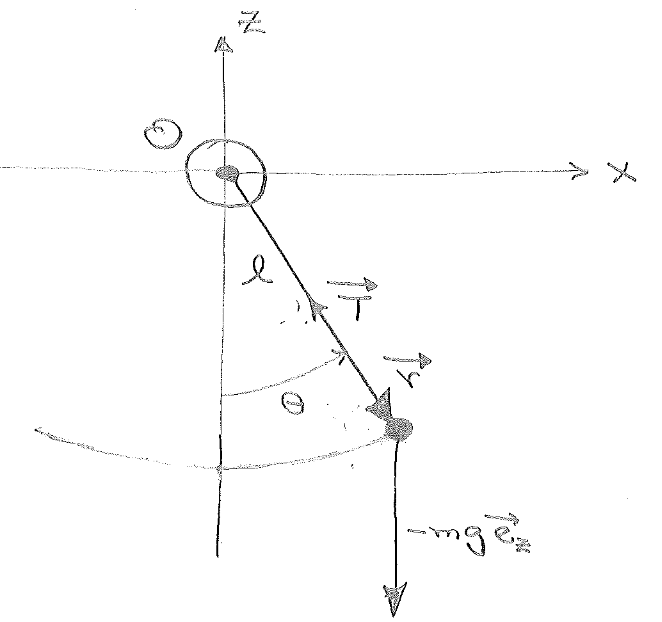
Lectures 1-2

D'Alembert's principle



Newton's 2nd Law: $\vec{F} = \dot{\vec{p}} = m\ddot{\vec{r}}$

1 Cartesian coordinates:

$$\vec{F} = (-mg + T\cos\theta)\vec{e}_z - T\sin\theta\,\vec{e}_x$$

$$\vec{r} = -l\cos\theta\,\vec{e}_z + l\sin\theta\,\vec{e}_x$$

$$\vec{v} = \dot{\vec{r}} = l\sin\theta\,\dot{\theta}\,\vec{e}_z + l\cos\theta\,\dot{\theta}\,\vec{e}_x$$

$$\vec{a} = \ddot{\vec{r}} = (+l\cos\theta\,\dot{\theta}^2 + l\sin\theta\,\ddot{\theta})\vec{e}_z + (-l\sin\theta\,\dot{\theta}^2 + l\cos\theta\,\ddot{\theta})\vec{e}_x$$

$$-mg + T\cos\theta = ml\cos\theta\,\dot{\theta}^2 + ml\sin\theta\,\ddot{\theta}$$

$$-T\sin\theta = -ml\sin\theta\,\dot{\theta}^2 + ml\cos\theta\,\ddot{\theta}$$

$$\Rightarrow \boxed{\begin{aligned} -\cancel{m}g\sin\theta &= \cancel{m}l\ddot{\theta} \\ T - mg\cos\theta &= ml\ddot{\theta}^2 \end{aligned}}$$

centripetal acceleration

(2) Polar coordinates

$$\vec{F} = (mg\cos\theta - T)\,\vec{e}_r - mg\sin\theta\,\vec{e}_\theta$$

$$\vec{r} = l\,\vec{e}_r$$

$$\vec{v} = \dot{\vec{r}} = l\,\dot{\vec{e}}_r = l\dot{\theta}\,\vec{e}_\theta$$

$$\vec{a} = \ddot{\vec{r}} = l\ddot{\theta}\,\vec{e}_\theta + l\dot{\theta}\,\underbrace{\dot{\vec{e}}_\theta}_{= -\dot{\theta}\vec{e}_r} = l\ddot{\theta}\,\vec{e}_\theta - l\dot{\theta}^2\,\vec{e}_r$$

$$\boxed{\begin{aligned} mg\cos\theta - T &= -ml\dot{\theta}^2 \\ -g\sin\theta &= l\ddot{\theta} \end{aligned}}$$

Problems:

(1) Vectors —
- tedious
- prone to sign errors
- differentiation of basis vectors
- more components than degrees of freedom

(2) Solve for unnecessary $\underbrace{\text{constraint force}}_{\text{unknown force specified by how it constrains the motion}}$ $\vec{T}$

Scalar approach: conservation of energy

$$E = T + V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

Conserved because $\underline{\underline{\text{work}}}$ done by force can be written in two ways

$$0 = \int (\dot{\vec{p}} - \vec{F}) \cdot \vec{ds}$$

(actual displacement) $\cdot\ \vec{v}\,dt$

$$= \int m\dot{\vec{v}} \cdot \vec{v}\,dt + \int \nabla V \cdot \vec{ds}$$

Why can we neglect the constraint force? It does no work.

$$= \int \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt + \int \nabla V \cdot \vec{ds}$$

Note we <u>cannot</u> leave out the constraint force in the equations of motion.

$$= \Delta T + \Delta V$$

$$0 = \frac{dE}{dt} \Rightarrow l\ddot{\theta} = -g\sin\theta$$

But conservation of energy not enough if more than one variable.

Somehow T and V contain all the information

Can we generalize? Yes. { D'Alembert's principle / Lagrange's equations

1. Works only if constraint forces do no work
2. Addresses above problems
3. Frees mechanics to use generalized coordinates (field theory)
4. Shifts focus to symmetries and conservation laws
5. Readies mechanics to be quantized

Will you ever use D'Alembert's principle?

Use two charged particles as example

N particles: $\vec{r}_1(t), \ldots, \vec{r}_N(t)$ ⟵ path in multi-particle configuration space

Newton's Second law: $\dot{\vec{p}}_i = \vec{F}_i^{(tot)} = \vec{F}_i^{(con)} + \vec{F}_i$

non-constraint forces are derivable from a potential; conservative if $\partial V/\partial t = 0$

$\vec{F}_i = -\nabla_{\vec{r}_i} V$

⟵ time derivative $d\vec{p}_i/dt$

(1) No constraints: $\dot{\vec{p}}_i = \frac{d}{dt}(m\vec{v}_i) = \frac{d}{dt}\nabla_{\vec{v}_i}\left(\frac{1}{2}m v_i^2\right)$

(1) Gradient formula

(2) Geometric definition

$$\dot{\vec{p}}_i = \frac{d}{dt}(m\vec{v}_i) = \frac{d}{dt}\nabla_{\vec{v}_i}\left(\sum_j \frac{1}{2} m v_j^2\right) = \frac{d}{dt}\nabla_{\vec{v}_i} T$$

function of velocities

$$\vec{F}_i = -\nabla_{\vec{r}_i} V$$ ⟵ function of positions

$\vec{F}$ derivable from a potential; conservative if $\frac{\partial V}{\partial t} = 0$

$$0 = \frac{d}{dt}\nabla_{\vec{v}_i} T + \nabla_{\vec{r}_i} V = \frac{d}{dt}\nabla_{\vec{v}_i}(T-V) - \nabla_{\vec{r}_i}(T-V)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j}$$

Lagrangian L

The point is that E (or T and V) contain all the information to reconstruct the equations of motion. But we have to consider how T and V change in all directions in configuration and velocity space, not just along the actual motion.

But we could have used $E = T + V$ just as well.

(2) Constraints

Holonomic constraints: $f_\alpha(\vec{r}_1, \ldots, \vec{r}_N) = 0, \quad \alpha = 1, \ldots, A$

text includes time dependence

↓

motion on a $(3N-A)$-d surface in configuration space

Generalized coordinates: $q_j, \quad j = 1, \ldots, n = 3N - A$

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_n)$$ ⟵ text includes time dependence

role of i and j

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Displacement along actual path

$\delta\vec{r}_i = \vec{v}_i\, dt$

$\dot{\vec{p}}_i \cdot \delta\vec{r}_i = m\dot{\vec{v}}_i \cdot \vec{v}_i\, dt = d\left(\frac{1}{2} m v_i^2\right) = dT$

$\vec{F}_i \cdot \delta\vec{r}_i = -\vec{\nabla}V \cdot \delta\vec{r}_i = -dV$

$\sum_i (\dot{\vec{p}}_i - \vec{F}_i) \cdot \delta\vec{r}_i = d(T+V) = 0$

D'Alembert's principle:

$$\boxed{0 = \sum_i (\dot{\vec{p}}_i - \vec{F}_i) \cdot \delta\vec{r}_i}$$

virtual displacement: any displacement consistent w/ constraints; doesn't assume $\delta\vec{r}_i = \vec{v}_i\, dt$. n independent virtual displacements; generalizes 1 actual displacement

forces of constraint can be omitted because they do no virtual work (beads on wire, rolling friction, ~~friction~~). Physics: no energy exchanged with constraints; otherwise they are dynamical variables

Assumption is $\sum_i \vec{F}_i^{(con)} \cdot \delta\vec{r}_i = 0$

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j$$

function of q_j and $\dot{q}_j$

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_j \delta q_j \underbrace{\left(-\sum_i \nabla_{\vec{r}_i} V \cdot \frac{\partial\vec{r}_i}{\partial q_j}\right)}_{-\frac{\partial V}{\partial q_j} \equiv Q_j = \left(\text{generalized force}\right)} = \sum_j \left(-\frac{\partial V}{\partial q_j}\right)\delta q_j$$

Easy part

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_j \delta q_j \left(\sum_i \underbrace{\frac{d}{dt}(m_i\vec{v}_i) \cdot \frac{\partial\vec{r}_i}{\partial q_j}}\right)$$

$$\frac{d}{dt}\left(m\vec{v}_i \cdot \underbrace{\frac{\partial\vec{r}_i}{\partial q_j}}_{\frac{\partial\vec{v}_i}{\partial\dot{q}_j}}\right) - m\vec{v}_i \cdot \underbrace{\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)}_{\sum_k \frac{\partial^2\vec{r}_i}{\partial q_k \partial q_j}\dot{q}_k = \frac{\partial\vec{v}_i}{\partial q_j}}$$

crucial steps

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 = \frac{1}{2}\underbrace{\sum_{j,k} \dot{q}_j \dot{q}_k}_{\text{cross terms}} \underbrace{\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}}_{\text{spatial dependence}}$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\tfrac{1}{2} m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\tfrac{1}{2} m_i v_i^2\right)$$

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_j \delta q_j \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right)$$

$$0 = \sum_j \delta q_j \underbrace{\left(\frac{d}{dt}\left(\frac{\partial \overbrace{(T-V)}^{L}}{\partial \dot{q}_j}\right) - \frac{\partial \overbrace{(T-V)}^{L}}{\partial q_j} \right)}_{=0}$$

T appears naturally here

insert V by assuming $\frac{\partial V}{\partial \dot{q}_j} = 0$

Lagrange's equations

$$\boxed{\begin{gathered} L = T - V = L(q_1,\dots,q_n;\dot{q}_1,\dots,\dot{q}_n,t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \end{gathered}}$$

Example: $T = \frac{1}{2} m (l\dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$

$V = -mgl\cos\theta$

$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$; $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

$\cancel{m} l^{\cancel{2}} \ddot{\theta} = -\cancel{m} g \cancel{l} \sin\theta$

Velocity-dependent forces?