Phys 503

Lectures 12-13

Dynamics of rigid-body motion

Dynamics

Repeat kinematic diagram: $\vec{r}_i = \vec{R} + \vec{r}_i{}'$

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i{}'$$

$$T_{tot} = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} M \dot{\vec{R}}^2 + \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i{}'^2 + \sum_i m_i \dot{\vec{r}}_i{}' \cdot \dot{\vec{R}}$$

$$\vec{L}_{tot} = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i = \vec{R} \times M \dot{\vec{R}} + \sum_i \vec{r}_i{}' \times m_i \dot{\vec{r}}_i{}' + \left(\sum_i m_i \vec{r}_i{}'\right) \times \dot{\vec{R}} + \vec{R} \times \left(\sum_i m_i \dot{\vec{r}}_i{}'\right)$$

(about O)

convenient choices:

① $\vec{R} = 0$: $O = O'$ (some point in body is also fixed in inertial space)

② $\sum_i m_i \vec{r}_i{}' = 0$: O' is CM

Deal only with $T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i{}'^2$ and $\vec{L} = \sum_i m_i \vec{r}_i{}' \times \dot{\vec{r}}_i{}'$

(drop the primes, but remember these vectors can be non-inertial because O' can be accelerating.)

Nothing about rotation so far; put in rigid body now

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$$

$$\dot{\vec{r}}_i = \left(\frac{d\vec{r}_i}{dt}\right)_{space} = \cancel{\left(\frac{d\vec{r}}{dt}\right)_{body}}^{\,0} + \vec{\omega} \times \vec{r}_i$$

$$= \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \vec{\omega} \times \vec{r}_i$$

$$= \frac{1}{2} \vec{\omega} \cdot \underbrace{\left(\sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \right)}_{\vec{L}}$$

vector relation that does not rely on body vs. space

$$\boxed{T = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i$$

$$= \sum_i m_i \vec{r}_i \times (\vec{\omega}\times\vec{r}_i)$$

$$= \sum_i m_i \left[\vec{\omega} r_i^2 - \vec{r}_i(\vec{r}_i\cdot\vec{\omega})\right]$$

$$= \underset{\text{any basis}}{\sum_{j,k} \vec{e}_j} \underbrace{\left(\sum_i m_i \left(r_i^2 \delta_{jk} - x_j x_k\right)\right)}_{\equiv I_{jk}} \omega_k$$

$$\boxed{\begin{aligned} I_{jk} &= \int dV\, \rho\left(r^2 \delta_{jk} - x_j x_k\right) \\ \vec{L} &= \sum_j \vec{e}_j I_{jk}\omega_k \end{aligned}}$$

$$I_{11} = \int dV\, \rho(r^2 - x^2) = \int dV\, \rho(y^2 + z^2) \longleftarrow \text{moment of inertia about the x-axis}$$

Tensors: I_{jk} are components of a matrix, but the they are different from A_{jk}. The I_{jk} are components of a 2-tensor, the 2-index analogue of a vector.

$$I'_{jk} = \int dV\, \rho\left(r'^2 \delta_{jk} - x'_j x'_k\right)$$

$$= A_{j\ell}' A_{km} \underbrace{\int dV \rho (r^2 \delta_{\ell m} - x_\ell x_m)}_{I_{\ell m}}$$

$I_{jk}' = A_{j\ell} A_{km} I_{\ell m} \longleftarrow$ each index transforms like a vector

<u>So</u>, just like a vector $\vec{G} = G_j \vec{e}_j = G_j' \vec{e}_j'$ is independent of basis, while its components change from one basis to another, we can make up a

2-tensor

$$\overleftrightarrow{I} = I_{jk} \overbrace{\vec{e}_j \otimes \vec{e}_k}^{\text{dyad — basis tensors}} = I_{jk}' \vec{e}_j' \otimes \vec{e}_k' = \int dV \rho (r^2 \overleftrightarrow{1} - \vec{r} \otimes \vec{r}),$$

tensor product unit tensor

which is independent of basis, while its components change from one basis to another.

Summary: $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \vec{\omega} \cdot \overleftrightarrow{I}$

$T = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$

$\overleftrightarrow{I}$ is special in two ways: ① it rotates with the body (components constant in body frame); ② it is (real) symmetric.

This means we prefer to formulate rigid-body dynamics in body frame; very odd because we want the motion of the body frame

① General time derivatives

$$\left(\frac{d\overleftrightarrow{H}}{dt}\right)_{space} = \frac{dH_{jk}}{dt}\,\vec{e}_j \otimes \vec{e}_k$$

$$= \underbrace{\frac{dH'_{jk}}{dt}\,\vec{e}\,'_j \otimes \vec{e}\,'_k}_{\left(\frac{d\overleftrightarrow{H}}{dt}\right)_{body}} + H'_{jk}\,\underbrace{\frac{d\vec{e}\,'_j}{dt}}_{\substack{= \epsilon_{jlm}\,\omega'_m\,\vec{e}\,'_l \\ l \ne j}} \otimes \vec{e}_k + H'_{jk}\,\vec{e}_j \otimes \underbrace{\frac{d\vec{e}\,'_k}{dt}}_{\substack{= \epsilon_{klm}\,\omega'_m\,\vec{e}\,'_l \\ l \ne k}}$$

$$= \left(\frac{d\overleftrightarrow{H}}{dt}\right)_{body} + \left(\epsilon_{jlm}\,\omega'_l\,H'_{mk} + \epsilon_{klm}\,\omega'_l\,H'_{jk}\right)\vec{e}\,'_j \otimes \vec{e}\,'_k$$

② Real symmetric $\Rightarrow$ find vectors such that

$$\overleftrightarrow{I} \cdot \vec{f}^{(j)} = I_j\,\vec{f}^{(j)} \iff I_{kl}\,f^{(j)}_l = I_j\,f^{(j)}_k$$

special body axes (principal axes)

Eigenvalue problem

Diagonalize a real symmetric matrix by an orthogonal transformation

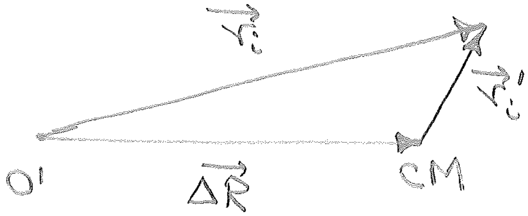
$$\vec{f}_j \cdot \overleftrightarrow{I} \cdot \vec{f}_k = I_j\,\delta_{jk}$$

Geometric interpretation of symmetric 2-tensor (cf. vector's magnitude and direction): diagonal basis (3 #'s) & diagonal elements (3 #'s).

orientation of $\overleftrightarrow{I}$

moment of inertia ellipsoid

Caution! Principal axes depend on O'.



$\vec{r}_i = \vec{r}_i' + \Delta\vec{R}$

$r_i^2 = r_i'^2 + |\Delta\vec{R}|^2 + 2\vec{r}_i' \cdot \Delta\vec{R}$

$$\overleftrightarrow{I}_{O'} = \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \vec{r}_i \otimes \vec{r}_i \right)$$

$$= \sum_i m_i \overleftrightarrow{1} \left(r_i'^2 + |\Delta\vec{R}|^2 + 2\vec{r}_i' \cdot \Delta\vec{R} \right)$$

$$- \sum_i m_i \left(\vec{r}_i' \otimes \vec{r}_i' + \Delta\vec{R} \otimes \Delta\vec{R} + \vec{r}_i' \otimes \Delta\vec{R} + \Delta\vec{R} \otimes \vec{r}_i' \right)$$

$$= \underbrace{\sum_i m_i \left(r_i'^2 \overleftrightarrow{1} - \vec{r}_i' \otimes \vec{r}_i' \right)}_{\overleftrightarrow{I}_{CM}} + M\left(|\Delta\vec{R}|^2 \overleftrightarrow{1} - \Delta\vec{R} \otimes \Delta\vec{R} \right)$$

$$\boxed{\overleftrightarrow{I}_{O'} = \overleftrightarrow{I}_{CM} + M\left(|\Delta\vec{R}|^2 \overleftrightarrow{1} - \Delta\vec{R} \otimes \Delta\vec{R} \right)}$$

Parallel-axis theorem

Change of principal axes

Euler's equations: Don't rush to use these equations if there is a simpler approach.

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L} = \overleftrightarrow{I} \cdot \left(\frac{d\vec{\omega}}{dt}\right)_{body} + \vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega})$$

remember our assumptions about O'

$$\boxed{\vec{N} = \overleftrightarrow{I} \cdot \left(\frac{d\vec{\omega}}{dt}\right)_{body} + \vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega})}$$

No assumption of body axes

Body component form

$$N_j = I_{jk}\dot{\omega}_k + \epsilon_{jkl}\,\omega_k I_{lm}\omega_m$$

Use principle axes

$$\begin{aligned} N_1 &= I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) \\ N_2 &= I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) \\ N_3 &= I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) \end{aligned}$$

This assumes using body principal axes

We solve for motion as seen in body frame.

Integration constants: 6

Torque-free rigid body

$\vec{N} = 0 \iff \vec{L}$ is fixed in space frame

Symmetric body: $I_1 = I_2 = I$

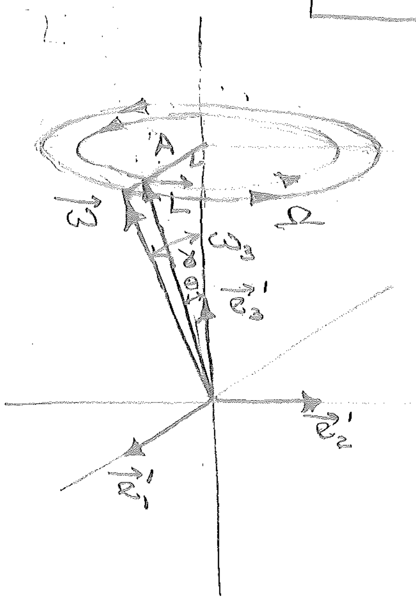
$$I_3 \dot{\omega}_3 = 0 \implies \omega_3 = \text{constant}$$

$$\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I - I_3) &= 0 \\ I \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I) &= 0 \end{aligned} \iff \begin{aligned} \dot{\omega}_1 &= -\Omega \omega_2 \\ \dot{\omega}_2 &= \Omega \omega_1 \end{aligned}$$

$$\Omega = \frac{I_3 - I}{I} \omega_3 = \frac{I_3 \omega_3}{I} - \omega_3$$

Solution: $\frac{d}{dt}(\omega_1 + i\omega_2) = i\Omega(\omega_1 + i\omega_2)$

$$\boxed{\begin{gathered} \omega_1 + i\omega_2 = (\omega_1(0) + i\,\underbrace{\omega_2(0)}_{=0})\, e^{i\Omega t} \qquad \tan\theta = \frac{IA}{I_3 \omega_3} = \frac{I}{I_3}\tan\gamma \\ \| \\ A = \omega_3 \tan\gamma = \frac{I_3 \omega_3}{I} \tan\theta = (\omega_3 + \Omega) \tan\theta \end{gathered}}$$



Body frame

θ and ω_3 are the 2 important variables

$$L^2 = I^2 A^2 + I_3^2 \omega_3^2 = I^2 (\omega_3 + \Omega)^2 \tan^2\theta + I_3^2 \omega_3^2$$

$$T = \frac{1}{2} I A^2 + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{1}{2} I (\omega_3 + \Omega)^2 \tan^2\theta + \frac{1}{2} I_3 \omega_3^2$$

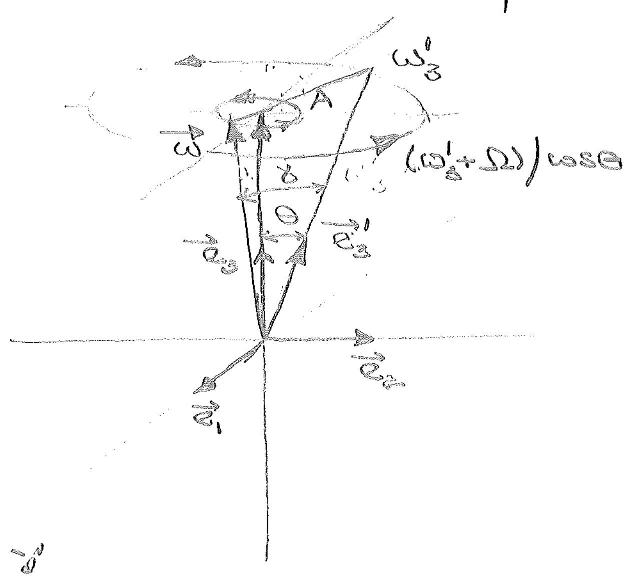
$I_3 > I$ ($\Omega > 0$)

$\vec{L}$ inside $\vec{\omega}$ ($\theta < \gamma$)

$I_3 < I$ ($\Omega < 0$)

$\vec{L}$ outside $\vec{\omega}$ ($\theta > \gamma$)

Space frame



$I_3 > I \quad (\Omega > 0)$ $\qquad$ $I_3 > I \quad (\Omega < 0)$

Euler angles: use primes

Let $\vec{e}_3$ be along $\vec{L}$ $\Rightarrow$ $\vec{e}_3 \cdot \vec{e}_3{}' = \cos\theta \Longrightarrow \theta = \text{const}$

$$\omega_3' = \dot{\psi} + \dot{\phi}\cos\theta$$

$$\omega_1' + i\omega_2' = \dot{\phi}\sin\theta\,(\sin\psi + i\cos\psi)$$
$$= i\dot{\phi}\sin\theta\, e^{-i\psi} = (\omega_3'+\Omega)\tan\theta\; e^{i\Omega t}$$

$$\boxed{\dot{\phi} = \frac{\omega_3' - \dot{\psi}}{\cos\theta} = \frac{\omega_3'+\Omega}{\cos\theta}\, e^{i(\psi + \Omega t - \pi/2)}}$$

Solution:
$$\boxed{\begin{aligned}\psi &= -\Omega t + \pi/2 \\ \phi &= \frac{\omega_3'+\Omega}{\cos\theta}\, t\end{aligned}}$$

precession of $\vec{L}$ about $\vec{e}_3{}'$

$\vec{e}_3{}'$ has angular velocity $\dfrac{\omega_3'+\Omega}{\cos\theta} = \dfrac{I_3\omega_3'}{I\cos\theta}$ about $\vec{L}$

Integration constants:
2 in direction of $\vec{L}$
1 in magnitude of $\vec{L}$ $\rightarrow \theta, \omega_3'$
1 in T
1 in $\omega_2(0) = 0$ ← choice of orientation for e_1'-e_2'
1 in $\phi(0) = 0$ ← zero of time

Space frame:

$$\omega_1 + i\omega_2 = \dot{\psi}\sin\theta\,(\sin\phi - i\cos\phi)$$

$$= -i\dot{\psi}\sin\theta\, e^{i\phi} = +i\Omega\sin\theta\,\exp\left(i\,\frac{\omega_3' + \Omega}{\cos\theta}\,t\right)$$

$$\omega_3 = \dot{\phi} + \dot{\psi}\cos\theta = \frac{\omega_3' + \Omega}{\cos\theta} - \Omega\cos\theta = \frac{\omega_3' + \Omega\sin^2\theta}{\cos\theta}$$

Direct approach: write Lagrangian in terms of Euler angles

$$L = T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}I_3\omega_3'^2 + \frac{1}{2}I(\omega_1'^2 + \omega_2'^2) \longleftarrow$$ have to use body frame to write this

$$\omega_3' = \dot{\psi} + \dot{\phi}\cos\theta$$

$$\omega_1' + i\omega_2' = \dot{\theta}\,\overbrace{(\cos\psi - i\sin\psi)}^{e^{-i\psi}}$$

$$+\dot{\phi}\sin\theta\,\underbrace{(\sin\psi + i\cos\psi)}_{ie^{-i\psi}}$$

$$= (\dot{\theta} + i\dot{\phi}\sin\theta)\,e^{-i\psi}$$

$$\omega_1'^2 + \omega_2'^2 = |\omega_1' + i\omega_2'|^2 = \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta$$

$$L = \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3' = \underbrace{L_3' = L\cos\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_3\underbrace{(\dot\psi + \dot\phi\cos\theta)}_{\omega_3'}\cos\theta + I\dot\phi\sin^2\theta = L_3 = L$$

$$= I_3\omega_3'\cos\theta + I\dot\phi\sin^2\theta \Rightarrow \theta \text{ is constant}$$

So now the two equations lead to the previous results.

Asymmetric body

$\vec{\omega}$ constant in body $\Rightarrow$ two components of $\vec{\omega}$ are zero

Slightly perturbed

$$\begin{aligned} \omega_3 &= \omega_{30} + \delta\omega_3 \\ \omega_1 &= \delta\omega_1 \\ \omega_2 &= \delta\omega_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} I_1 \delta\dot{\omega}_1 &= \omega_{30}\,\delta\omega_2 (I_2 - I_3) \\ I_2 \delta\dot{\omega}_2 &= \omega_{30}\,\delta\omega_1 (I_3 - I_1) \\ I\,\delta\dot{\omega}_3 &= 0 \end{aligned}$$

$$\delta\dot{\omega}_1 = -\omega_{30} \frac{I_3 - I_2}{I_1} \delta\omega_2$$

$$\delta\dot{\omega}_2 = +\omega_{30} \frac{I_3 - I_1}{I_2} \delta\omega_1$$

$$\Rightarrow \quad \delta\ddot{\omega}_1 = -\underbrace{\omega_{30}^2 \frac{I_3 - I_2}{I_1} \frac{I_3 - I_1}{I_2}}_{\Omega^2} \delta\omega_1$$

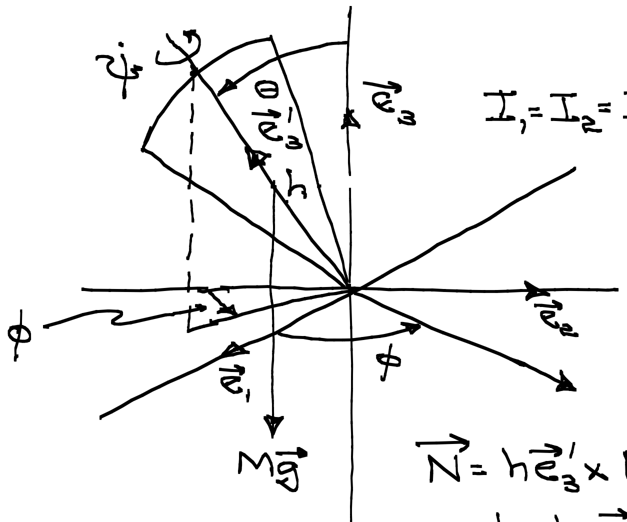
$$\delta\omega_1 = A \cos \Omega t$$

Stability: Ω real $\Leftrightarrow$ $\Omega^2 > 0$ $\qquad$ $I_3 > I_1, I_2$ or $I_3 < I_1, I_2$

Conventional ordering: $I_3 > I_2 > I_1$

$$\begin{array}{ccc} I_3 & I_2 & I_1 \\ \downarrow & \downarrow & \downarrow \\ \text{stable} & \text{unstable} & \text{stable} \end{array}$$

Symmetric top in a gravitational field with tip fixed



Body frame components of $\vec{\omega}$

$$\omega_1' = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\omega_2' = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\omega_3' = \dot\phi \cos\theta + \dot\psi$$

$$\vec{N} = h\vec{e}_3' \times M\vec{g} = -Mgh\,\vec{e}_3' \times \vec{e}_3 = Mgh\sin\theta\, \vec{f}_1$$
$\perp$ to $\vec{e}_3$ and $\vec{e}_3'$

$$T = \frac{1}{2} I (\omega_1'^2 + \omega_2'^2) + \frac{1}{2} I_3 \omega_3'^2 = \frac{1}{2} I (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi\cos\theta + \dot\psi)^2$$
$$V = Mgh\cos\theta$$

$$L = T - V = \frac{1}{2} I (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi\cos\theta + \dot\psi)^2 - Mgh\cos\theta$$

ϕ and ψ are cyclic, so $p_\phi = L_3$ and $p_\psi = L_3'$ are conserved:

$$\begin{aligned} L_3 = p_\phi = \frac{\partial L}{\partial \dot\phi} &= I\dot\phi \sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta \\ &= \dot\phi(I\sin^2\theta + I_3\cos^2\theta) + I_3\dot\psi\cos\theta \\ &= L_3'\cos\theta + I\dot\phi\sin^2\theta \end{aligned}$$

$$L_3' = p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\phi\cos\theta + \dot\psi) = I_3\omega_3'$$

Energy (Jacobi integral) is also conserved:

$$E = T + V = \frac{1}{2} I (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi\cos\theta + \dot\psi)^2 + Mgh\cos\theta$$

With three conserved quantities, the problem is reduced to quadratures:

$$L_3' = I_3\omega_3' = p_\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$
$$L_3 = p_\phi = I\dot\phi\sin^2\theta + L_3'\cos\theta$$
$$E = \frac{1}{2} I (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{L_3'^2}{2I_3} + Mgh\cos\theta$$

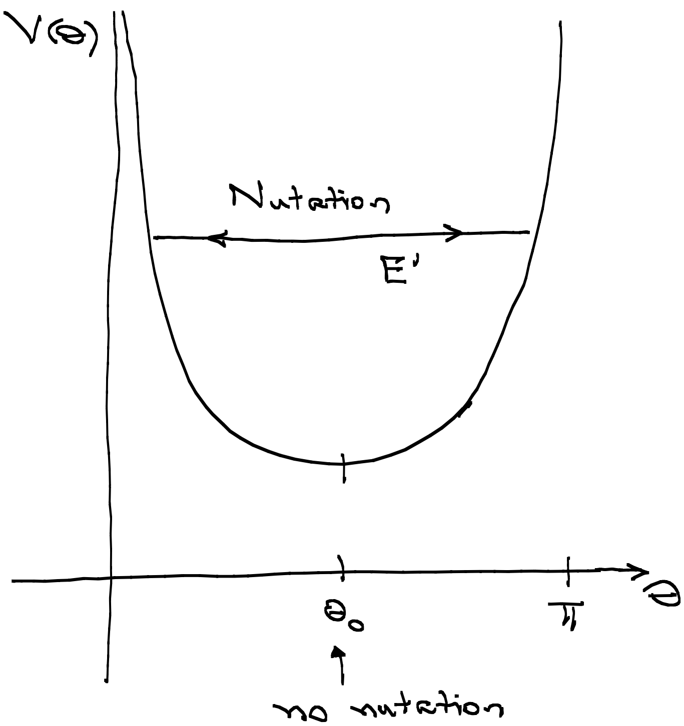
① $$\dot{\phi} = \frac{L_3 - L_3' \cos\theta}{I \sin^2\theta}$$

② $$\dot{\psi} = \frac{L_3'}{I_3} - \dot{\phi}\cos\theta = \frac{L_3'}{I_3} - \frac{L_3 - L_3'\cos\theta}{I\sin^2\theta}\cos\theta$$

③ $$E' = E - \frac{L_3'^2}{2I_3} = \frac{1}{2}I\dot{\theta}^2 + \underbrace{\frac{(L_3 - L_3'\cos\theta)^2}{2I\sin^2\theta} + Mgh\cos\theta}_{V(\theta)}$$

$V(\theta) \leftarrow$ effective potential

Formally, solve ③ for $t(\theta)$, invert to get $\theta(t)$, and plug into ① and ② to get $\phi(t)$ and $\psi(t)$.



If no torque, $L_3 = L$ and

$$I_3\omega_3' = L_3' = L\cos\theta = L_3\cos\theta$$

which implies that θ is a constant, since L_3 and L are constants. So everything reduces to

$$\dot{\phi} = \frac{L}{I} = \frac{I_3\omega_3'}{I\cos\theta} = \frac{\omega_3' + \Omega}{\cos\theta}$$

$$\dot{\psi} = \omega_3' - \frac{I_3\omega_3'}{I} = -\Omega$$

This argument doesn't work when there is torque; L_3 and L_3' are conserved, but they are <u>not</u> related by $L_3' = L_3\cos\theta$.