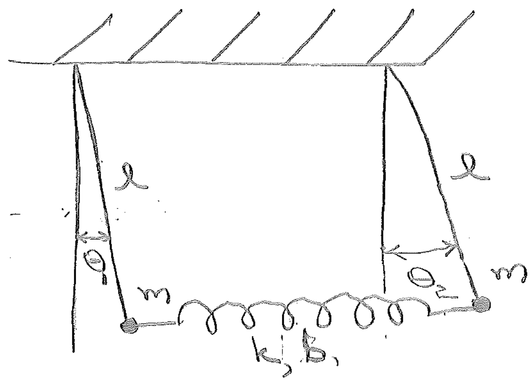
Phys 503

Lectures 14 – 16

Small oscillations

Example: coupled pendulum



$$T = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$V = mgl(1 - \cos\theta_1 + 1 - \cos\theta_2)$$
$$+ \frac{1}{2} k \left(\left[(b + l\sin\theta_2 - l\sin\theta_1)^2 + l^2(\cos\theta_2 - \cos\theta_1)^2 \right]^{1/2} - b \right)^2$$

Small-oscillation approximation

$$V = \frac{1}{2} mgl(\theta_1^2 + \theta_2^2) + \frac{1}{2} k l^2 (\theta_2 - \theta_1)^2$$

$$L = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2} mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2} k l^2 (\theta_2 - \theta_1)^2$$

Equations of motion:

① $\frac{\partial L}{\partial \dot{\theta}_1} = m l^2 \dot{\theta}_1$; $\frac{\partial L}{\partial \theta_1} = -mgl\theta_1 + kl^2(\theta_2 - \theta_1)$

$$\ddot{\theta}_1 + \frac{g}{l}\theta_1 = \frac{k}{m}(\theta_2 - \theta_1)$$

$$\omega^2 = \frac{2k}{m}$$

② $\ddot{\theta}_2 + \frac{g}{l}\theta_2 = -\frac{k}{m}(\theta_2 - \theta_1)$

Normal coordinates: $\mathcal{J}_1 = \frac{1}{2}(\theta_2 + \theta_1)$, $\mathcal{J}_2 = \theta_2 - \theta_1$

$$\theta_1 = \mathcal{J}_1 - \frac{1}{2}\mathcal{J}_2, \quad \theta_2 = \mathcal{J}_1 + \frac{1}{2}\mathcal{J}_2$$

$$\ddot{\mathcal{J}}_1 + \underbrace{\left(\frac{g}{l}\right)}_{\omega_1^2} \mathcal{J}_1 = 0$$

$$\ddot{\mathcal{J}}_2 + \underbrace{\left(\frac{g}{l} + \frac{2k}{m}\right)}_{\omega_2^2} \mathcal{J}_2 = 0$$

What about L?

$$T = \frac{1}{2}\underbrace{(2m)}_{\text{total mass}} l^2 \dot{\mathcal{J}}_1^2 + \frac{1}{2}\underbrace{\left(\frac{m}{2}\right)}_{\text{reduced mass}} l^2 \dot{\mathcal{J}}_2^2$$

$$V = \frac{1}{2}(2m) g l \mathcal{J}_1^2 + \frac{1}{2}\left[\left(\frac{m}{2}\right) g l + k l^2\right] \mathcal{J}_2^2$$

$$L = \frac{1}{2}(2m) l^2 \left(\dot{\mathcal{J}}_1^2 - \underbrace{\left(\frac{g}{l}\right)}_{\omega_1^2} \mathcal{J}_1^2 \right)$$
$$+ \frac{1}{2}\left(\frac{m}{2}\right) l^2 \left(\dot{\mathcal{J}}_2^2 - \underbrace{\left(\frac{g}{l} + \omega^2\right)}_{\omega_2^2} \mathcal{J}_2^2 \right)$$

Do this in general. Decouple Lagrangian into normal coordinates. Works for any <u>linear</u> system (Lagrangian quadratic in coordinates): actual linear systems or linearized about equilibrium.

symmetric ($m_{jk} = m_{kj}$)

$$L = \frac{1}{2} m_{jk}\, \dot{q}_j \dot{q}_k - V \qquad (n \text{ coordinates})$$

no linear in velocity terms

functions of coordinates

Equations of motion:

$$\frac{\partial L}{\partial \dot{q}_j} = m_{jk} \dot{q}_k \; ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = m_{jk} \ddot{q}_k + \frac{\partial m_{jk}}{\partial q_l} \dot{q}_k \dot{q}_l$$

$$\frac{\partial L}{\partial q_j} = \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot{q}_k \dot{q}_l - \frac{\partial V}{\partial q_j}$$

$$\Longrightarrow \quad m_{jk} \ddot{q}_k + \frac{\partial m_{jk}}{\partial q_l} \dot{q}_k \dot{q}_l - \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot{q}_k \dot{q}_l = - \frac{\partial V}{\partial q_j}$$

Equilibrium: $\dot{q}_j = 0 \Rightarrow \dfrac{\partial V}{\partial q_j} = 0$ for all j

Stable vs. unstable

Expand about equilibrium: $q_j = q_{j0} + \eta_j$

$$T = \frac{1}{2} \underbrace{m_{jk}\big|_0}_{\equiv\, T_{jk}} \dot{\eta}_j \dot{\eta}_k = \frac{1}{2} T_{jk} \dot{\eta}_j \dot{\eta}_k$$

$$V = \underbrace{V\big|_0}_{\text{irrelevant constant}} + \cancelto{0}{\frac{\partial V}{\partial q_j}\bigg|_0} \eta_j + \frac{1}{2} \underbrace{\frac{\partial^2 V}{\partial q_j \partial q_k}\bigg|_0}_{\equiv\, V_{jk}} \eta_j \eta_k = \frac{1}{2} V_{jk} \eta_j \eta_k$$

$$\boxed{L = \frac{1}{2} T_{jk} \dot{\eta}_j \dot{\eta}_k - \frac{1}{2} V_{jk} \eta_j \eta_k}$$

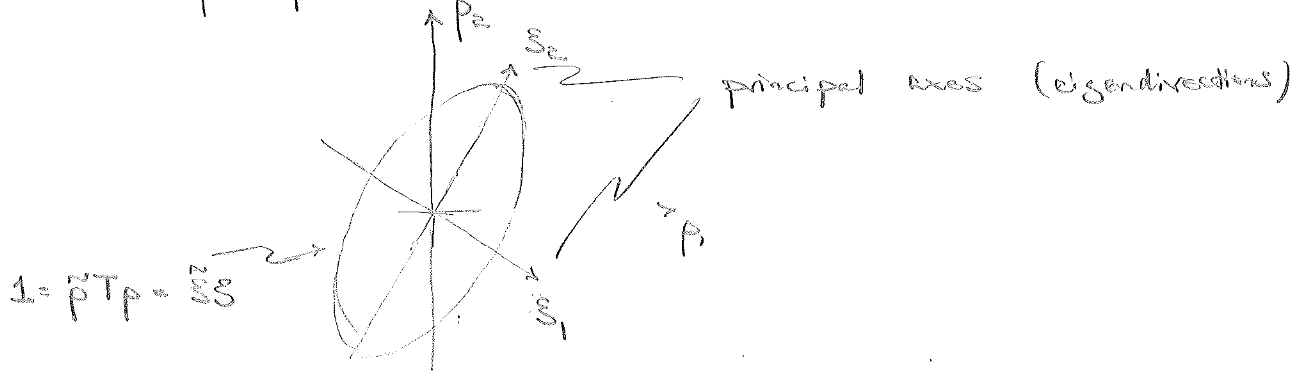
Equations of motion?

Use matrix notation:

$$L = \frac{1}{2}\tilde{\dot{\eta}}\, T\, \dot{\eta} - \frac{1}{2}\tilde{\eta}\, V\, \eta$$

real, symmetric $n \times n$ matrices

Can diagonalize T and V separately by orthogonal transformations, but how both? Because have more freedom than orthogonal transformations. Since $\tilde{p} T p > 0$ $\forall p$ (T is positive-definite), can look at surface $\tilde{p} T p = 1$



But instead of diagonalizing by orthogonal transformation, instead regard $\tilde{p} T p$ as length of p: Ellipsoid is then a sphere. The "natural" coordinates for the sphere are $\xi = T^{1/2}\eta$. Still have the freedom to do orthogonal transformations in new coordinates.

$$\xi = T^{1/2}\eta \Rightarrow \tilde{\dot{\xi}} = \tilde{\dot{\eta}}\, T^{1/2} \Rightarrow \tilde{\dot{\xi}}\dot{\xi} = \tilde{\dot{\eta}}\, T\, \dot{\eta}$$

$$\xi_j = (T^{1/2})_{jk}\eta_k$$

$$\Rightarrow \boxed{L = \frac{1}{2}\tilde{\dot{\xi}}\dot{\xi} - \frac{1}{2}\tilde{\xi}\, \underbrace{T^{-1/2} V T^{-1/2}}_{V'}\, \xi}$$

$$V'_{jk} = (T^{-1/2})_{jl} V_{lm} (T^{-1/2})_{mk}$$

Now diagonalize V': $\quad V' b = \lambda b$

b: eigenvector, λ: eigenvalue

$$\Longleftrightarrow \quad Va = \lambda T a, \qquad b = T^{1/2} a$$

$$b_j = (T^{1/2})_{jk} a_k$$

$$(V' - \lambda 1) b = 0 \quad \Longleftrightarrow \quad (V - \lambda T) a = 0 \leftarrow \boxed{\text{how get this from eqns. of motion}}$$

$\lambda = \omega^2$

Secular equation: $\quad \det(V' - \lambda 1) = 0 \quad \Longleftrightarrow \quad \det(V - \lambda T) = 0$ $\quad$ nth degree polynomial in λ

n eigenvalues λ_l and eigenvectors $b_l = T^{1/2} a_l$

V' is real, symmetric $\Longrightarrow$ λ's real, a's real and orthogonal

If V' Hermitian, $\quad V' b_l = \lambda_l b_l \Leftrightarrow b_l^\dagger V' = \lambda_l^* b_l^\dagger$

$$\left.\begin{aligned} b_l^\dagger V' b_m &= \lambda_l^* b_l^\dagger b_m \\ b_l^\dagger V' b_m &= \lambda_m b_l^\dagger b_m \end{aligned}\right\} \Longrightarrow 0 = (\lambda_l^* - \lambda_m) b_l^\dagger b_m$$

$l = m$: $\quad \lambda_l^* = \lambda_l \Longrightarrow \lambda_l$ is real

$l \neq m$: $\quad$ if $\lambda_l \neq \lambda_m$, then $b_l^\dagger b_m = 0$

Degenerate subspaces (can choose orthogonal)

V' real $\Longrightarrow$ the equations for eigenvectors are linear equations w/ real coefficients $\Longrightarrow$ eigenvectors are real

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$$B = (b_1 \ldots b_n) = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = T^{1/2} A \qquad A = (a_1 \ldots a_n) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = T^{-1/2} B$$

$$b_\ell = \begin{pmatrix} b_{1\ell} \\ \vdots \\ b_{n\ell} \end{pmatrix} \qquad a_\ell = \begin{pmatrix} a_{1\ell} \\ \vdots \\ a_{n\ell} \end{pmatrix}$$

Eigenvalues $\lambda_\ell = \omega_\ell^2$, $\ell = 1, \ldots, n$, are real (sign); eigenvectors b_ℓ can be chosen to be an orthonormal basis:

$$\tilde{b}_\ell b_m = \delta_{\ell m} \iff \tilde{a}_\ell T a_m = \delta_{\ell m}$$

$$b_{j\ell} b_{jm} = \delta_{\ell m} \qquad a_{j\ell} T_{jk} a_{km} = \delta_{\ell m}$$

$$\tilde{B} B = 1 \qquad \tilde{A} T A = 1$$

$\boxed{\text{Note that } A^{-1} = \tilde{A} T}$

$$b_\ell = T^{1/2} a_\ell \iff b_{j\ell} = (T^{1/2})_{jk} a_{k\ell} \iff B = T^{1/2} A$$

$$A = T^{-1/2} B$$

$$V' b_\ell = \lambda_\ell b_\ell \iff V'_{jk} b_{k\ell} = \lambda_\ell b_{j\ell}$$

$$\tilde{b}_\ell V' b_m = \lambda_\ell \delta_{\ell m} \iff b_{j\ell} V'_{jk} b_{km} = \lambda_\ell \delta_{\ell m}$$

$$\iff \tilde{B} V' B = \Lambda = \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_n^2 \end{pmatrix}$$

$$\iff \tilde{A} V A = \Lambda$$

$\boxed{a_\ell \text{ are normal modes}}$

Normal coördinates: $\eta = A \zeta \iff \xi = B \zeta$, $\quad \zeta = \tilde{B} \xi = \tilde{A} T \eta$

$$L = \tfrac{1}{2} \dot{\tilde{\xi}} \dot{\xi} - \tfrac{1}{2} \tilde{\xi} V' \xi = \tfrac{1}{2} \dot{\tilde{\zeta}} \dot{\zeta} - \tfrac{1}{2} \tilde{\zeta} \Lambda \zeta$$

$$= \tfrac{1}{2} \sum_j \left(\dot{\zeta}_j^2 - \omega_j^2 \zeta_j^2 \right)$$

$\boxed{[\zeta_j] = \sqrt{M} L}$

$$\eta = A\zeta \iff \eta_j = a_{jk}\zeta_k = \zeta_k a_{jk}$$

$$\eta_j = \zeta_1 a_{j1} + \zeta_2 a_{j2} + \dots \zeta_n a_{jn}$$

or

$$\boxed{\eta = \zeta_1 a_1 + \zeta_2 a_2 + \dots \zeta_n a_n}$$

$T = ml^2 I \qquad T^{1/2} = \sqrt{m}\,l\, I$

Initial example:

$$\eta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \qquad V = \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl+kl^2\end{pmatrix}$$

$$\xi = T^{1/2}\eta = \sqrt{m}\,l\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \qquad V' = \begin{pmatrix} g/l + k/m & -k/m \\ -k/m & g/l + k/m\end{pmatrix}$$

$$b_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$a_1 = T^{-1/2} b_1 = \frac{1}{\sqrt{2m}\,l}\begin{pmatrix}1\\1\end{pmatrix} \qquad a_2 = T^{-1/2} b_2 = \frac{1}{\sqrt{2m}\,l}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\lambda_1 = \omega_1^2 = g/l \qquad \lambda_2 = \omega_2^2 = g/l + 2k/m$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad A = \frac{1}{\sqrt{2m}\,l}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \tilde{A}T\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \frac{\sqrt{m}\,l}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \frac{\sqrt{m}\,l}{\sqrt{2}}\begin{pmatrix}\theta_1+\theta_2\\ \theta_1-\theta_2\end{pmatrix}$$

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \underline{\zeta_1 a_1 + \zeta_2 a_2 = \frac{1}{2}(\theta_1+\theta_2)\begin{pmatrix}1\\1\end{pmatrix}} + \frac{1}{2}(\theta_1-\theta_2)\begin{pmatrix}1\\-1\end{pmatrix}$$

General solution:

$$\zeta_j(t) = \zeta_j(0)\cos\omega_j t + \frac{\dot{\zeta}_j(0)}{\omega_j}\sin\omega_j t$$

$$\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_n(t) \end{pmatrix} = \cos\Lambda^{1/2}t \begin{pmatrix} \zeta_1(0) \\ \vdots \\ \zeta_n(0) \end{pmatrix} + \Lambda^{-1/2}\sin\Lambda^{1/2}t \begin{pmatrix} \dot{\zeta}_1(0) \\ \vdots \\ \dot{\zeta}_n(0) \end{pmatrix}$$

$$= \left(\cos\Lambda^{1/2}t\right)\zeta(0) + \left(\Lambda^{-1/2}\sin\Lambda^{1/2}t\right)\dot{\zeta}(0)$$

$$= \left(\cos\Lambda^{1/2}t\right)\tilde{A}T\eta(0) + \left(\Lambda^{-1/2}\sin\Lambda^{1/2}t\right)\tilde{A}T\dot{\eta}(0)$$

$$\boxed{\eta(t) = A\zeta(t) = A\left(\cos\Lambda^{1/2}t\right)\tilde{A}T\eta(0) + A\left(\Lambda^{-1/2}\sin\Lambda^{1/2}t\right)\tilde{A}T\dot{\eta}(0)}$$

We can write this formally in a very neat way using $\Lambda = \tilde{B}V'B \Rightarrow \Lambda^{1/2} = \tilde{B}\sqrt{V'}B$

$$\cos\Lambda^{1/2}t = \tilde{B}\left(\cos\sqrt{V'}t\right)B$$

$$\Rightarrow \quad \Lambda^{-1/2}\sin\Lambda^{1/2}t = \tilde{B}\left((\sqrt{V'})^{-1}\sin\sqrt{V'}t\right)B$$

$$A\left(\cos\Lambda^{1/2}t\right)\tilde{A}T = A\tilde{B}\left(\cos\sqrt{V'}t\right)B\tilde{A}T = T^{-1/2}\left(\cos\sqrt{V'}t\right)T^{1/2}$$

$$\Rightarrow \quad A\left(\Lambda^{-1/2}\sin\Lambda^{1/2}t\right)\tilde{A}T = T^{-1/2}(\sqrt{V'})^{-1}\left(\sin\sqrt{V'}t\right)T^{1/2}$$

$$\Rightarrow \boxed{\underbrace{T^{1/2}\eta(t)}_{\xi(t)} = \left(\cos\sqrt{V'}t\right)\underbrace{T^{1/2}\eta(0)}_{\xi(0)} + (\sqrt{V'})^{-1}\sin\left(\sqrt{V'}t\right)\underbrace{T^{1/2}\dot{\eta}(0)}_{\dot{\xi}(0)}}$$

The matrix solution for $\xi(t)$ is exactly what you would write down from the matrix equation for $\xi(t)$: $\ddot{\xi} + V'\xi = 0$