Phys 503

Lectures 17–19

Hamiltonian Dynamics

Raw idea: curves generated by surfaces (motion generated by a scalar function; cf. 1st lectures)

$\vec{x}(\lambda) = x_j(\lambda)\vec{e}_j$; tangent vector $\dfrac{d\vec{x}}{d\lambda} = \dfrac{dx_j}{d\lambda}\vec{e}_j$

$f(\vec{x})$; gradient $\nabla f = \dfrac{\partial f}{\partial x_j}\vec{e}_j$ $\left(df = \dfrac{\partial f}{\partial x_j} dx_j \right)$

$$\frac{d\vec{x}}{d\lambda} = \nabla f \iff \frac{dx_j}{d\lambda} = \frac{\partial f}{\partial x_j} \iff \nabla f = f_{,j}\, dx_j = \dot{x}_j\, dx_j$$

1. Curl-free paths (field lines of conservative field)
2. Picture $\qquad \nabla f = \dfrac{d\vec{x}}{d\lambda} \iff \dot{x}_j = f_{,j}$
3. 1st-order de's $\qquad$ Official: $\delta_{jk}\dot{x}^k = f_{,j} \iff g(\,,\dot{\vec{x}}) = df$

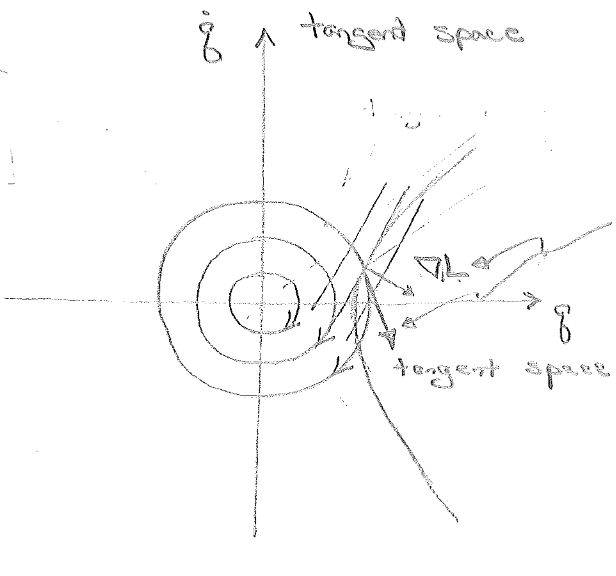
Lagrangian formulation: $L(q_1,\dots,q_n; \dot{q}_1,\dots,\dot{q}_n; t) = L(q_j, \dot{q}_j; t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,\dots,n$$

Example: HO $\quad L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$

Arena: $q_j, \dot{q}_j$

Tangent vector: $\dot{q}_j, \ddot{q}_j$



$$\nabla L = \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial q_j} dq_j = \dot{q}_j\, d\dot{q}_j + \ddot{q}_j\, dq_j$$

$\underbrace{\qquad}_{\text{switch}}$

$\dfrac{\partial L}{\partial \dot{q}_j} = \dot{q}_j \quad \dfrac{\partial L}{\partial q_j} = \ddot{q}_j$

tangent bundle

$$dL = \frac{\partial L}{\partial q_j}\,dq_j + \frac{\partial L}{\partial \dot q_j}\,d\dot q_j + \frac{\partial L}{\partial t}\,dt$$

$$\frac{\partial L}{\partial q_j} = \dot p_j \qquad \frac{\partial L}{\partial \dot q_j} = p_j$$

1-form in the cotangent space to tangent bundle

$$= \dot p_j\, dq_j + \boxed{p_j\, d\dot q_j} + \frac{\partial L}{\partial t}\,dt$$

$p_j dq^j$ is a 1-form in cotangent space to configuration space

dL gives not tangent vector $\dot q_j, \ddot q_j$, but rather p_j and $\dot p_j$ (which are proportional to $\dot q_j$ and $\ddot q_j$ for $L = T - V$ of standard sort, but not for more general systems), so get n 2nd order equations.

↓

Suggests shifting arena to q_j, p_j ← phase space

Legendre transformation: shifts from L, which has natural variables $q_i\ \dot q_j$ to quantity with natural variables q_j, p_j

$$dL = \dot p_j\, dq_j + d(p_j \dot q_j) - \dot q_j\, dp_j + \frac{\partial L}{\partial t}\,dt$$

$$\boxed{\begin{aligned} d(\underbrace{p_j \dot q_j - L}) &= -\dot p_j\, dq_j + \dot q_j\, dp_j - \frac{\partial L}{\partial t}\,dt \\ &\equiv H = H(q_1, \dots, q_n; p_1, \dots, p_n; t) = H(q_j, p_j; t) \end{aligned}}$$

1-form in the cotangent space to the cotangent bundle

$$\boxed{\dot p_j = -\frac{\partial H}{\partial q_j}, \quad \dot q_j = \frac{\partial H}{\partial p_j}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

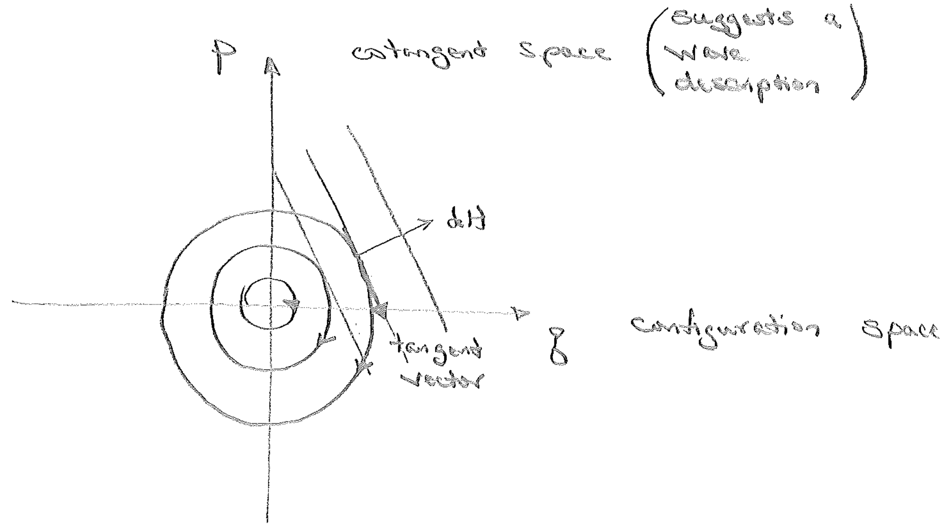
Hamilton's equations

Example: HO $\quad p = \dot{q}, \quad H = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$

Canonical coordinates and momenta

Arena: q_j, p_j

Tangent vector: $\dot{q}_j, \dot{p}_j$



Cotangent bundle
phase space

dH does generate tangent vector to motion in phase space

① 2n 1st order equations:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{is "inverse" of} \quad p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \quad \text{is same equation as} \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

how remember sign (HO)

② H is a function on phase space, i.e., of canonical coordinates. It can generate equations only when written in canonical form. More than just a number (often energy); motion is generated by dH, which is "orthogonal" to energy surface in which motion occurs.

③ Procedure for obtaining H from L

ⓐ $$T = \underbrace{m_0 + m_j \dot{q}_j + \tfrac{1}{2} m_{jk} \dot{q}_j \dot{q}_k}_{\text{fcns. of } q_j}$$

$$V = V(q_j)$$

$$L = T - V$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = m_j + m_{jk}\dot{q}_k \iff \dot{q}_j = (m^{-1})_{jk}(p_k - m_k)$$

$$H = p_j \dot{q}_j - L = \cancel{m_j \dot{q}_j} + m_{jk}\dot{q}_j \dot{q}_k - m_0 - \cancel{m_j \dot{q}_j} - \tfrac{1}{2} m_{jk}\dot{q}_j \dot{q}_k + V$$

almost always a good idea to write in terms of generalized velocities first

$$H = \underbrace{\tfrac{1}{2} m_{jk}\dot{q}_j \dot{q}_k - m_0 + V}_{\text{Is this the Hamiltonian? No!}} \stackrel{?}{=} T + V \quad \begin{pmatrix} m_j = 0 \\ m_0 = 0 \end{pmatrix}$$

$$m_{jk}\dot{q}_j \dot{q}_k = (p_j - m_j)\dot{q}_j = (p_j - m_j)(m^{-1})_{jk}(p_k - m_k)$$

$$\boxed{H = \tfrac{1}{2}(p_j - m_j)(m^{-1})_{jk}(p_k - m_k) - m_0 + V}$$

Good idea to check equations of motion

ⓑ Charged particle in EM field:

$$L = \tfrac{1}{2} m \dot{\vec{x}}^2 - q\,\phi(\vec{x},t) + \frac{q}{c}\dot{\vec{x}} \cdot \vec{A}(\vec{x},t)$$

$$L = \frac{1}{2} m \dot{x}_j \dot{x}_j - q\phi(\vec{x},t) + \frac{q}{c}\dot{x}_j A_j(\vec{x},t)$$

Good idea to check eqs. of motion

$$H = \frac{1}{2m}\left(p_j - \frac{q}{c}A_j\right)\left(p_j - \frac{q}{c}A_j\right) + q\phi = \underbrace{\frac{1}{2}m\dot{x}_j\dot{x}_j}_{T} + \underbrace{q\phi}_{V}$$

Symplectic geometry

Tangent vector $\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{pmatrix} \equiv \dot{\vec{\eta}}$

Gradient of H $\begin{pmatrix} \partial H/\partial q_1 \\ \vdots \\ \partial H/\partial q_n \\ \partial H/\partial p_1 \\ \vdots \\ \partial H/\partial p_n \end{pmatrix} \equiv \nabla_{\vec{\eta}} H = \frac{\partial H}{\partial \vec{\eta}}$

$$\dot{\eta}_J = J_{JK}\frac{\partial H}{\partial \eta_K}$$

$$\dot{\vec{\eta}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\equiv J}\frac{\partial H}{\partial \vec{\eta}}$$

This gives trajectories in terms of ∇, but with J tossed in

$$J^2 = -1, \quad \tilde{J} = -J = J^{-1}$$

$$\boxed{\frac{\partial H}{\partial \vec{\eta}} = -J\dot{\vec{\eta}}}$$

$$\boxed{\frac{\partial H}{\partial \eta_J} = -J_{JK}\dot{\eta}^K \iff dH = \gamma(\ ,\dot{\vec{\eta}})}$$

Canonical 1-form $\tilde{\theta} = p_j\, dq^j$

Canonical 2-form

$$\gamma = d\tilde{\theta} = dp_j \wedge dq^j$$

$$= dp_j \otimes dq^j - dq^j \otimes dp_j$$

$$= \gamma_{JK}\, d\eta^J \otimes d\eta^K$$

$$\|\gamma_{JK}\| = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -J = \tilde{J} = J^{-1}$$

$$\gamma_{JK} = -\gamma_{KJ}$$

Lowering and raising:

$$\langle \lrcorner\vec{V}, \vec{W}\rangle = \gamma(\vec{W},\vec{V}) = \gamma_{JK}W^J V^K$$

$$\Rightarrow (\lrcorner\vec{V})_J = \gamma_{JK}V^K$$

If $\vec{V} = \dot{q}^j \frac{\partial}{\partial q^j} + \dot{p}_j \frac{\partial}{\partial p_j}$

$$\boxed{\lrcorner\vec{V} = -\dot{p}_j\, dq^j + \dot{q}_j\, dp_j = dH}$$

Since $\|\gamma_{JK}\|$ is invertible, can define $\Gamma\tilde{\omega}$ by $\lrcorner\Gamma\tilde{\omega} = \tilde{\omega}$, i.e.,

$$\gamma_{JK}(\Gamma\tilde{\omega})^K = \omega_J .$$

Extending γ to 1-forms,

$$\gamma(\tilde{\tau},\tilde{\omega}) = \gamma(\Gamma\tilde{\omega},\Gamma\tilde{\tau})$$

$$= \langle \tilde{\tau}, \Gamma\tilde{\omega}\rangle$$

$$= \tau_J(\Gamma\tilde{\omega})^J$$

$$= \gamma^{JK}\tau_J\omega_K$$

$$\Rightarrow (\Gamma\tilde{\omega})^J = \gamma^{JK}\omega_K$$

$\Rightarrow \|\gamma^{JK}\| = J$ is the inverse of $\|\gamma_{JK}\| = -J$.

$$\lrcorner\vec{V} = dH \Rightarrow \gamma_{JK}V^K = (dH)_J \iff \vec{V} = \Gamma dH \Rightarrow V^J = \gamma^{JK}(dH)_K$$

Ⓐ

Hamilton's equations: $\lrcorner \vec{v} = dH \iff \gamma(\vec{w}, \vec{v}) = \langle dH, \vec{w} \rangle$

can add a gradient to H

Canonical transformations: Coordinate transformations that leave the form of Hamilton's equations invariant $\iff$ leave the canonical two-form γ in standard form.

$$\gamma = \gamma_{JK}\, dy^J \otimes dy^K = \gamma_{J'K'}\, dy^{J'} \otimes dy^{K'}$$

$$\gamma_{JK} = \underbrace{\frac{\partial y^{J'}}{\partial y^J}}_{M^{J'}{}_J} \frac{\partial y^{K'}}{\partial y^K} \gamma_{J'K'} = M^{J'}{}_J\, M^{K'}{}_K\, \gamma_{J'K'}$$

Canonical coordinates have $\gamma_{JK} = -J_{JK} \Rightarrow \gamma = dp^j \wedge dq^j$

$$J_{JK} = M^{L'}{}_J M^{M'}{}_K J_{LM} \Rightarrow J = \tilde{M} J M$$

$$M^{K'}{}_{M'} J_{JK} = M^{L'}{}_J J_{LM} \Rightarrow J M^{-1} = \tilde{M} J$$

We have $\gamma = dp_j \wedge dq^j = dp_{j'} \wedge dq^{j'}$

$= d(p_j dq^j) = d(p_{j'} dq^{j'})$

$$\Rightarrow p_j dq^j = p_{j'} dq^{j'} + dF$$

generating function

Poisson brackets:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \underbrace{\frac{\partial u}{\partial \eta^J} v^J}_{\langle du, \vec{v}\rangle = \langle du, \Gamma dH\rangle = \gamma(du, dH) \equiv [u, H]} = [u, H] + \frac{\partial u}{\partial t}$$

$$[u, v] \equiv \gamma(du, dv) = \langle du, \Gamma dv\rangle = \gamma^{JK} u_{,J} v_{,K} = \frac{\partial u}{\partial q^j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q^j}$$

Hamilton's equations from Hamilton's principle

$$I = \int dt\, L(q,\dot q,t) = \int dt\,\big(p_j \dot q_j - H(q,p,t)\big)$$ Action integral

$$\boxed{\text{Hamilton's principle: } 0 = \delta I \text{ gives equations of motion}}$$

Fiducial path is described by $q_j(t)$ $(\Rightarrow \dot q_j(t))$ and $p_j(t)$; varied path by $q_j'(t) = q_j(t) + \delta q_j(t)$ $\big(\Rightarrow \dot q_j'(t) = \dot q_j(t) + \delta \dot q_j(t)\big)$ and $p_j'(t) = p_j(t) + \delta p_j(t)$

Remind what these mean (phase-space drawing)

$$0 = \delta \int_{t_1}^{t_2} dt\,\big(p_j \dot q_j - H(q,p,t)\big)$$

$$= \int_{t_1}^{t_2} dt\,\Big(\delta p_j\, \dot q_j + \underbrace{p_j\, \delta \dot q_j}_{\frac{d}{dt}(p_j \delta q_j) - \delta q_j \dot p_j} - \frac{\partial H}{\partial q_j}\delta q_j - \frac{\partial H}{\partial p_j}\delta p_j\Big)$$

$$= \underbrace{\cancel{p_j \delta q_j \Big|_{t_1}^{t_2}}}_{0\ (\text{B.C.'s})} + \int_{t_1}^{t_2} dt\,\Big[-\delta q_j\Big(+\dot p_j + \frac{\partial H}{\partial q_j}\Big) + \delta p_j\Big(\dot q_j - \frac{\partial H}{\partial p_j}\Big)\Big]$$

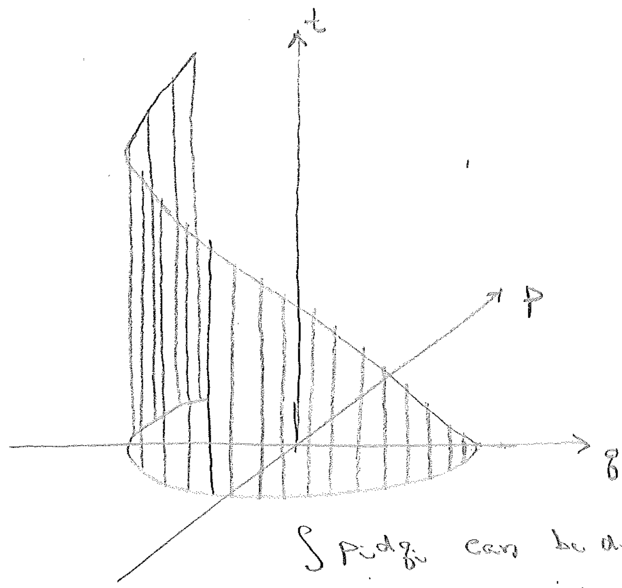
$$\Longrightarrow \boxed{\dot p_j = -\frac{\partial H}{\partial q_j} \quad \text{and} \quad \dot q_j = \frac{\partial H}{\partial p_j}}$$

This tells one the necessary BC's: $\delta q_j(t_1) = \delta q_j(t_2) = 0$

One can always impose additional B.C's on the variation; they restrict the varied paths without changing the equations of motion. If use $\delta p_j(t_1) = \delta p_j(t_2) = 0$, can add $\frac{d}{dt}F(q, p, t)$ to H without changing equations of motion (result generally has no Lagrangian formulation).

Uses of $\int p_j dq_j$ "Action"

$$I = \int dt(p_j \dot{q}_j - H) = \int (p_j dq_j - H\,dt)$$



$\int p_j dq_j$ can be done on phase space independent of parametrization by t

different symbol because $q_j'(\theta)$ and $q_j(\theta)$ may refer to different times

Fiducial path: $q_j(\theta)$, $p_j(\theta)$

Varied path: $q_j'(\theta) = q_j(\theta) + \Delta q_j(\theta)$, $p_j'(\theta) = p_j(\theta) + \Delta p_j(\theta)$

θ an arbitrary parameter; might have nothing to do with t

$$\Delta\left(\int_1^2 p_j dq_j\right) = \Delta\left(\int_{\theta_1}^{\theta_2} d\theta\, p_j \frac{dq_j}{d\theta}\right)$$

$$= \int_{\theta_1}^{\theta_2} d\theta \left(\Delta p_j \frac{dq_j}{d\theta} + p_j \frac{d\Delta q_j(\theta)}{d\theta}\right)$$

$$\Delta\left(\int p_j\, dq_j\right) = p_j \Delta q_j \Big|_1^2 + \int_{\theta_1}^{\theta_2} d\theta \left(\Delta p_j \frac{dq_j}{d\theta} - \Delta q_j \frac{dp_j}{d\theta}\right)$$

Use this in two ways

(1) Variational principle for path on phase space, independent of parametrization by t:

Assume fiducial path comes from a solution of Hamilton's equations (this only makes sense if $\frac{\partial H}{\partial t} = 0 \Rightarrow$ fiducial path lies in a surface of constant H.)

$$\int_{\theta_1}^{\theta_2} d\theta \left(\Delta p_j \frac{dq_j}{d\theta} - \Delta q_j \frac{dp_j}{d\theta}\right)$$

$$= \int_{t_1}^{t_2} dt \left(\Delta p_j \dot{q}_j - \Delta q_j \dot{p}_j\right) \longleftarrow \text{go to } t\text{-parametrization of fiducial path}$$

$$= \int_{t_1}^{t_2} dt \underbrace{\left(\frac{\partial H}{\partial p_j}\Delta p_j + \frac{\partial H}{\partial q_j}\Delta q_j\right)}_{\Delta H} \longleftarrow \text{uses } \partial H/\partial t = 0$$

$$= \int_{\theta_1}^{\theta_2} d\theta \frac{dt}{d\theta} \Delta H(\theta)$$

$$\boxed{\Delta\left(\int p_j\, dq_j\right) = p_j \Delta q_j \Big|_1^2 + \int_{\theta_1}^{\theta_2} d\theta \frac{dt}{d\theta} \Delta H(\theta)}$$

Compare fiducial path only to varied paths that have the same initial and final coordinates $(\Delta q_j(\theta_1) = \Delta q_j(\theta_2) = 0)$ and lie in the same surface of constant H $(\Delta H = 0)$

$$\boxed{\Delta\left(\int p_j \, dq_j\right) = 0}$$

Variational principle for path

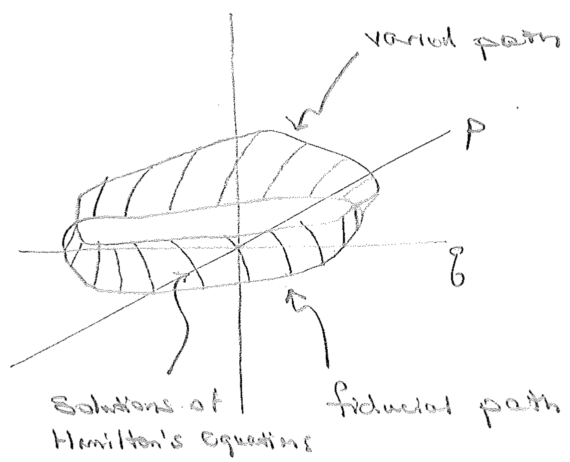
Principle of least (?) action

Example: $L = T - V$, $T = \frac{1}{2} m_{jk} \dot{q}_j \dot{q}_k \Rightarrow p_j = m_{jk} \dot{q}_k$

$\Rightarrow H = T + V$, $p_j \dot{q}_j = 2T$

$$\Delta\left(\int T \, dt\right) = 0 .$$

(2) Conservation of action



(1) Fiducial path is closed

(2) Varied path obtained by pushing fiducial path forward in time by $\Delta t(\theta)$ along solutions of Hamilton's equations

$$\Delta p_j(\theta) = \dot{p}_j \, \Delta t(\theta) = -\frac{\partial H}{\partial q_j} \Delta t(\theta)$$

$$\Delta q_j(\theta) = \dot{q}_j \, \Delta t(\theta) = +\frac{\partial H}{\partial p_j} \Delta t(\theta)$$

$$\Delta\left(\oint p_j\, dq_j\right) = \underbrace{\cancel{p_j \Delta q_j \Big|_1^2}}_{0 \text{ (closed curve)}} + \int_{\theta_1}^{\theta_2} d\theta \underbrace{\left(\Delta p_j \frac{dq_j}{d\theta} - \Delta q_j \frac{dp_j}{d\theta}\right)}$$

$\uparrow$ Closed path

$$= \left(-\frac{\partial H}{\partial q_j}\frac{dq_j}{d\theta} - \frac{\partial H}{\partial p_j}\frac{dp_j}{d\theta}\right)\Delta t(\theta)$$

$$= -\frac{dH}{d\theta}\Delta t(\theta)$$

$\uparrow$ derivative of H along fiducial path in initial constant-t surface

$$\frac{dH}{d\theta} = \frac{\partial H}{\partial q_j}\frac{dq_j}{d\theta} + \frac{\partial H}{\partial p_j}\frac{dp_j}{d\theta} + \frac{\partial H}{\partial t}\cancel{\frac{dt}{d\theta}}$$

Ex: H.O. $\quad q = A\cos(\omega t - \phi)$ $\quad E = \frac{1}{2} m\omega^2 A^2$

$\dot{q} = -A\omega \sin(\omega t - \phi)$

$$\oint p\,dq = \int_0^\tau dt\, p\dot{q} = \int_0^\tau m\dot{q}^2 dt = m\omega^2 A^2 \underbrace{\int_0^\tau dt \sin^2(\omega t - \phi)}_{\frac{1}{2}\tau}$$

$$\oint p\,dq = E\tau = \frac{2\pi E}{\omega} \propto \left(\begin{matrix}\#\text{ of}\\ \text{quanta}\end{matrix}\right)$$

$$\Delta\Big(\underbrace{\oint_C p_j\, dq_j}\Big) = -\oint dH\, \Delta t(\theta) = 0$$

if $\Delta t(\theta) = \Delta t$
or
if H is constant on fiducial path.

$$= \int_S dp_j\, dq_j \quad \text{(Stokes's theorem)}$$

$$\oint_C p_j\, dq_j = \int_S dq_j\, dp_j$$

$$\Uparrow$$

$$\oint_C f(x,y)\underbrace{\vec{e}_x \cdot d\vec{l}}_{dx} = \int_S \underbrace{\nabla\times(f(x,y)\vec{e}_x)\cdot(-\vec{e}_z\, dS)}_{-\frac{\partial f}{\partial y}} = +\int_S \frac{\partial f}{\partial y}\, dx\, dy$$