}_y}$$

6.4. Goldstein 8.23

(a)



$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}Br\,\vec{e}_\varphi$

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(r) + \frac{e}{c}\underbrace{\dot{\vec{r}}\cdot\vec{A}}$$

$$\frac{1}{2}\dot{\vec{r}}\cdot\vec{B}\times\vec{r} = \frac{1}{2}\vec{B}\cdot\vec{r}\times\dot{\vec{r}}$$

$$= \frac{1}{2m}\vec{B}\cdot\vec{L}$$

$$= \frac{1}{2m}Bmr^2\dot{\theta}$$

$$= \frac{1}{2}Br^2\dot{\theta}$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{e}{2c}Br^2\dot{\theta} - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{e}{2c}Br^2 = mr^2\left(\dot{\theta} + \frac{eB}{2mc}\right)$$

$$\Rightarrow \quad \dot{\theta} = \frac{p_\theta}{mr^2} - \frac{eB}{2mc}$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= m\dot{r}^2 + mr^2\dot{\theta}^2 + \frac{e}{2c}Br^2\dot{\theta} - L$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - p_\theta \frac{eB}{2mc} + \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2 + V(r)}$$

Notice that $p_\theta = \underbrace{mr^2\dot{\theta}}_{\ell} + mr^2\frac{eB}{2mc}$ is conserved

(b) Use angular coordinate $\chi = \theta + \omega t = \theta + \frac{eB}{2mc}t$

$$\dot{\theta} = \dot{\chi} - \frac{eB}{2mc}$$

$$L = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}mr^2\dot{\theta}^2 + \frac{eB}{2c}r^2\dot{\theta}} - V(r)$$

$$= \frac{1}{2}mr^2\left(\dot{\chi}^2 - \frac{eB}{mc}\dot{\chi} + \left(\frac{eB}{2mc}\right)^2\right)$$

$$+ \frac{eB}{2c}r^2\left(\dot{\chi} - \frac{eB}{2mc}\right)$$

$$= \frac{1}{2}mr^2\dot{\chi}^2 - \cancel{\frac{eB}{2c}r^2\dot{\chi}} + \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2$$

$$+ \cancel{\frac{eB}{2c}r^2\dot{\chi}} - \left(\frac{eB}{2mc}\right)^2 r^2 m$$

$$= \frac{1}{2}mr^2\dot{\chi}^2 - \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\chi}^2 - \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2 - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\chi = \frac{\partial L}{\partial \dot{\chi}} = mr^2\dot{\chi}$$

$$H = p_r\dot{r} + p_\chi\dot{\chi} - L$$

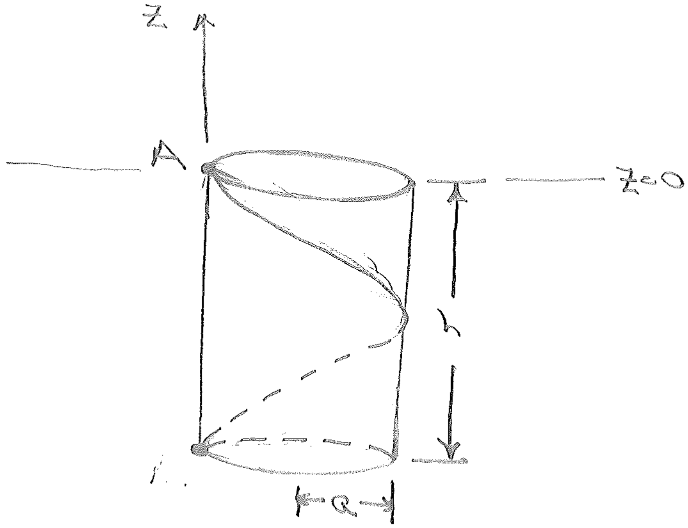
$$= m\dot{r}^2 + mr^2\dot{\chi}^2 - L$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\chi}^2 + \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2 + V(r)$$

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\chi^2}{2mr^2} + \frac{1}{2}mr^2\left(\frac{eB}{2mc}\right)^2 + V(r)}$$

Notice that $p_\chi = mr^2\dot{\chi} = mr^2\left(\dot{\theta} + \frac{eB}{2mc}\right) = p_\theta$ is conserved

6.5. Goldstein 8.24



$M = (\text{mass of cylinder}) = m$

$= \pi a^2 h \rho$

$I = \int \rho \, dV \, r^2$

$= \int \rho \, 2\pi h r \, dr \, r^2$

$= 2\pi h \rho \int_0^a r^3 \, dr$

$= \frac{\pi}{2} h \rho a^4$

$= \frac{1}{2} m a^2$

Helix: $z = -h\theta/2\pi$

θ is the azimuthal angle around cylinder, relative to A.

(a) Let ϕ be the angle through which the cylinder has rotated relative to inertial space.

$$T_{cylinder} = \frac{1}{2} I \dot{\phi}^2 = \frac{1}{4} m a^2 \dot{\phi}^2$$

The inertial cylindrical coördinates of the mass point m are $\theta + \phi$ and $z = -h\theta/2\pi$, so

$$T_m = \frac{1}{2} m \left(a^2 (\dot{\theta} + \dot{\phi})^2 + \dot{z}^2 \right)$$

$$= \frac{1}{2} m \left(a^2 (\dot{\theta} + \dot{\phi})^2 + \frac{h^2}{4\pi^2} \dot{\theta}^2 \right)$$

$$= \frac{1}{2} m \left(a^2 + \frac{h^2}{4\pi^2} \right) \dot{\theta}^2 + \frac{1}{2} m a^2 \dot{\phi}^2 + m a^2 \dot{\theta} \dot{\phi}$$

$$V_m = mgz = -\frac{mgh}{2\pi}\theta$$

$$L = \overbrace{T_{cylinder} + T_m}^{T} - V$$

$$= \frac{3}{4}ma^2\dot{\phi}^2 + \frac{1}{2}m\left(a^2+\frac{h^2}{4\pi^2}\right)\dot{\theta}^2 + ma^2\dot{\theta}\dot{\phi} + \frac{mgh}{2\pi}\theta$$

$$\boxed{\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \frac{3}{2}ma^2\dot{\phi} + ma^2\dot{\theta} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m\left(a^2+\frac{h^2}{4\pi^2}\right)\dot{\theta} + ma^2\dot{\phi} \end{aligned}}$$

$$T = \frac{1}{2}m_{jk}\dot{q}_j\dot{q}_k$$

$$\|m_{jk}\| = \begin{array}{c c} & \begin{array}{cc} \phi & \theta \end{array} \\ \begin{array}{c} \phi \\ \theta \end{array} & \begin{pmatrix} \frac{3}{2}ma^2 & ma^2 \\ ma^2 & m\left(a^2+\frac{h^2}{4\pi^2}\right) \end{pmatrix} \end{array} = ma^2\begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 1+\alpha \end{pmatrix}$$

$$\boxed{\alpha \equiv \frac{h^2}{4\pi^2 a^2}}$$

$$\|(m^{-1})_{jk}\| = \frac{1}{ma^2}\,\frac{2}{1+3\alpha}\begin{array}{c c} \begin{array}{cc} p_\phi & p_\theta \end{array} & \\ \begin{pmatrix} 1+\alpha & -1 \\ -1 & \frac{3}{2} \end{pmatrix} & \begin{array}{c} p_\phi \\ p_\theta \end{array} \end{array}$$

$$H = \frac{1}{2}(m^{-1})_{jk}p_j p_k + V$$

$$H = \frac{1}{\cancel{2}ma^2}\,\frac{\cancel{2}}{1+3\alpha}\left((1+\alpha)p_\phi^2 + \frac{3}{2}p_\theta^2 - 2p_\theta p_\phi\right) - \frac{mgh}{2\pi}\theta$$

$$\boxed{H = \frac{1}{ma^2(1+3\alpha)}\left((1+\alpha)p_\phi^2 + \frac{3}{2}p_\theta^2 - 2p_\theta p_\phi\right) - \frac{mgh}{2\pi}\theta}$$

Solution: Write L as

$$L = \underbrace{\frac{1}{2}ma^2\left(\frac{3}{2}\dot\phi^2 + (1+\alpha)\dot\theta^2 + 2\dot\theta\dot\phi\right)}_{T} + \frac{mgh}{2\pi}\theta$$

$$E = T+V = \frac{1}{2}ma^2\left(\frac{3}{2}\dot\phi^2 + (1+\alpha)\dot\theta^2 + 2\dot\theta\dot\phi\right) - \frac{mgh}{2\pi}\theta$$

$$= 0 \qquad \left(\begin{matrix}\text{IC's are} \\ \dot\theta = \dot\phi = 0 \text{ and } \theta = 0\end{matrix}\right)$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = ma^2\left(\frac{3}{2}\dot\phi + \dot\theta\right) = 0 \quad\Rightarrow\quad \dot\phi = -\frac{2}{3}\dot\theta$$

↑ Conserved because ϕ is cyclic

$$0 = \frac{1}{2}ma^2\left(\frac{2}{3}\dot\theta^2 + (1+\alpha)\dot\theta^2 - \frac{4}{3}\dot\theta^2\right) - \frac{mgh}{2\pi}\theta$$

$$= \frac{1}{2}ma^2\left(\frac{1}{3}+\alpha\right)\dot\theta^2 - \frac{mgh}{2\pi}\theta \quad\longleftarrow$$

Like a particle in a uniform gravitational field

$$a^2\left(\frac{1}{3}+\alpha\right)\dot{\theta}^2 = \frac{gh}{\pi}\,\theta$$

$$\dot{\theta}^2 = \underbrace{\frac{gh}{\pi a^2\left(\frac{1}{3}+\alpha\right)}}_{\beta^2}\,\theta \quad\Longrightarrow\quad \dot{\theta} = \beta\sqrt{\theta}$$

$$\frac{d\theta}{\sqrt{\theta}} = \beta\, dt \quad\Longrightarrow\quad 2\sqrt{\theta} = \beta t$$

$$\theta = \frac{1}{4}\beta^2 t^2 = \frac{gh}{4\pi a^2\left(\frac{1}{3}+\alpha\right)}\,t^2 = \boxed{\frac{gh}{\frac{4}{3}\pi a^2 + \frac{h^2}{\pi}}\,t^2 = \theta}$$

6.6. Goldstein 8.9

Constraints: $\psi_k(q,p,t) = 0$

Notice that λ_k is not a canonical coördinate or momentum because there is no term $\eta_k \dot{\lambda}_k$ or $\lambda_k \dot{\eta}_k$

Hamilton's principle:

$$0 = \delta\left(\int_{t_1}^{t_2} dt \left[p_j \dot{q}_j - H(p,q,t) - \sum_k \lambda_k(t)\, \psi_k(q,p,t) \right] \right)$$

$$= \int_{t_1}^{t_2} dt \Big[\delta p_j\, \dot{q}_j + p_j\, \delta \dot{q}_j - \frac{\partial H}{\partial q_j}\delta q_j - \frac{\partial H}{\partial p_j}\delta p_j$$

$$- \sum_k \lambda_k \left(\frac{\partial \psi_k}{\partial q_j}\delta q_j + \frac{\partial \psi_k}{\partial p_j}\delta p_j \right)$$

$$- \sum_k \delta\lambda_k\, \psi_k \Big]$$

$$= p_j\, \delta q_j \Big|_{t_1}^{t_2}$$

$$+ \int_{t_1}^{t_2} dt \Big[\delta q_j \left(-\dot{p}_j - \frac{\partial H}{\partial q_j} - \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j} \right)$$

$$+ \delta p_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} - \sum_k \lambda_k \frac{\partial \psi_k}{\partial p_j} \right)$$

$$- \sum_k \delta\lambda_k\, \psi_k \Big]$$

$$\Longrightarrow \boxed{\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} - \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j} \\ \psi_k &= 0 \end{aligned}}$$

This is just as though there were a "super-Hamiltonian" $H' = H + \sum_k \lambda_k \psi_k$

If one treats $t = q_{n+1}$ as an additional canonical coordinate and $p_{n+1} = -H(q,p,t)$ as its conjugate momentum, then Hamilton's principle reads

Parametrize path on phase space by θ

$t = q_{n+1}$ becomes a function of θ

p_{n+1} is the momentum conjugate to t

$$0 = \delta \int d\theta \left[p_j \frac{dq_j}{d\theta} + \underbrace{p_{n+1}}_{-H} \frac{dt}{d\theta} - \lambda \underbrace{\left(H(q,p,t) + p_{n+1}\right)} \right]$$

An arbitrary parameter along the path in $2(n+1)$-dimensional phase space

Once p_{n+1} becomes a canonical momentum, it is freed from any connection to H, so the connection must be enforced by a Lagrange multiplier λ.

Variables to be varied independently are $\underbrace{q_1,\dots,q_n, t, p_1,\dots,p_n}, p_{n+1}, \lambda$

$$0 = \delta \int d\theta \left[\sum_{j=1}^{n+1} p_j \frac{dq_j}{d\theta} - \lambda \underbrace{\left(H + p_{n+1}\right)}_{\mathcal{H} = \text{(super-Hamiltonian)}} \right]$$

Constraint: $p_{n+1} = -H(q,p,t) \longleftarrow$ Variation wrt λ

Hamilton's equations:

In these partial derivatives the independent variables are $q_1,\dots,q_n, t, p_1,\dots,p_n, p_{n+1}, \lambda$

$$\frac{dq_j}{d\theta} = \lambda \frac{\partial \mathcal{H}}{\partial p_j} \quad \text{and} \quad \frac{dp_j}{d\theta} = -\lambda \frac{\partial \mathcal{H}}{\partial q_j}, \quad j = 1, \dots, n+1$$

For $j = n+1$: $\dfrac{dt}{d\theta} = \lambda \underbrace{\dfrac{\partial \mathcal{H}}{\partial p_{n+1}}}_{1} = \lambda$ and $\underbrace{\dfrac{dp_{n+1}}{d\theta}}_{= -\frac{dH}{d\theta}} = -\lambda \underbrace{\dfrac{\partial \mathcal{H}}{\partial t}}_{= \frac{\partial H}{\partial t}}$

$$\Longrightarrow \boxed{\frac{dt}{d\theta} = \lambda \quad \text{and} \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

For $j=1,\dots,n$: $\dfrac{dq_j}{d\theta} = \dfrac{dt}{d\theta}\dfrac{\partial \mathcal{H}}{\partial p_j} \Longrightarrow$

$$\boxed{\begin{aligned} \frac{dq_j}{dt} &= \frac{\partial H}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial q_j} \end{aligned}}$$

$\dfrac{dp_j}{d\theta} = -\dfrac{dt}{d\theta}\dfrac{\partial \mathcal{H}}{\partial q_j} \Longrightarrow$

6.7. Rotating frame with rotating axes $\vec{e}_j$

Constant angular velocity $\vec{\omega} = \omega_j \vec{e}_j$

$$0 = \left(\frac{d\vec{\omega}}{dt}\right)_{space} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \cancelto{0}{\vec{\omega}\times\vec{\omega}}$$

$$\dot{\omega}_j \vec{e}_j \Rightarrow \dot{\omega}_j = 0$$

Particle of mass m with position vector $\vec{x} = x_j \vec{e}_j$ moving in a potential $V(x_1, x_2, x_3)$.

(a) Lagrangian $L = T - V$

$$T = \frac{1}{2} m \vec{W}\cdot\vec{W} = \frac{1}{2} m w^j w^j$$

$$\vec{W} = \begin{pmatrix}\text{inertial}\\ \text{velocity}\end{pmatrix} = \left(\frac{d\vec{x}}{dt}\right)_{space} = \left(\frac{d\vec{x}}{dt}\right)_{rot} + \vec{\omega}\times\vec{x}$$

$$w^j = \dot{x}_j + \epsilon_{jkl}\omega_k x_l$$
$$\vec{W} = \vec{V} + \vec{\omega}\times\vec{x}$$

$$L = \frac{1}{2} m w^j w^j - V(x_1, x_2, x_3)$$

$$\boxed{\begin{aligned} L &= \frac{1}{2} m \Big(\dot{x}_j + \underbrace{\epsilon_{jkl}\omega_k x_l}_{(\vec{\omega}\times\vec{x})_j}\Big)\left(\dot{x}_j + \epsilon_{jmn}\omega_m x_n\right) - V(x_1, x_2, x_3) \\ &= \frac{1}{2} m \left(\vec{V}\cdot\vec{V} + 2\vec{V}\cdot\vec{\omega}\times\vec{x} + \vec{\omega}\times\vec{x}\cdot\vec{\omega}\times\vec{x}\right) - V \end{aligned}}$$

(b) $$p_j = \frac{\partial L}{\partial \dot{x}_j} = \frac{\partial T}{\partial \dot{x}_j} = \underbrace{\frac{\partial T}{\partial w_j}}_{= m w_j} \underbrace{\frac{\partial w_j}{\partial \dot{x}_j}}_{= \delta_{jk}} = m w_j$$

$$\boxed{p_j = m w_j = m(\dot{x}_j + \epsilon_{jkl}\,\omega_k x_l)}$$

$$\boxed{\vec{p} = p_j \vec{e}_j = m\vec{w} \text{ is the particle's momentum vector.}}$$

(c) $$H = p_j \dot{x}_j - L$$

$$= m\dot{x}_j\dot{x}_j + m\dot{x}_j\,\epsilon_{jkl}\,\omega_k x_l \quad = 0$$

$$- \frac{1}{2} m \dot{x}_j \dot{x}_j - \frac{1}{2} m \dot{x}_j\,\epsilon_{jmn}\,\omega_m x_n - \frac{1}{2} m \dot{x}_j\,\epsilon_{jkl}\, x_k \omega_l$$

$$- \frac{1}{2} m\,\epsilon_{jkl}\,\omega_k x_l\,\epsilon_{jmn}\,\omega_m x_n + V$$

$$H = \frac{1}{2} m \dot{x}_j \dot{x}_j - \frac{1}{2} m\,\epsilon_{jkl}\,\omega_k x_l\,\epsilon_{jmn}\,\omega_m x_n + V$$

$$= \frac{1}{2} m \vec{v}\cdot\vec{v} - \frac{1}{2} m\, \vec{\omega}\times\vec{x}\cdot\vec{\omega}\times\vec{x} + V$$

$$\boxed{H = \frac{1}{2m}(p_j - m\epsilon_{jkl}\,\omega_k x_l)(p_j - m\epsilon_{jmn}\,\omega_m x_n) - \frac{1}{2} m\,\epsilon_{jkl}\,\omega_k x_l\,\epsilon_{jmn}\,\omega_m x_n + V(x_1, x_2, x_3)}$$

$$= \frac{1}{2m}(\vec{p} - m\vec{\omega}\times\vec{x})\cdot(\vec{p} - m\vec{\omega}\times\vec{x}) - \frac{1}{2} m\, \vec{\omega}\times\vec{x}\cdot\vec{\omega}\times\vec{x} + V$$

(d) $E = T + V = \frac{1}{2} m \vec{w} \cdot \vec{w} + V = \frac{\vec{p} \cdot \vec{p}}{2m} + V$

$H = E - \vec{p} \cdot \vec{\omega} \times \vec{x}$!!

$= E - m \vec{v} \cdot \vec{\omega} \times \vec{x} - m \vec{\omega} \times \vec{x} \cdot \vec{\omega} \times \vec{x}$

$H = E - \vec{\omega} \cdot \vec{x} \times \vec{p} = E - \vec{\omega} \cdot \vec{L}$

angular momentum

H is conserved because $\frac{\partial H}{\partial t} = 0$.

E is conserved when $V = 0$ since the particle experiences no forces

These are consistent because $\vec{L}$ is conserved when $V = 0$.

(e) More generally, E is conserved iff the potential is time independent in the nonrotating frame $\Leftrightarrow$ $V(x_1, x_2, x_3)$ is rotationally symmetric about $\vec{\omega}$ $\Leftrightarrow$ $\vec{\omega} \cdot \vec{L}$ being conserved.