Phys 503

Homework #7

Solution Set

7.1. Goldstein 9.7

(a) The fundamental differential relation for a canonical transformation is

① $$p_j\,dq_j - P_j\,dQ_j = dF$$

$$= dF_1(q,Q,t)$$

the differential is taken w/ t fixed

② $$p_j\,dq_j + Q_j\,dP_j = d(F_1 + P_jQ_j) = dF_2(q,P,t)$$

③ $$-q_j\,dp_j - P_j\,dQ_j = d(F_1 - p_jq_j) = dF_3(p,Q,t)$$

④ $$-q_j\,dp_j + Q_j\,dP_j = d(F_1 - p_jq_j + P_jQ_j) = dF_4(p,P,t)$$

$$\boxed{\begin{aligned} F_2(q,P,t) &= F_1(q,Q,t) + P_jQ_j \\ F_3(p,Q,t) &= F_1(q,Q,t) - p_jq_j \\ F_4(p,P,t) &= F_1(q,Q,t) - p_jq_j + P_jQ_j \end{aligned}}$$

(b) Identity transformation: $F_2(q,P) = q_jP_j \Rightarrow$
$$p_j = \frac{\partial F_2}{\partial q_j} = P_j$$
$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j$$

$$F_3(p,Q) = F_2(q,P) - P_jQ_j - p_jq_j$$

$$= q_jP_j - P_jQ_j - p_jq_j$$

write in terms of p and Q $\rightarrow$ $= Q_jp_j - p_jQ_j - p_jQ_j$

$$\Longrightarrow \quad F_3(p,Q) = -p_j Q_j \quad \Longrightarrow \quad \begin{aligned} q_j &= -\frac{\partial F_3}{\partial p_j} = Q_j \\ P_j &= -\frac{\partial F_3}{\partial Q_j} = p_j \end{aligned}$$

Can't use F_1 or F_4 because q, Q or p, P is <u>not</u> a complete set of coördinates on phase space.

Exchange transformation: $F_1(q,Q) = q_j Q_j \Rightarrow$

$$p_j = \frac{\partial F_1}{\partial q_j} = Q_j$$

$$P_j = -\frac{\partial F_1}{\partial Q_j} = -q_j$$

$$F_4(p,P) = F_1(q,Q) - p_j q_j + P_j Q_j$$

$$= q_j Q_j - p_j q_j + P_j Q_j$$

Write in terms of p and P

$$= -P_j p_j + p_j P_j + P_j p_j$$

$$\Longrightarrow \quad F_4(p,P) = p_j P_j \quad \Longrightarrow \quad \begin{aligned} q_j &= -\frac{\partial F_4}{\partial p_j} = -P_j \\ Q_j &= \frac{\partial F_4}{\partial P_j} = p_j \end{aligned}$$

Can't use F_2 or F_3 because q, P or p, Q is <u>not</u> a complete set of coördinates on phase space.

(c) Orthogonal point transformation: $Q_j = A_{jk} q_k$

$\uparrow$ orthogonal matrix

(a) Use an $F_2(q,P)$ generating function so that

$$p_j = \frac{\partial F_2}{\partial q_j} \quad \text{and} \quad Q_j = \frac{\partial F_2}{\partial P_j} = A_{jk} q_k$$

re-labels indices

$$\Rightarrow \quad F_2(q,P) = A_{jk} q_k P_j + f(q) \overset{\downarrow}{=} A_{kl} q_l P_k$$

arbitrary function of q

$$\Rightarrow \quad p_j = \frac{\partial F_2}{\partial q_j} = A_{kj} P_k + \frac{\partial f}{\partial q_j}$$

$$\Rightarrow \quad \tilde{A}_{jk} P_k = p_j - \frac{\partial f}{\partial q_j}$$

In matrix language,

$$\tilde{A} \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} = \begin{pmatrix} p_1 - \partial f/\partial q_1 \\ \vdots \\ p_n - \partial f/\partial q_n \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} = \tilde{A}^{-1} \begin{pmatrix} p_1 - \partial f/\partial q_1 \\ \vdots \\ p_n - \partial f/\partial q_n \end{pmatrix}$$

$\tilde{A}^{-1} = A$ (since A is orthogonal)

$$\boxed{P_j = A_{jk}\left(p_k - \frac{\partial f}{\partial q_k}\right)}$$

$f(q)$ is arbitrary

(b) We could also use the canonical Poisson bracket. Letting $Q_j = A_{jk} q_k$ and $P_j = F_j(q,p)$, we have

$$\delta_{jk} = [Q_j, P_k] = A_{jl} \underbrace{[q_l, F_k(q,p)]}_{\frac{\partial q_l}{\partial q_m}\frac{\partial F_k}{\partial p_m} - \cancel{\frac{\partial q_l}{\partial p_m}}\frac{\partial F_k}{\partial q_m}} = A_{jm} \frac{\partial F_k}{\partial p_m}$$

$$\frac{\partial q_l}{\partial q_m} = \delta_{lm}$$

$$\Longrightarrow \frac{\partial F_k}{\partial p_l} = A_{kl}$$

$$\Rightarrow \quad F_k = A_{k\ell} p_\ell + h_k(q) = A_{k\ell}(p_\ell + g_\ell(q)) = P_k$$

(3) $$0 = [P_j, P_k] = A_{j\ell} A_{km} [p_\ell + g_\ell, p_m + g_m]$$

$$\Longrightarrow \quad 0 = [p_j + g_j, p_k + g_k]$$

$$= [p_j, g_k] + [g_j, p_k]$$

$$= \underbrace{[p_j, g_k]}_{\cancel{\frac{\partial p_j}{\partial q_\ell}\frac{\partial g_k}{\partial p_\ell}}^{0} - \frac{\partial p_j}{\partial p_\ell}\frac{\partial g_k}{\partial q_\ell} = -\delta_{j\ell}\frac{\partial g_k}{\partial q_\ell} = -\frac{\partial g_k}{\partial q_j}} - [p_k, g_j]$$

$$0 = -\frac{\partial g_k}{\partial q_j} + \frac{\partial g_j}{\partial q_k}$$

$$\Longrightarrow \quad g_k = \frac{\partial f}{\partial q_k} \quad \text{for some } f(q)$$

So $$\boxed{P_k = A_{k\ell}\left(p_\ell + \frac{\partial f}{\partial q_\ell}\right)}$$

7.2. Goldstein 9.23

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right), \quad p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right)$$

$$y = \frac{1}{\alpha}\left(+\sqrt{2P_1}\cos Q_1 + Q_2\right), \quad p_y = \frac{\alpha}{2}\left(-\sqrt{2P_1}\sin Q_1 + P_2\right)$$

Method 1:

First, notice that

$$\frac{1}{2}\left(\alpha x + \frac{2}{\alpha}p_y\right) = P_2 \equiv p_2$$

$$\frac{1}{2}\left(\alpha x - \frac{2}{\alpha}p_y\right) = \sqrt{2P_1}\sin Q_1 \equiv q_1$$

$$\frac{1}{2}\left(\alpha y + \frac{2}{\alpha}p_x\right) = \sqrt{2P_1}\cos Q_1 \equiv p_1$$

$$\longrightarrow \quad P_1 = \frac{1}{2}(p_1^2 + q_1^2), \quad \cot Q_1 = \frac{p_1}{q_1}$$

$$\frac{1}{2}\left(\alpha y - \frac{2}{\alpha}p_x\right) = Q_2 \equiv q_2$$

We do the proof in two steps: (i) show that the transformation from (x, y, p_x, p_y) to (q_1, q_2, p_1, p_2) is canonical; (ii) show that the transformation from (q_1, q_2, p_1, p_2) to (Q_1, Q_2, P_1, P_2) is canonical.

(i) This transformation is linear, so it is easy to check the preservation of the fundamental Poisson brackets (....,

$$[q_1, q_2] = \frac{1}{4}\Big(-2\underbrace{[x, p_x]}_{1} - 2\underbrace{[p_y, y]}_{-1}\Big) = 0$$

$$[q_1, p_1] = \frac{1}{4}\Big(2\underset{\overset{\shortparallel}{1}}{[x, p_x]} - 2\underset{\overset{\shortparallel}{-1}}{[p_y, y]}\Big) = 1$$

$$[q_1, p_2] = 0$$

$$[q_2, p_1] = 0$$

$$[q_2, p_2] = \frac{1}{4}\Big(2\underset{\overset{\shortparallel}{1}}{[y, p_y]} - 2\underset{\overset{\shortparallel}{-1}}{[p_x, x]}\Big) = 1$$

$$[p_1, p_2] = \frac{1}{4}\Big(2\underset{\overset{\shortparallel}{1}}{[y, p_y]} + 2\underset{\overset{\shortparallel}{-1}}{[p_x, x]}\Big) = 0$$

(ii) We can read off the generating function for this transformation from the text's discussion of the identity transformation and the harmonic oscillator (w/ $m\omega = 1$):

$$\boxed{F' = \frac{1}{2} q_1^2 \cot Q_1 + q_2 P_2}$$

generating function of mixed type

F_2 generating function for identity transformation on 2nd set of canonical variables

F_1 generating function for a harmonic oscillator [Eq (9-37)] on 1st set of canonical variables

The fundamental differential relation for a canonical transformation is

$$p_j dq_j - P_j dQ_j = dF = p_1 dq_1 + p_2 dq_2 - P_1 dQ_1 - P_2 dQ_2$$

$$= p_1 dq_1 + p_2 dq_2 - P_1 dQ_1 + d(P_2 Q_2) + Q_2 dP_2$$

$$\Longrightarrow \quad p_1 dq_1 + p_2 dq_2 - P_1 dQ_1 + Q_2 dP_2 = d(\underbrace{F + P_2 Q_2}_{F'}) = dF'$$

$$\therefore \quad p_1 = \frac{\partial F'}{\partial q_1} = q_1 \cot Q_1$$

$$p_2 = \frac{\partial F'}{\partial q_2} = P_2$$

$$P_1 = -\frac{\partial F'}{\partial Q_1} = -\frac{1}{2} q_1^2 \underbrace{\frac{\partial \cot Q_1}{\partial Q_1}}_{-\frac{1}{\sin^2 Q_1}} = \frac{1}{2} \frac{q_1^2}{\sin^2 Q_1}$$

$$\longrightarrow \quad q_1 = \sqrt{2P_1} \sin Q_1, \quad p_1 = \sqrt{2P_1} \cos Q_1$$

$$Q_2 = \frac{\partial F'}{\partial P_2} = q_2$$

Method 2: Direct check of fundamental Poisson brackets

$$[x, y] = \frac{\partial x}{\partial Q_j} \frac{\partial y}{\partial P_j} - \frac{\partial x}{\partial P_j} \frac{\partial y}{\partial Q_j}$$

$$= \underbrace{\frac{1}{\alpha} \sqrt{2P_1} \cos Q_1}_{\frac{\partial x}{\partial Q_1}} \cdot \underbrace{\frac{1}{\alpha} \frac{1}{\sqrt{2P_1}} \cos Q_1}_{\frac{\partial y}{\partial P_1}} + \underbrace{0}_{\frac{\partial x}{\partial Q_2}} \cdot \underbrace{0}_{\frac{\partial y}{\partial P_2}}$$

$$- \underbrace{\frac{1}{\alpha} \frac{1}{\sqrt{2P_1}} \sin Q_1}_{\frac{\partial x}{\partial P_1}} \cdot \underbrace{\frac{1}{\alpha} \sqrt{2P_1} (-\sin Q_1)}_{\frac{\partial y}{\partial Q_1}} - \underbrace{\frac{1}{\alpha}}_{\frac{\partial x}{\partial P_2}} \cdot \underbrace{\frac{1}{\alpha}}_{\frac{\partial y}{\partial Q_2}}$$

$$[x,y] = \frac{1}{\alpha^2}\cos^2 Q_1 + \frac{1}{\alpha^2}\sin^2 Q_1 - \frac{1}{\alpha^2} = \boxed{0 = [x,y]}$$

$$[x,P_x] = \frac{\partial x}{\partial Q_j}\frac{\partial P_x}{\partial P_j} - \frac{\partial x}{\partial P_j}\frac{\partial P_x}{\partial Q_j}$$

$$= \underbrace{\frac{1}{\alpha}\sqrt{2P_1}\cos Q_1}_{\frac{\partial x}{\partial Q_1}} \cdot \underbrace{\frac{\alpha}{2}\frac{1}{\sqrt{2P_1}}\cos Q_1}_{\frac{\partial P_x}{\partial P_1}} + \underset{\frac{\partial x}{\partial Q_2}}{\underset{\uparrow}{0}} \times \underset{\frac{\partial P_x}{\partial P_2}}{\underset{\uparrow}{0}}$$

$$- \underbrace{\frac{1}{\alpha}\frac{1}{\sqrt{2P_1}}\sin Q_1}_{\frac{\partial x}{\partial P_1}} \cdot \underbrace{-\frac{\alpha}{2}\sqrt{2P_1}\sin Q_1}_{\frac{\partial P_x}{\partial Q_1}} - \underset{\frac{\partial x}{\partial P_2}}{\underset{\uparrow}{\frac{1}{\alpha}}} \cdot \underset{\frac{\partial P_x}{\partial Q_2}}{\underset{\uparrow}{-\frac{\alpha}{2}}}$$

$$= \frac{1}{2}\cos^2 Q_1 + \frac{1}{2}\sin^2 Q_1 + \frac{1}{2}$$

$$\boxed{[x,P_x] = 1}$$

⋮ I hope you see the procedure

Particle of charge q moving in plane $\perp$ to constant magnetic field $\vec{B} = B\vec{e}_z$. A corresponding vector potential is $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$.

$$\therefore \vec{A} = \frac{1}{2}\vec{e}_j\,\epsilon_{jkl}B_k x_l = \frac{1}{2}B\vec{e}_j\,\epsilon_{3lj}x_l$$

$$\boxed{\vec{A} = \frac{1}{2}B\left(x\vec{e}_2 - y\vec{e}_1\right)}$$

Check:

$$\nabla\times\vec{A} = \vec{e}_j\,\epsilon_{jkl}\,A_{l,k}$$

$$= \vec{e}_j\,\epsilon_{jkl}\underbrace{\left(\tfrac{1}{2}\epsilon_{lmn}B_m x_n\right)_{,k}}_{\frac{1}{2}\epsilon_{l3k}B}$$

$$= \frac{1}{2}B\,\vec{e}_j\underbrace{\epsilon_{jkl}\,\epsilon_{3kl}}_{2\delta_{j3}}$$

$$= B\vec{e}_3$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{q}{c}(\dot{x}A_x + \dot{y}A_y)$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{qB}{2c}(-\dot{x}y + \dot{y}x)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2c}y$$

$$p_y = m\dot{y} + \frac{qB}{2c}x$$

$$H = p_x\dot{x} + p_y\dot{y} - L$$

$$= m\dot{x}^2 - \cancel{\frac{qB}{2c}\dot{x}y} + m\dot{y}^2 + \cancel{\frac{qB}{2c}\dot{y}x} - \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$+ \frac{qB}{2c}(\cancel{\dot{x}y} - \cancel{\dot{y}x})$$

$$H = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2m}\left(\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2\right)$$

This could be gotten from general considerations

This could be gotten directly from

$$H = \frac{1}{2m}\left(p_j - \frac{q}{c}A_j\right)\left(p_j - \frac{q}{c}A_j\right)$$

With $\alpha = \sqrt{\frac{qB}{c}}$, we can write

$$\sqrt{2P_1}\sin Q_1 = -\frac{1}{\alpha}\left(p_y - \frac{\alpha^2}{2}x\right) = -\frac{1}{\alpha}\left(p_y - \frac{qB}{2c}x\right)$$

$$\sqrt{2P_1}\cos Q_1 = \frac{1}{\alpha}\left(p_x + \frac{\alpha^2}{2}y\right) = \frac{1}{\alpha}\left(p_x + \frac{qB}{2c}y\right)$$

$$\therefore\ H = \frac{1}{\cancel{2}m}\alpha^2\left(\cancel{2}P_1\cos^2 Q_1 + \cancel{2}P_1\sin^2 Q_1\right)$$

$$\boxed{H = \frac{qB}{mc}P_1}$$

Hamiltonian in action-angle form

$$\dot{Q}_2 = \frac{\partial H}{\partial P_2} = 0 \qquad \Longrightarrow \qquad Q_2 = \alpha_2 = \text{const}$$

$$\dot{P}_2 = -\frac{\partial H}{\partial Q_2} = 0 \qquad \qquad P_2 = \beta_2 = \text{const}$$

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = \frac{qB}{mc} = \omega \Longrightarrow Q_1 = \omega t + \alpha_1$$

$$\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = 0 \Longrightarrow P_1 = \beta_1$$

$$x = \sqrt{\frac{c}{qB}}\left(\sqrt{2\beta_1}\sin(\omega t + \alpha_1) + \beta_2\right)$$

$$y = \sqrt{\frac{c}{qB}}\left(\sqrt{2\beta_1}\cos(\omega t + \alpha_1) + \alpha_2\right)$$

Cyclotron motion about $\vec{B}$

Center of cyclotron motion

How do we get this motion directly

$$m \frac{d\vec{v}}{dt} = \frac{q}{c} \vec{v} \times \vec{B} = -\frac{qB}{c} \vec{e}_z \times \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\omega \vec{e}_z \times \vec{v} \longleftarrow$$ $\vec{v}$ rotates about z-axis with angular velocity $-\vec{\omega}$.

Solution: $\vec{v} = \vec{v}_0 \cos\omega t - \vec{e}_z \times \vec{v}_0 \sin\omega t$

$$\frac{d\vec{v}}{dt} = -\omega \underbrace{\left[+\vec{v}_0 \sin\omega t + \vec{e}_z \times \vec{v}_0 \cos\omega t\right]}_{\vec{e}_z \times \left(\vec{v}_0 \cos\omega t - \vec{e}_z \times \vec{v}_0 \sin\omega t\right) = \vec{e}_z \times \vec{v}} = -\omega \vec{e}_z \times \vec{v}$$

$$\frac{d\vec{x}}{dt} = \vec{v} = \vec{v}_0 \cos\omega t - \vec{e}_z \times \vec{v}_0 \sin\omega t$$

$$\boxed{\vec{x}(t) = \frac{1}{\omega}\left[\vec{v}_0 \sin\omega t + \vec{e}_z \times \vec{v}_0 \cos\omega t\right] + \vec{x}_c}$$

7.3. Goldstein 9.25

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$$

(a)

$$\dot{q} = \frac{\partial H}{\partial p} = p q^4$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\left(-\frac{1}{q^3} + 2p^2 q^3\right) = -2p^2 q^3 + \frac{1}{q^3}$$

$$p = \frac{\dot{q}}{q^4} = -\frac{1}{3}\frac{d}{dt}\left(\frac{1}{q^3}\right)$$

$$\dot{p} = \frac{\ddot{q}}{q^4} - 4\frac{\dot{q}^2}{q^5} = -2p^2 q^3 + \frac{1}{q^3} = -2\frac{\dot{q}^2}{q^8} q^3 + \frac{1}{q^3}$$

$$\therefore \frac{\ddot{q}}{q^4} - 2\frac{\dot{q}^2}{q^5} - \frac{1}{q^3} = 0$$

$$\Longrightarrow \boxed{\ddot{q} - 2\frac{\dot{q}^2}{q} - q = 0}$$

Let $u = q^n \Longrightarrow \dot{u} = n q^{n-1}\dot{q}$ and $\ddot{u} = n q^{n-1}\ddot{q} + n(n-1) q^{n-2}\dot{q}^2$

$$\ddot{u} + u = n q^{n-1}\ddot{q} + n(n-1) q^{n-2}\dot{q}^2 + q^n$$

$$= n q^{n-1}\left(\ddot{q} + (n-1)\frac{\dot{q}^2}{q} + \frac{q}{n}\right) = 0 \quad \text{if} \quad n = -1$$

The equation of motion is equivalent to

$$\frac{d^2}{dt^2}\left(\frac{1}{q}\right) + \frac{1}{q} = 0$$

(b) Seek new coördinates with $Q = \frac{1}{q}$. Use a generating function

$$F_2(q,P) = \frac{P}{q} \implies \begin{aligned} Q &= \frac{\partial F_2}{\partial P} = \frac{1}{q} \\ p &= \frac{\partial F_2}{\partial q} = -\frac{P}{q^2} \end{aligned}$$

New canonical coordinates: $\boxed{Q = \frac{1}{q}, \quad P = -pq^2}$

Hamiltonian: $\boxed{H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) = \frac{1}{2}\left(P^2 + Q^2\right)}$

Direct check of fundamental Poisson bracket: ($^{\text{there's only}}_{\text{one}}$):

$$[Q,P] = \underset{\uparrow \atop -\frac{1}{q^2}}{\frac{\partial Q}{\partial q}} \underset{\uparrow \atop -q^2}{\frac{\partial P}{\partial p}} - \underset{\uparrow \atop 0}{\frac{\partial Q}{\partial p}} \underset{\uparrow \atop -2pq}{\frac{\partial P}{\partial q}} = 1$$

7.4. Goldstein 9.39

(a) The objective here is to minimize any actual calculating of Poisson brackets.

Use in the following

$[f(\vec{p}), g(\vec{p})] = 0$

$[f(\vec{x}), g(\vec{x})] = 0$

$$H = \frac{\vec{p}^2}{2m} + V(r) = \frac{p_j p_j}{2m} - \frac{k}{r}$$

$$\vec{A} = \vec{p} \times \vec{L} - \frac{mk\vec{r}}{r} \Leftrightarrow A_j = \epsilon_{jkl} p_k L_l - mk\frac{x_j}{r}$$

$$[A_j, H] = \epsilon_{jkl} [p_k L_l, H] - mk \left[\frac{x_j}{r}, H\right]$$

$$= p_k [\cancel{L_l}, H] + [p_k, H] L_l$$

0 (since $\vec{L}$ is conserved)

$$= -k L_l \left[p_k, \frac{1}{r}\right]$$

$$= \frac{1}{r}[x_j, H] + x_j \left[\frac{1}{r}, H\right]$$

$$= \frac{1}{r}\left[x_j, \frac{p_k p_k}{2m}\right] + x_j \left[\frac{1}{r}, \frac{p_k p_k}{2m}\right]$$

$$= \frac{1}{r}\frac{p_k}{m} \underbrace{[x_j, p_k]}_{\delta_{jk}} + x_j \frac{p_k}{m}\left[\frac{1}{r}, p_k\right]$$

$$= \frac{1}{mr} p_j + \frac{1}{m} x_j p_k \left[\frac{1}{r}, p_k\right]$$

$$= +k \underbrace{\epsilon_{jkl} L_l}_{} \left[\frac{1}{r}, p_k\right] - \frac{k}{r} p_j - k x_j p_k \left[\frac{1}{r}, p_k\right]$$

$$= \epsilon_{jkl} \epsilon_{lmn} x_m p_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) x_m p_n$$

$$= x_j p_k - x_k p_j$$

$$[A_j, H] = k(\cancel{x_j p_k} - x_k p_j)\left[\frac{1}{r}, p_k\right] - \frac{k}{r} p_j - k\,\cancel{x_j p_k}\left[\frac{1}{r}, p_k\right]$$

$$= -k x_k p_j \underbrace{\left[\frac{1}{r}, p_k\right]}_{} - \frac{k}{r} p_j$$

This is the only bracket we must calculate

Direct evaluation: $$\frac{\partial r^{-1}}{\partial x_j}\underbrace{\frac{\partial p_k}{\partial p_j}}_{\delta_{jk}} - \frac{\partial r^{-1}}{\partial p_j}\cancel{\frac{\partial p_k}{\partial x_j}} = \frac{\partial r^{-1}}{\partial x_k}$$

0

$$= \frac{\partial}{\partial x_k}\left(\frac{1}{\sqrt{x_\ell x_\ell}}\right)$$

$$= -\frac{1}{\cancel{2}}\left(\frac{\cancel{2}x_k}{(x_\ell x_\ell)^{3/2}}\right)$$

$$= -\frac{x_k}{r^3} = \left[\frac{1}{r}, p_k\right]$$

Indirect evaluation: $$\left[\frac{1}{r}, p_k\right] = -\frac{1}{k}\left[-\frac{k}{r}, p_k\right]$$

$$= \frac{1}{k}[p_k, H]$$

$$= \frac{1}{k}\dot{p}_k \text{ — force}$$

$$= \frac{1}{k}\left(-\frac{k}{r^3}x_k\right)$$

$$= -\frac{x_k}{r^3}$$

$$[A_j, H] = +k p_j \underbrace{\frac{x_k x_k}{r^3}}_{\frac{1}{r}} - \frac{k}{r} p_j = 0$$

(b) This evaluation is independent of H.

$$[A_i, L_j] = \epsilon_{ikl}\,[p_k L_l, L_j] - mk\left[\frac{x_i}{r}, L_j\right]$$

$$= p_k [L_l, L_j] + L_l [p_k, L_j]$$

$$= p_k \epsilon_{ljm} L_m + \epsilon_{jmn} L_l [p_k, x_m p_n]$$

$$= \epsilon_{ljm} p_k L_m + \epsilon_{jmn} L_l p_n \underbrace{[p_k, x_m]}_{-\delta_{km}}$$

$$= \epsilon_{ljm} p_k L_m - \epsilon_{jkn} L_l p_n$$

$$= \frac{1}{r}[x_i, L_j] + x_i\left[\frac{1}{r}, L_j\right]$$

$$= \frac{1}{r}\epsilon_{jkl}[x_i, x_k p_l] + x_i \epsilon_{jkl}\left[\frac{1}{r}, x_k p_l\right]$$

$$= \frac{1}{r}\epsilon_{jkl} x_k \underbrace{[x_i, p_l]}_{\delta_{il}} + x_i \epsilon_{jkl} x_k \underbrace{\left[\frac{1}{r}, p_l\right]}_{-\frac{x_l}{r^3}}$$

$$= \frac{1}{r}\epsilon_{jki} x_k - \frac{1}{r^3}\cancel{x_i \epsilon_{jkl} x_k x_l} \to 0$$

$$= \frac{1}{r}\epsilon_{ijk} x_k$$

$$[A_i, L_j] = \underbrace{\epsilon_{ikl}\,\epsilon_{ljm}}\, p_k L_m - \underbrace{\epsilon_{ikl}\,\epsilon_{jkm}}\, L_l p_m - \frac{mk}{r}\,\epsilon_{ijk}\, x_k$$

$$\epsilon_{ikl}\,\epsilon_{jml} = \delta_{ij}\,\delta_{km} - \delta_{im}\,\delta_{kj}$$

$$\epsilon_{kil}\,\epsilon_{kjm} = \delta_{ij}\,\delta_{lm} - \delta_{im}\,\delta_{lj}$$

$$= \delta_{ij}\,\cancel{p_k L_k} - p_j L_i - \delta_{ij}\,\cancel{p_l L_l} + p_i L_j - \frac{mk}{r}\,\epsilon_{ijk}\, x_k$$

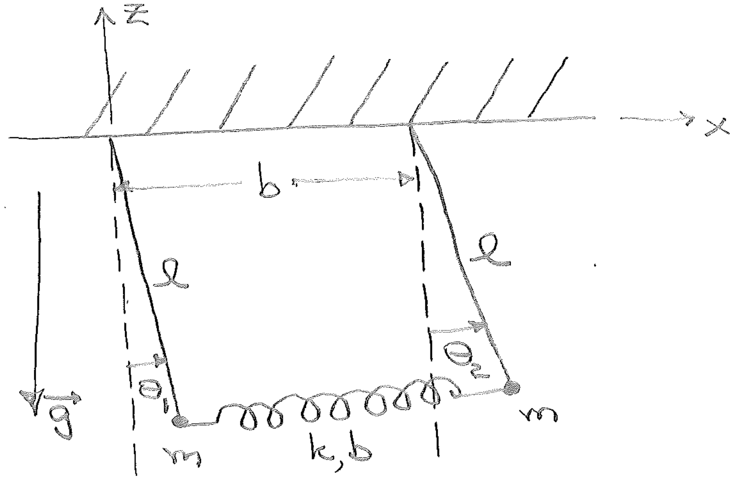
$$= \underbrace{p_i L_j - p_j L_i} - \frac{mk}{r}\,\epsilon_{ijk}\, x_k$$

$$(\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\, p_l L_m = \epsilon_{ijk}\,\epsilon_{klm}\, p_l L_m$$

$$= \epsilon_{ijk}\underbrace{\left(\epsilon_{klm}\, p_l L_m - \frac{mk}{r}\, x_k\right)}_{A_k}$$

$$\boxed{\therefore\ [A_i, L_j] = \epsilon_{ijk}\, A_k}$$

7.5.



(a) $x_1 = \ell \sin\theta_1 = \ell\theta_1$

$z_1 = \ell(1-\cos\theta_1) = \frac{1}{2}\ell\theta_1^2$

$x_2 = b + \ell\sin\theta_2 = b + \ell\theta_2$

$z_2 = \ell(1-\cos\theta_2) = \frac{1}{2}\ell\theta_2^2$

$T = \frac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2)$

$V = mg(z_1 + z_2) + \frac{1}{2}k(d-b)^2$

$d = x_2 - x_1 = b + \ell(\theta_2 - \theta_1)$

$= \frac{1}{2}mg\ell(\theta_1^2 + \theta_2^2) + \frac{1}{2}k\ell^2(\theta_2-\theta_1)^2$

$$\boxed{L = T - V = \frac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2) - \frac{1}{2}mg\ell(\theta_1^2+\theta_2^2) - \frac{1}{2}k\ell^2(\theta_2-\theta_1)^2}$$

(b) $\theta_1^2 + \theta_2^2 = \frac{1}{2}(\theta_1+\theta_2)^2 + \frac{1}{2}(\theta_1-\theta_2)^2$

$\dot\theta_1^2 + \dot\theta_2^2 = \frac{1}{2}(\dot\theta_1+\dot\theta_2)^2 + \frac{1}{2}(\dot\theta_1-\dot\theta_2)^2$

$$L = \frac{1}{2}\frac{ml^2}{2}(\dot\theta_1+\dot\theta_2)^2 - \frac{1}{2}\frac{mgl}{2}(\theta_1+\theta_2)^2$$
$$+ \frac{1}{2}\frac{ml^2}{2}(\dot\theta_1-\dot\theta_2)^2 - \frac{1}{2}\frac{mgl}{2}(\theta_1-\theta_2)^2 - \frac{1}{2}kl^2(\theta_1-\theta_2)^2$$

Normal coordinates:
$$\boxed{\begin{aligned} Q_1 &= \sqrt{\tfrac{m}{2}}\,l\,(\theta_1+\theta_2) \\ Q_2 &= \sqrt{\tfrac{m}{2}}\,l\,(-\theta_1+\theta_2) \end{aligned}}$$

Q_1 ← Swinging together

Q_2 ← swinging oppositely

$$\boxed{A = \sqrt{\frac{m}{2}}\,l\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \qquad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$L = \frac{1}{2}\left(\dot Q_1^2 - \frac{g}{l}Q_1^2\right) + \frac{1}{2}\left(\dot Q_2^2 - \left(\frac{g}{l}+\frac{2k}{m}\right)Q_2^2\right)$$

$$\boxed{\omega_1^2 = \frac{g}{l}, \qquad \omega_2^2 = \frac{g}{l} + \frac{2k}{m}}$$

(c) $$p_j = \frac{\partial L}{\partial \dot\theta_j} = ml^2\dot\theta_j$$

$$P_j = \frac{\partial L}{\partial \dot Q_j} = \dot Q_j = A_{jk}\dot\theta_k = \frac{A_{jk}}{ml^2}p_k$$

$$\boxed{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{1}{ml^2}A\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{\sqrt{2m}\,l}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}$$

(d) The matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is orthogonal, so if we write

$$\sqrt{m}\,l \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{\text{the inverse of this matrix is its transpose}} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\,l \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \tilde{A} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Find a generating function $F_2(\theta, P)$

$$p_j = \frac{\partial F_2}{\partial \theta_j} \quad \text{and} \quad Q_j = \frac{\partial F_2}{\partial P_j}$$

$$(p_1 \quad p_2) = \left(\frac{\partial F_2}{\partial \theta_1} \quad \frac{\partial F_2}{\partial \theta_2} \right) = (P_1 \quad P_2) A$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \partial F_2 / \partial P_1 \\ \partial F_2 / \partial P_2 \end{pmatrix} = A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\Longrightarrow \boxed{F_2(\theta, P) = (P_1 \quad P_2) A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = P_j A_{jk} \theta_k}$$

You could also proceed from

$$Q_j = A_{jk}\Theta_k = \frac{\partial F_2}{\partial P_j}$$

$$\Rightarrow \quad F_2 = P_j A_{jk} \Theta_k + f(\Theta_i, \Theta_2)$$

$$\Rightarrow \quad P_j = \frac{\partial F_2}{\partial \Theta_j} = P_k A_{kj} + \cancelto{0}{\frac{\partial f}{\partial \Theta_j}}$$

Choose $f = 0$

$$\boxed{F_2 = P_j A_{jk} \Theta_k}$$

7.6.

$$H = qp$$

(a) $\boxed{\dot{p} = -\frac{\partial H}{\partial q} = -p \qquad \dot{q} = \frac{\partial H}{\partial p} = q} \Longrightarrow \boxed{p(t) = p_0 e^{-t} \qquad q(t) = q_0 e^{t}}$

(b) $X = \frac{1}{\sqrt{2}}(q - p) \qquad \Longleftrightarrow \qquad q = \frac{1}{\sqrt{2}}(X + \Pi)$

$\Pi = \frac{1}{\sqrt{2}}(q + p) \qquad\qquad\qquad p = \frac{1}{\sqrt{2}}(-X + \Pi)$

Show this is a canonical transformation

① Method 1: preservation of fundamental Poisson bracket

$$\boxed{[X, \Pi] = \frac{1}{2}[q - p, q + p] = \frac{1}{2}\Big([q,p] \underbrace{- [p,q]}_{+[q,p]}\Big) = [q,p] = 1}$$

② Method 2: Find a generating function $F_2(q, \Pi)$

$$\begin{cases} p = \sqrt{2}\,\Pi - q = \dfrac{\partial F_2}{\partial q} \\ X = \sqrt{2}\,q - \Pi = \dfrac{\partial F_2}{\partial \Pi} \end{cases} \Longrightarrow \boxed{F_2 = \sqrt{2}\,q\Pi - \frac{1}{2}q^2 - \frac{1}{2}\Pi^2}$$

$$K = H = qp = \frac{1}{2}(\Pi^2 - X^2)$$

time independent canonical transformation

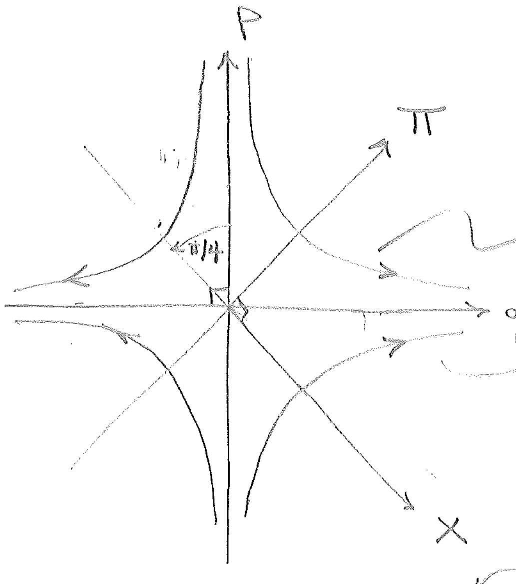
$$\boxed{\begin{aligned} \dot{\Pi} &= -\frac{\partial H}{\partial X} = X \\ \dot{X} &= \frac{\partial H}{\partial \Pi} = \Pi \end{aligned}} \implies \ddot{X} = \dot{\Pi} = X$$

Solution: $\boxed{\begin{aligned} X(t) &= X_0 \cosh t + \Pi_0 \sinh t \\ \Pi(t) &= X_0 \sinh t + \Pi_0 \cosh t \end{aligned}}$

Check: $q = \frac{1}{\sqrt{2}}(X + \Pi) = \underbrace{\frac{1}{\sqrt{2}}(X_0 + P_0)}_{q_0}\underbrace{(\cosh t + \sinh t)}_{e^t} = q_0 e^t$

$$p = \frac{1}{\sqrt{2}}(-X + \Pi) = \underbrace{\frac{1}{\sqrt{2}}(-X_0 + \Pi_0)}_{p_0}\underbrace{(\cosh t - \sinh t)}_{e^{-t}} = p_0 e^{-t}$$

(c)



hyperbolae $pq = p_0 q_0$ or

$\Pi^2 - X^2 =$ constant

unit mass

The particle has $m = 1$, kinetic energy $\Pi^2/2$, and moves in a potential $V(X) = -\frac{1}{2}X^2$.

(d) $q = e^{Q}$, $p = f(P)\,e^{-Q}$

What $f(P)$ makes this a canonical transformation?

Method 1: preservation of fundamental Poisson brackets

$$1 = [q,p] = \underbrace{\frac{\partial q}{\partial Q}}_{e^{Q}}\underbrace{\frac{\partial p}{\partial P}}_{f'(P)e^{-Q}} - \cancel{\frac{\partial q}{\partial P}}^{\,0}\frac{\partial p}{\partial Q} = f'(P) \Rightarrow f(P) = P + (\text{Constant})$$

Choose the (Constant) = 0, so $\boxed{f(P) = P}$

$$\boxed{q = e^{Q}, \quad p = Pe^{-Q}}$$

Method 2: Find a $F_2(q,P)$ generating function

$$\begin{cases} p = \dfrac{\partial F_2}{\partial q} = f(P)/q \\ Q = \dfrac{\partial F_2}{\partial P} = \log q \end{cases}$$

$$\frac{1}{q} = \frac{\partial^2 F_2}{\partial q \partial P} = \frac{\partial^2 F_2}{\partial P \partial q} = \frac{f'(P)}{q} \Rightarrow f'(P) = 1 \Rightarrow f(P) = P$$

$$\boxed{F_2(q,P) = P\log q}$$

Method 3: Hamilton's characteristic function $W(q,P)$

$$q\frac{\partial W}{\partial q} = H\left(q, \frac{\partial W}{\partial q}\right) = P \Rightarrow W(q,P) = P\log q = F_2(q,P)$$

Notice that (Q,P) cover only half the phase plane, since $q = e^{Q} \geq 0$.

$$K = H = qp = \boxed{P = H}$$

$\uparrow$

time-independent canonical transformation

$$\therefore \quad \dot{P} = -\frac{\partial H}{\partial Q} = 0 \qquad \Longrightarrow \qquad P = P_0 \qquad \longrightarrow \qquad q = q_0 e^{t}$$

$$\dot{Q} = \frac{\partial H}{\partial P} = 1 \qquad\qquad Q = t + Q_0 \qquad\qquad p = p_0 e^{-t}$$