Lectures 3-4

Hamilton's principle

Hamilton's principle

$$L = L(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t) = T - V$$

ordinary function

for mechanical systems such that all forces, except constraint forces, are derivable from a potential

$$\text{Action} \quad I = I[q_1(t), \dots, q_n(t)] \equiv \int_{t_1}^{t_2} dt\, L$$

Note dimensions

functional

$$\boxed{\text{Actual motion from } t_1 \text{ to } t_2 \text{ is such that the action has a stationary value.}}$$

Math: $$0 = \delta I = \delta \int_{t_1}^{t_2} dt\, L(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)$$

B.C.: q_j fixed at endpoints

$$\delta q_j(t_1) = \delta q_j(t_2) = 0$$

Physics - what does it mean?

more advanced — Variational principle tells us the BC's.

Calculus of variations:

$$q_j(t, \alpha) = q_j(t, 0) + \alpha \eta_j(t), \qquad \text{BC: } \eta_j(t_1) = \eta_j(t_2) = 0$$

$$\dot{q}_j(t, \alpha) = \dot{q}_j(t, 0) + \alpha \dot{\eta}_j(t)$$

$$0 = \underbrace{\frac{dI}{d\alpha}\, d\alpha}_{\delta I} = \int dt \left(\sum_j \frac{\partial L}{\partial q_j} \underbrace{\frac{\partial q_j}{\partial \alpha}\, d\alpha}_{\eta_j} + \sum_j \frac{\partial L}{\partial \dot q_j} \underbrace{\frac{\partial \dot q_j}{\partial \alpha}\, d\alpha}_{\dot\eta_j} \right)$$

Integrate by parts

$$= \int dt \sum_j \underbrace{\eta_j\, d\alpha}_{\delta q_j} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j} \right) \right] + \underbrace{\sum_j \frac{\partial L}{\partial \dot q_j} \underbrace{\eta_j\, d\alpha}_{\delta q_j} \Big|_{t_1}^{t_2}}_{=0}$$

$$= \int dt \sum_j \underbrace{\eta_j\, d\alpha}_{\delta q_j} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j} \right) \right]$$

$$\Longrightarrow \boxed{0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j} \right) - \frac{\partial L}{\partial q_j}}$$

(Euler)- Lagrange equation

Discuss examples: kinetic energies can be tricky
① hoop rolling on cylinder, ② particle sliding on wedge

Holonomic constraints: Good idea to write things in terms of inertial reference frame

Generalized coordinates: q_j, $j = 1, \dots, N$

Constraints: $0 = f_\alpha(q_1, \dots, q_N)$, $\alpha = 1, \dots, A$

$N - A = n$ independent coordinates

Extremize $\int dt \left(\overbrace{L + \sum_\alpha \lambda_\alpha(t) f_\alpha}^{L'} \right)$

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j} \right) - \frac{\partial L}{\partial q_j} = \boxed{+ \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}}, \quad j = 1, \dots, N \\ &f_\alpha(q_1, \dots, q_N) = 0, \quad \alpha = 1, \dots, A \end{aligned}}$$

Generalized unstrained forces: Adding potentials $-\lambda_\alpha f_\alpha$ enforces constraints and these are the forces.

Constraints do no work because $\sum_j \frac{\partial f_\ell}{\partial q_j} \delta q_j = 0$ for any δq_j that lies in the surface $f_\ell = 0$.

Example: Pendulum

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + m g r \cos\theta$$

Constraint $\quad r - \ell = f(r,\theta) = 0$

Extremize $\quad \int dt \, \underbrace{\left(L + \lambda (r - \ell) \right)}_{\tilde{L}}$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{r}}\right) = \frac{\partial \tilde{L}}{\partial r}$$

$$m\ddot{r} = m r \dot{\theta}^2 + m g \cos\theta + \lambda$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\theta}}\right) = \frac{\partial \tilde{L}}{\partial \theta}$$

$$\underbrace{m r^2 \dot{\theta}}_{m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})} = \underbrace{-m g r \sin\theta}_{}$$

Constraint: $r = \ell$

$$\ell \ddot{\theta} = -g \sin\theta$$

$$+\lambda = +\lambda \frac{\partial f}{\partial r} = -m g \cos\theta - m \ell \dot{\theta}^2 = -T$$

What if one uses $\phi = g(r,\theta) = r^2 - \ell^2$

$$\int dt \, \underbrace{\left(L + \lambda (r^2 - \ell^2) \right)}_{\tilde{L}}$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{r}}\right) = \frac{\partial \tilde{L}}{\partial r}$$

$$m\ddot{r} = m r \dot{\theta}^2 + m g \cos\theta + 2r\lambda$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\theta}}\right) = \frac{\partial \tilde{L}}{\partial \theta}$$

$$\underbrace{m r^2 \dot{\theta}}_{m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})} = -m g r \sin\theta$$

Constraint: $r = \ell$

$$\lambda \frac{\partial \phi}{\partial r} = 2r\lambda = -m g \cos\theta - m r \dot{\theta}^2 = -T$$

Conservation laws:

1

$$\frac{dL}{dt} = \sum_j \underbrace{\left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right)} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) = \frac{\partial L}{\partial t}$$

If no explicit time dependence,

$$\boxed{\frac{dh}{dt} = 0, \qquad h \equiv \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = \begin{pmatrix}\text{Jacobi's}\\ \text{integral}\end{pmatrix}}$$

If $L = T - V$ and $T = \sum_{j,k} \frac{1}{2} m_{jk}\dot{q}_j\dot{q}_k$, $\frac{\partial L}{\partial \dot{q}_j} = \sum_k m_{jk}\dot{q}_k$,

$\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = \sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$, so $h = T + V$

2 Generalized Canonical Conjugate momentum $p_j = \frac{\partial L}{\partial \dot{q}_j}$ EM: $p_j = m\dot{x}_j + \frac{q}{c}A_j$

Lagrange equations: $\dot{p}_j = \frac{\partial L}{\partial q_j} = \begin{pmatrix}\text{generalized}\\ \text{force}\end{pmatrix}$?

$$\boxed{\text{If } L \text{ is independent of } q_j \text{ (cyclic)}, \ \dot{p}_j = 0}$$

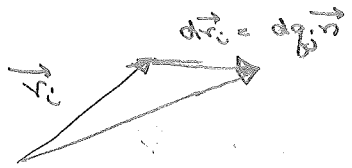
→ Relation to Symmetries

Assume $T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$, $V = V(q_1, \ldots, q_n)$

1. Translations: Suppose q_j corresponds to translating entire system (CM) along $\vec{n}$.

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_i \underbrace{\nabla_{\dot{\vec{r}}_i} T}_{= m_i \dot{\vec{r}}_i} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \vec{n} = \vec{P} \cdot \vec{n}$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} = \vec{n}$$

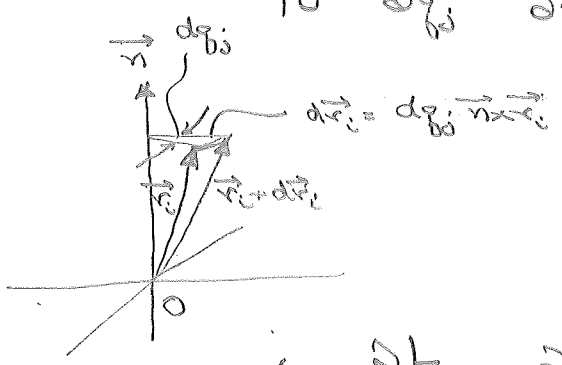


$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j} = \sum_i \underbrace{\left(-\nabla_{\vec{r}_i} V\right)}_{= \vec{F}_i} \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{= \vec{n}} = \vec{F} \cdot \vec{n}$$

T cannot depend on CM position

2. Rotations: Suppose q_j corresponds to rotating entire system about $\vec{n}$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_i \underbrace{\nabla_{\dot{\vec{r}}_i} T}_{= m_i \dot{\vec{r}}_i} \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{= \vec{n} \times \vec{r}_i} = \vec{n} \cdot \sum_i \vec{r}_i \times (m_i \dot{\vec{r}}_i) = \vec{n} \cdot \vec{L}$$



$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j} = \sum_i \underbrace{\left(-\nabla_{\vec{r}_i} V\right)}_{= \vec{F}_i} \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{= \vec{n} \times \vec{r}_i} = \vec{n} \cdot \sum_i \vec{r}_i \times \vec{F}_i = \vec{n} \cdot \vec{N}$$

T cannot depend on orientation

Ask students to define ∇, $\nabla\cdot$, $\nabla\times$ in Cartesian coordinates

(A)

Motion of a charged particle:

Efficient vector calculus:

Right-handed orthonormal Cartesian basis:

$$\vec{e}_1 = \vec{e}_x, \quad \vec{e}_2 = \vec{e}_y, \quad \vec{e}_3 = \vec{e}_z$$

$$\vec{e}_j \cdot \vec{e}_k = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

$$\vec{e}_j \times \vec{e}_k = \sum_l \epsilon_{jkl} \vec{e}_l = \vec{e}_l \epsilon_{ljk}$$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$

$\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$

all others zero

summation convention

totally antisymmetric symbol → consequences of antisymmetry → cyclic property

Vector: $\vec{A} = A_j \vec{e}_j$

Vector algebra: $\vec{A} \cdot \vec{B} = A_j B_k \underbrace{\vec{e}_j \cdot \vec{e}_k}_{\delta_{jk}} = A_j B_j$

$$\vec{A} \times \vec{B} = A_j B_k \underbrace{\vec{e}_j \times \vec{e}_k}_{\epsilon_{jkl} \vec{e}_l} = \vec{e}_l \epsilon_{ljk} A_j B_k = \vec{e}_j \underbrace{\epsilon_{jkl} A_k B_l}_{(\vec{A} \times \vec{B})_j}$$

$$\vec{A} \cdot \vec{B} \times \vec{C} = \epsilon_{jkl} A_j B_k C_l$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = A_j B_k C_l \, \vec{e}_j \times (\underbrace{\vec{e}_k \times \vec{e}_l}_{\vec{e}_m \epsilon_{mkl}})$$

$$= \vec{e}_n \epsilon_{njm} \epsilon_{mkl}$$

$$= \vec{e}_n \epsilon_{mnj} \epsilon_{mkl}$$

$$= \vec{e}_n (\delta_{nk} \delta_{jl} - \delta_{nl} \delta_{jk})$$

$\epsilon_{njk} \epsilon_{nlm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$

$\epsilon_{jkl} \epsilon_{jkm} = 2 \delta_{lm}$

$$= \vec{e}_n (A_j B_n C_j - A_j B_j C_n)$$

$$= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Vector calculus: $\nabla = \vec{e}_j \frac{\partial}{\partial x_j}$

$$\nabla f = \vec{e}_j \frac{\partial f}{\partial x_j} = \vec{e}_j f_{,j}$$

$$\nabla \cdot \vec{A} = \vec{e}_j \frac{\partial}{\partial x_j} \cdot \vec{e}_k A_k = A_{j,j}$$

$$\nabla \times \vec{A} = \vec{e}_j \frac{\partial}{\partial x_j} \times \vec{e}_k A_k = \vec{e}_\ell \epsilon_{\ell jk} A_{k,j} = \vec{e}_j \epsilon_{jk\ell} A_{\ell,k}$$

Charged particle in EM field:

$$\dot{\vec{p}} = m \frac{d^2 \vec{r}}{dt^2} = q\vec{E} + \frac{q}{c} \vec{v} \times \vec{B}$$

$$\vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}$$

$$E_j = -\phi_{,j} - \frac{1}{c} A_{j,t}, \quad B_j = \epsilon_{jk\ell} A_{\ell,k}$$

$$\vec{v} \times \vec{B} = \vec{e}_j \epsilon_{jk\ell} v_k B_\ell$$

$$= \vec{e}_j \underbrace{\epsilon_{jk\ell} \epsilon_{\ell mn}}_{\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}} v_j A_{n,m}$$

$$= \vec{e}_j (v_k A_{k,j} - v_k A_{j,k})$$

$$m \ddot{x}_j = -q\phi_{,j} - \frac{q}{c} A_{j,t} + \frac{q}{c} v_k (A_{k,j} - A_{j,k})$$

Lagrangian: $L = \frac{1}{2} m v^2 - q\phi + \frac{q}{c} \vec{v} \cdot \vec{A}$

$$= \frac{1}{2} m \dot{x}_k \dot{x}_k - q\phi(\vec{r}, t) + \frac{q}{c} \dot{x}_k A_k(\vec{r}, t)$$

$$\frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + \frac{q}{c} A_j(\vec{r},t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = m\ddot{x}_j + \frac{q}{c} A_{j,t} + \frac{q}{c} A_{j,k}\dot{x}_k$$

$$\parallel$$

$$\frac{\partial L}{\partial x_j} = -q\phi_{,j} + \frac{q}{c}\dot{x}_k A_{k,j}$$

$$m\ddot{x}_j = -q\phi_{,j} - \frac{q}{c} A_{j,t} + \frac{q}{c}\dot{x}_k\left(A_{k,j} - A_{j,k}\right)$$