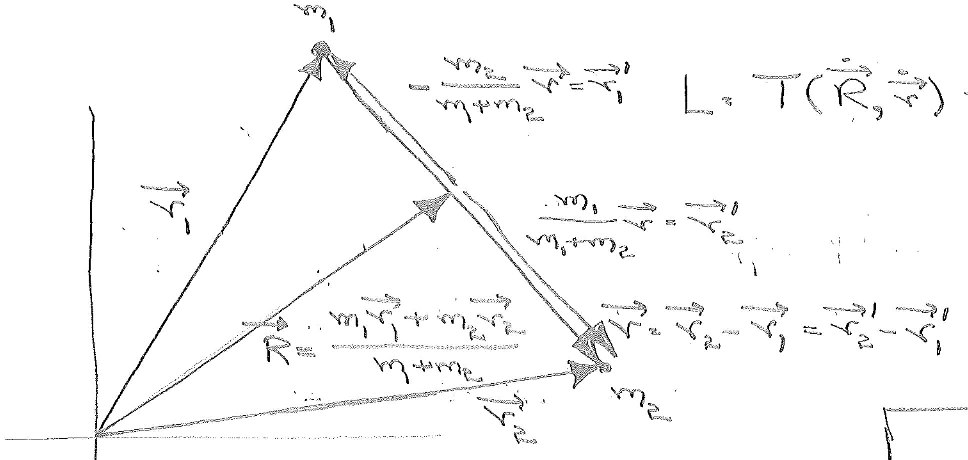
Lectures 5-8

Central-force motion

Reduction to 1-body problem



$$L = T(\dot{\vec{R}}, \dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}})$$

$$\vec{r}_1 = \vec{R} + \vec{r}_1{}' = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \vec{r}_2{}' = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r}$$

$$\boxed{\begin{aligned} \vec{p}_1{}' &= m_1\dot{\vec{r}}_1{}' = -\mu\dot{\vec{r}} = -\vec{p} \\ \vec{p}_1 &= \frac{m_1}{m_1+m_2}\vec{P} + \vec{p}_1{}' = \frac{m_1}{m_1+m_2}\vec{P} - \vec{p} \\ \vec{p}_2{}' &= m_2\dot{\vec{r}}_2{}' = \mu\dot{\vec{r}} = +\vec{p} \\ \vec{p}_2 &= \frac{m_2}{m_1+m_2}\vec{P} + \vec{p}_2{}' = \frac{m_2}{m_1+m_2}\vec{P} + \vec{p} \end{aligned}}$$

$$T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$$ ← refer kinetic energy to inertial frame

$$\dot{\vec{R}}^2 - \frac{2m_2}{m_1+m_2}\dot{\vec{R}}\cdot\dot{\vec{r}} + \left(\frac{m_2}{m_1+m_2}\right)^2\dot{\vec{r}}^{\,2}$$

$$\dot{\vec{R}}^2 + \frac{2m_1}{m_1+m_2}\dot{\vec{R}}\cdot\dot{\vec{r}} + \left(\frac{m_1}{m_1+m_2}\right)^2\dot{\vec{r}}^{\,2}$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\underbrace{\frac{m_1 m_2}{m_1+m_2}}_{\mu = \left(\text{reduced mass}\right)}\dot{\vec{r}}^{\,2} \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$L = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(\vec{r}, \dot{\vec{r}})$$

$$= L_{cm}(\dot{\vec{R}}, \cancel{\vec{R}}) + L_{rel}(\vec{r}, \dot{\vec{r}})$$ ← equations of motion separate

Point transformation

$\vec{R}$ cyclic : (translational invariance)

$$\Rightarrow \quad \vec{P} = \partial L / \partial \dot{\vec{R}} = (m_1 + m_2)\dot{\vec{R}} = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = (\text{total momentum})$$

is conserved $\qquad = \vec{p}_1 + \vec{p}_2$

Center-of-mass motion separates from relative motion, so can forget about it — relative coordinate is not an inertial coordinate.

Central-force motion: $U = V(r)$ $\longleftarrow$ equivalent to single particle of mass m with a center of force at origin

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - V(r)$$

① $\vec{p} = \partial L / \partial \dot{\vec{r}} = m \dot{\vec{r}} = \vec{p}_2' = -\vec{p}_1' = \frac{1}{2}(\vec{p}_2' - \vec{p}_1') = \dfrac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2}$

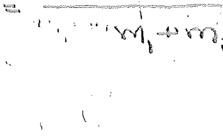
Notice that
$0 = \vec{p}_1' + \vec{p}_2'$
$\downarrow$
zero-momentum frame

② $\vec{L} = \vec{r} \times \vec{p} = (\vec{r}_2' - \vec{r}_1') \times \vec{p} = \vec{r}_2' \times \vec{p}_2' + \vec{r}_1' \times \vec{p}_1' = \begin{pmatrix} \text{angular momentum} \\ \text{about CM} \end{pmatrix}$

$\vec{L}$ is conserved (rotational invariance) (explain why explicitly by writing Lagrangian in spherical coordinates)

$\Longrightarrow$ motion in a plane orthogonal to $\vec{L}$

$$L = \underbrace{\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2}_{} - V(r)$$



(2 conserved components of $\vec{L}$ reduce from 3 to 2 coordinates — 6 integration constants reduced to 4 — 3rd component of $\vec{L}$ still available)

(3) θ cyclic $\Longrightarrow$ $p_\theta = \dfrac{\partial L}{\partial \dot\theta} = mr^2\dot\theta = l =$ constant

Conserved component of $\vec{L}$ orthogonal to plane.

1 integration constant

Kepler's 2nd law: equal areas in equal times

$$dA = \frac{1}{2} r (r\,d\theta) \iff \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{l}{2m} = \text{constant}$$

(4) $p_r = \dfrac{\partial L}{\partial \dot r} = m\dot r = \begin{pmatrix}\text{radial}\\ \text{momentum}\end{pmatrix}$

$$m\ddot r = \frac{\partial L}{\partial r} = \underbrace{mr\dot\theta^2}_{(\text{centrifugal force}) = \frac{l^2}{mr^3}} \overbrace{\boxed{-\frac{\partial V}{\partial r}}}^{f(r)}$$

(5) Conservation of energy:

$$E = T + V = \frac{1}{2} m\dot r^2 + \overbrace{\frac{1}{2} m r^2 \dot\theta^2}^{l^2/2mr^2} + V(r)$$

derive from Jacobi integral

(6) Formal solution:

$$\dot r = \frac{dr}{dt} = \sqrt{\frac{2}{m}}\left(E - V(r) - \frac{l^2}{2mr^2}\right)^{1/2}$$

$$\boxed{t = \int_{r_0}^{r} \frac{dt}{dr}\,dr = \int_{r_0}^{r} dr' \frac{1}{\sqrt{\frac{2}{m}}\left(E - V(r') - \frac{l^2}{2mr'^2}\right)^{1/2}}}$$

$$\boxed{\theta(t) - \theta_0 = \int_0^t \frac{d\theta}{dt}\, dt = \int_0^t dt' \frac{\ell}{mr^2(t')}}$$

Where are 4 integration constants? What do they mean?

Shifting origin of time changes r_0; rotation changes θ_0.

for $r \neq$ there is a special angle

Effective potential; classification of orbits

$$E = \underbrace{\tfrac{1}{2} m\dot{r}^2}_{\substack{\text{radial}\\ \text{kinetic}\\ \text{energy}}} + \underbrace{\frac{\ell^2}{2mr^2}}_{\substack{\text{angular}\\ \text{kinetic energy}\\ \text{or}\\ \text{centrifugal}\\ \text{potential}}} + V(r)$$

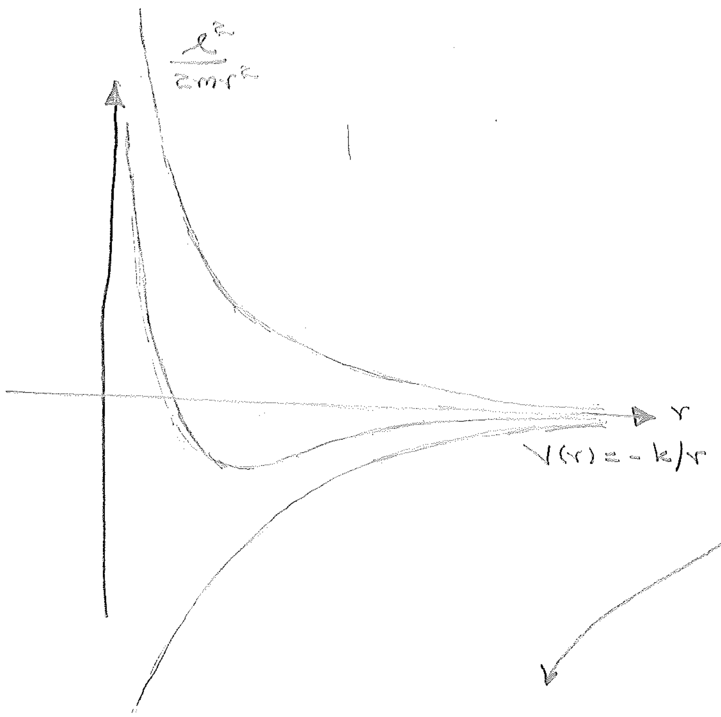
angular kinetic energy $\frac{1}{2}mr^2\dot{\theta}^2$

$$\underbrace{\frac{\ell^2}{2mr^2} + V(r)}_{V'(r) = \binom{\text{effective}}{\text{potential}}}$$

$$f'(r) = -\frac{\partial V'}{\partial r} = \underbrace{+\frac{\ell^2}{mr^3}}_{\text{centrifugal force}} \underbrace{- \frac{\partial V}{\partial r}}_{f(r)}$$

WATCH OUT!

Effective radial Lagrangian $L' = \frac{1}{2}m\dot{r}^2 - V'(r) = \frac{1}{2}m\dot{r}^2 - \frac{\ell^2}{2mr^2} - V(r)$. Notice that this is not what you get by plugging $\dot{\theta} = \ell/mr^2$ into the Lagrangian. When $\frac{1}{2}mr^2\dot{\theta}^2$ goes from being a kinetic term to an effective potential term, its sign must change. Why not? It is important what a Lagrangian is even for paths not taken.



Classification of orbits

4 integration constants?

Motion in θ- direction?

$\boxed{\text{Circular orbits? Stability?}}$

Other examples - maximum

must be attractive

$$0 = -\frac{\partial V'}{\partial r}\bigg|_{r=r_0} = f'(r_0) \Rightarrow f(r_0) = -\frac{\partial V}{\partial r}\bigg|_{r=r_0} = -\frac{\ell^2}{mr_0^3}$$

(r_0 in terms of ℓ)

$$E = V(r_0) + \frac{\ell^2}{2mr_0^2}$$

(E in terms of ℓ)

Power-law force: $f(r) = -\frac{k}{r^{n+1}}$, $V(r) = -\frac{k}{nr^n}$

($k>0$, $k<0$)

$$f(r_0) = -\frac{k}{r_0^{n+1}} = -\frac{\ell^2}{mr_0^3} \Rightarrow r_0^{n-2} = \frac{km}{\ell^2}$$

Key thing: what happens at small and large r?

What about $n=2$?

$n>2$, $0<n<2$, $n=0$, $n<0$

$$E = -\frac{k}{nr_0^n} + \frac{\ell^2}{2mr_0^2} = \frac{1}{r_0^2}\Big(\frac{\ell^2}{2m} - \underbrace{\frac{k}{nr_0^{n-2}}}_{\ell^2/nm}\Big)$$

$$= \frac{\ell^2}{mr_0^2}\left(\frac{1}{2} - \frac{1}{n}\right)$$

Stability: $\left.\dfrac{\partial^2 V'}{\partial r^2}\right|_{r=r_0} = -\left.\dfrac{\partial f'}{\partial r}\right|_{r=r_0} > 0$

$$0 > \left.\frac{\partial f'}{\partial r}\right|_{r=r_0} = \left.\frac{\partial f}{\partial r}\right|_{r=r_0} - \frac{\boxed{3}\,l^2}{m r_0^4} = \left.\frac{\partial f}{\partial r}\right|_{r=r_0} + \frac{3 f(r_0)}{r_0}$$

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} < -\frac{3 f(r_0)}{r_0}$$

Easier to think in terms of $-f$:

$$\left.\frac{\partial(-f)}{\partial r}\right|_{r=r_0} > -3\,\frac{-f(r_0)}{r_0}$$

$$\boxed{\left.\frac{r}{-f}\frac{\partial(-f)}{\partial r}\right|_{r=r_0} = \left.\frac{\partial \ln(-f)}{\partial \ln r}\right|_{r=r_0} > -3}$$

interplay between force and centrifugal force (what does it mean?)

Power-law force: $f = -\dfrac{k}{r^{n+1}} \Longrightarrow \ln(-f) = \ln k - (n+1)\ln r$

Why use logs?

Goldstein (3-43') and power-law stability are wrong

$$\frac{\partial \ln(-f)}{\partial \ln r} = -(n+1) > -3$$

$$\Longrightarrow \boxed{n < 2}$$

$$-f(r_0) = \frac{l^2}{m r_0^3} \Longleftrightarrow \ln(-f(r_0)) = \ln l^2 - \ln m - 3\ln r_0$$

$$\Longrightarrow \frac{d\ln(-f)}{d\ln r_0} + 3 = \frac{d\ln l^2}{d\ln r_0} > 0$$

Stability

$$\Longleftrightarrow \frac{dl^2}{dr_0} > 0$$

Particle in orbit at r_0 w/ l. Give θ-kick changing l by $\Delta l > 0$ and E by $\Delta E > 0$. Kick increases centrifugal force, so particle accelerates outward from $r = r_0$. Kicked orbit can be regarded as perturbation of circular orbit w/ $l + \Delta l$ at $r_0 + \Delta r_0$. Orbit is stable iff $\Delta r_0 > 0 \Rightarrow \Delta l / \Delta r_0 > 0$

Perturbation theory for perturbed orbit: get oscillation frequency as well as stability condition

First get equation for shape of orbit:

$$\frac{d\theta}{dt} = \frac{l}{mr^2} \Longrightarrow \frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$m\ddot{r} = f'(r) = \frac{l^2}{mr^3} + f(r)$$

$$\parallel$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}\right)r = -\frac{l^2}{m}u^2\frac{d^2u}{d\theta^2} \Longrightarrow -\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = f(1/u)$$

$$\boxed{u = 1/r}$$

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta} = -\frac{du}{d\theta}$$

Orbit equation: $\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}f(1/u) \equiv J(u)}$

Circular orbit: $u = u_0 = J(u_0) \iff \boxed{f(r_0) = -\frac{l^2}{mr_0^3}}$

Perturbed orbit: $u = u_0 + x$

$$J(u) = J(u_0) + \left.\frac{dJ}{du}\right|_{u=u_0} x + \frac{1}{2}\left.\frac{d^2J}{du^2}\right|_{u=u_0} x^2 + \ldots$$

Keep only linear term

To go beyond linear order, you need expand u as $u = u_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ and equate terms with same power of ϵ in the resulting equation.

$$\frac{d^2x}{d\theta^2} + u_0 + x = J(u_0) + \left.\frac{dJ}{du}\right|_{u=u_0} x$$

$$\boxed{\frac{d^2x}{d\theta^2} + \underbrace{\left(1 - \left.\frac{dJ}{du}\right|_{u_0}\right)}_{\beta^2} x = 0 \quad \Rightarrow x = a\cos\beta\theta}$$

Stability: $\beta^2 > 0$

$$\frac{dJ}{du} = \underbrace{2\frac{m}{l^2u^3}f(u)}_{-\frac{2}{u}J(u)} - \underbrace{\frac{m}{l^2u^2}}_{-\frac{J(u)}{f(1/u)}} \left.\frac{df}{dr}\right|_{r=1/u}\left(-\frac{1}{u^2}\right)$$

$$\left.\frac{dJ}{du}\right|_{u=u_0} = -2 - \frac{1}{u_0 f(1/u_0)}\left.\frac{df}{dr}\right|_{r=1/u_0}$$

$$= -2 - \frac{r_0}{f(r_0)}\left.\frac{df}{dr}\right|_{r=r_0}$$

$$= -2 - \left.\frac{d\ln(-f)}{d\ln r}\right|_{r=r_0}$$

$$\boxed{\beta^2 = 3 + \left.\frac{d\ln(-f)}{d\ln r}\right|_{r=r_0} = \frac{d\ln l^2}{d\ln r_0} > 0}$$

Textbook argument on closed orbits

$3-(n+1) = 2-n$

$n=1$: 1 $(\beta=1)$

$n=-2$: 4 $(\beta=2)$

Kepler problem: $V = -\dfrac{k}{r}$, $f = -\dfrac{k}{r^2}$

$\dfrac{l^2}{mk} = r_0$ is the radius of the circular orbit with angular momentum l

Orbital shape

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f(1/u) = \frac{mk}{l^2}$$

Solution: $$u = \underbrace{\frac{mk}{l^2}}_{\frac{1}{r_0}}\left(1 + e\cos(\theta - \theta')\right)$$

θ': orientation of orbit (choose $\theta' = 0$, measure θ from periastron)

e: must be related to E and l

(4th constant: initial position in orbit; does not appear in shape)

Find e in terms of E and l:

$$E = \frac{1}{2}m\dot{r}^2 + V'(r) = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

$$\frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$\dot{r} = \frac{l}{mr^2}\frac{dr}{d\theta} = -\frac{l}{m}\frac{du}{d\theta}$$

$$E = \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}u^2 - ku$$

$$= \frac{\cancel{l^2} m^{\cancel{2}} k^2}{2\cancel{m}\, l^{\cancel{4}2}}\left(e^2\sin^2\theta + e^2\cos^2\theta + 2e\cos\theta + 1\right) - \frac{mk^2}{l^2} - \frac{mk^2}{l^2}e\cos\theta$$

$$\boxed{E = \frac{mk^2}{2l^2}\left(e^2 - 1\right)}$$

$$\Longrightarrow \boxed{e = \sqrt{1 + \frac{2l^2E}{mk^2}} = \sqrt{1 - \frac{E}{E_0}}}$$

$E_0 = -\dfrac{k}{r_0} + \dfrac{l^2}{2mr_0^2} \overset{\text{virial theorem}}{=} -\dfrac{mk^2}{2l^2}$ is energy of circular orbit with angular momentum l

Classification of orbits in E or e

Isotropic oscillator: $V(r) = \frac{1}{2}kr^2$, $f(r) = kr \Rightarrow \dfrac{d^2u}{d\theta^2} + u = -\dfrac{m}{l^2u^2}f(1/u) = -\dfrac{mk}{l^2u^3}$

This problem is easy to solve in Cartesian coordinates

$\ddot{x} + \omega^2 x = 0 \Rightarrow x = x_0\cos\omega t$

$\ddot{y} + \omega^2 y = 0 \quad\; y = y_0\sin\omega t$

$\Rightarrow \tan\theta = \dfrac{y}{x} = \dfrac{y_0}{x_0}\tan\omega t$

$r = \sqrt{x^2+y^2} = \left(\dfrac{\cos^2\theta}{x_0^2} + \dfrac{\sin^2\theta}{y_0^2}\right)^{-1/2}$

$E = \frac{1}{2}m\omega^2(x_0^2 + y_0^2)$

$l = m\omega x_0 y_0$

$e=0$: circle
$0<e<1$: ellipse
$e=1$: parabola
$e>1$: hyperbola

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + e\cos\theta\right)$$

Changes sign for repulsive force

Semi-latus rectum (α) $\quad r_0 = \dfrac{\ell^2}{mk} = \begin{pmatrix}\text{radius of circular orbit} \\ \text{with ang mom } \ell\end{pmatrix} = r(\theta = \pi/2)$

eccentricity $e = \sqrt{1 + \dfrac{2\ell^2 E}{mk^2}} = \sqrt{1 - \dfrac{E}{E_0}}$

$E_0 = mk^2/2\ell^2$

Geometric versions of 2 constants of motion

Semimajor axis: $\quad a = \dfrac{r_0}{1-e^2} = -\dfrac{\ell^2/mk}{2\ell^2 E/mk^2} = -\dfrac{k}{2E}$

$$\boxed{E = -\frac{k}{2a}}$$

Kepler's laws:

① Equal areas in equal times

$$dA = \frac{1}{2}r(r\,d\theta) \Longrightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{\ell}{2m}$$

true for any central force

② Elliptical orbits w/ Sun at focus $\longleftarrow -\frac{1}{r}$ potential

③ 1-2-3 rule:

$$\frac{\ell}{2m}\tau = A = \pi ab = \pi a\sqrt{a r_0} = \pi a^{3/2}\frac{\ell}{\sqrt{mk}}$$

$$\sqrt{\frac{k}{m}} = \underbrace{\frac{2\pi}{\tau}}_{\Omega} a^{3/2} \Longrightarrow \boxed{\frac{k}{m} = \Omega^2 a^3} = GM$$

Newton: $k = G m_1 m_2 = GMm$

Coulomb: $k = Ze^2$

What if $k<0$? Solution is still $u = \frac{mk}{l^2}(1 + e\cos\theta)$,

but e must be negative, so that u can be positive.

Redefining e to be positive, we get

$$\boxed{\begin{aligned} \frac{1}{r} &= u = \frac{mk}{l^2}(1 - e\cos\theta) = -\frac{mk}{l^2}(e\cos\theta - 1) \\ e &= \sqrt{1 + \frac{2l^2E}{mk^2}} \end{aligned}}$$

Runge-Lenz vector:

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \dot{\vec{p}}\times\vec{L}$$

$$= f(r)\,\vec{e}_r \times (\vec{r}\times m\dot{\vec{r}})$$

Draw pic

$$= m f(r)\, r\, \underbrace{\vec{e}_r\times(\vec{e}_r\times\dot{\vec{r}})}$$

$$\vec{e}_r(\vec{e}_r\cdot\dot{\vec{r}}) - \dot{\vec{r}} = \underbrace{\vec{e}_r\dot{r} - \dot{\vec{r}}} = -r\left(\frac{\dot{\vec{r}}}{r} - \vec{e}_r\frac{\dot{r}}{r}\right)$$

$-$ (angular part of velocity)

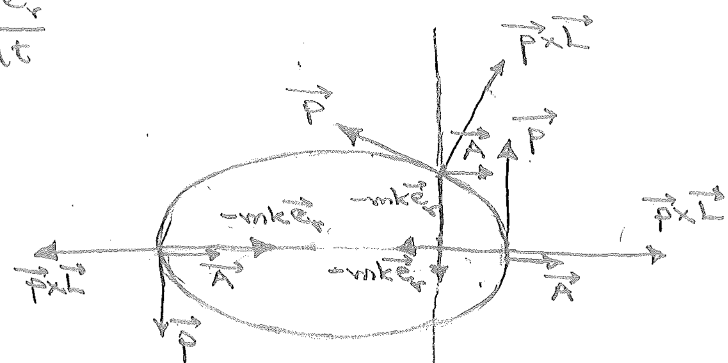
$$= -r\left(\underbrace{\frac{\dot{\vec{r}}}{r} - \vec{e}_r\frac{\dot{r}}{r}}\right)$$

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{d\vec{e}_r}{dt}$$

$$= -r\,\frac{d\vec{e}_r}{dt}$$

$$= -m\underbrace{f(r)\,r^2}_{-k}\frac{d\vec{e}_r}{dt}$$

$$= \frac{d}{dt}(mk\,\vec{e}_r)$$



What if there were precession?

$$\frac{d}{dt}\Big(\underbrace{\vec{p}\times\vec{L} - mk\,\vec{e}_r}_{\vec{A}}\Big) = 0$$

$\vec{p}\times\vec{L}$ lies in plane of orbit, so $\vec{A}$ does, too.

Define $\vec{A}\cdot\vec{r} = Ar\cos\theta = \underbrace{\vec{r}\cdot\vec{p}\times\vec{L}}_{\vec{L}\cdot\vec{r}\times\vec{p} = l^2} - mkr = l^2 - mkr$

$$\Rightarrow\quad r(A\cos\theta + mk) = l^2 \quad\Rightarrow\quad \frac{1}{r} = \frac{mk}{l^2}\Big(1 + \underbrace{\frac{A}{mk}}_{=\,e}\cos\theta\Big)$$

conservation of $\vec{A}$ gives the shape, with $|\vec{A}|$ giving e; still left with direction of $\vec{A}$

$\vec{A}$ gives one additional constant of motion: in essence, the direction towards periastron. This direction can be special only if all orbits close (otherwise there is no unique periastron direction).

Symmetry?

The analogue of the Runge-Lenz vector for a quadratic potential ($f = -m\omega^2 r$, $V = \frac{1}{2} m\omega^2 r^2$) is the tensor

$$\overleftrightarrow{A} = m^2\omega^2 \vec{r} \otimes \vec{r} + \vec{p} \otimes \vec{p}$$

$$\left(\frac{d\overleftrightarrow{A}}{dt} = m\omega^2 (\vec{p} \otimes \vec{r} + \vec{r} \otimes \vec{p}) - m\omega^2 (\vec{p} \otimes \vec{r} + \vec{r} \otimes \vec{p}) \right).$$

In its 6 independent components, $\overleftrightarrow{A}$ contains the 5 constants of motion (2 for plane of orbit, 2 energies (or E and l) in the plane of orbit, and the orthonormal axes that define the major and minor axes of the ellipse).

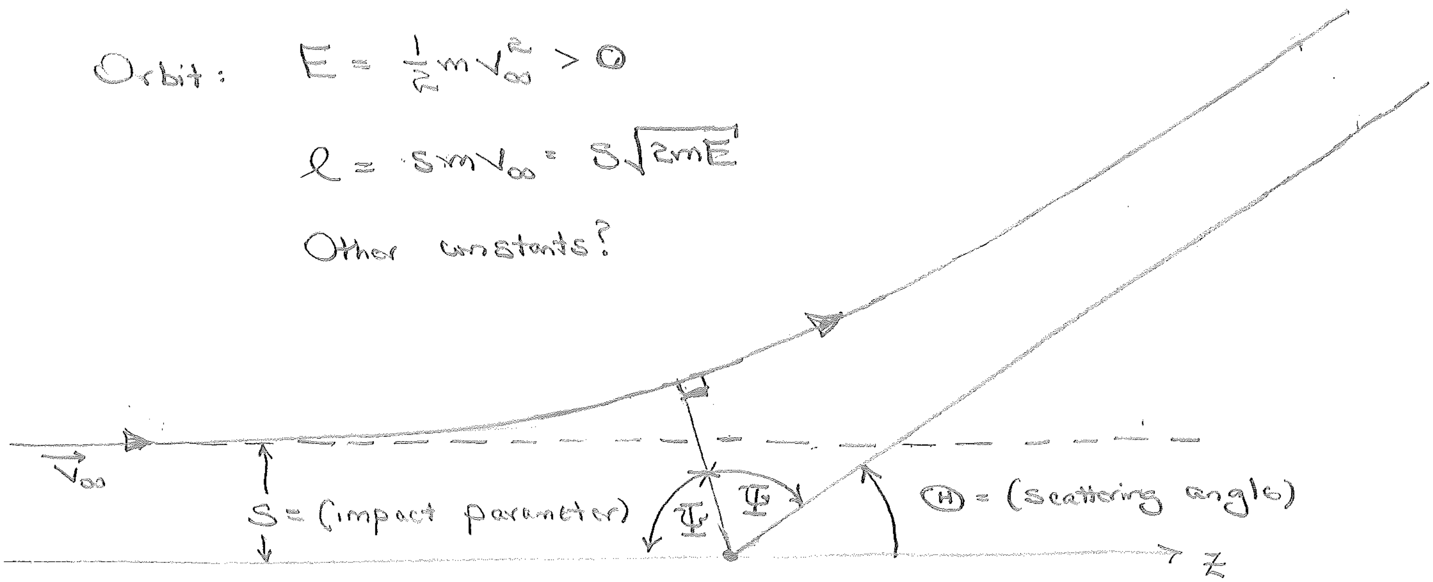
Scattering in a central-force field

Assume $V \to 0$ at $r = \infty$; work in relative coordinates

Orbit: $E = \frac{1}{2} m v_\infty^2 > 0$

$\ell = s m v_\infty = s\sqrt{2mE}$

Other constants?



$\Theta - \pi = 2\Psi$

Find $\Theta(s)$ for fixed E.

Example: Coulomb scattering

$$V = -\frac{k}{r}, \quad f = -\frac{k}{r^2}, \quad k = -ZZ'e^2$$

Trajectory: $$\frac{1}{r} = \frac{mk}{\ell^2}\left(1 + \epsilon\cos(\theta - \theta')\right)$$

$$= \frac{mZZ'e^2}{\ell^2}\left(\epsilon\cos(\theta - \theta') - 1\right), \quad \theta' = \Theta + \Psi$$

$$\epsilon = \sqrt{1 + \frac{2\ell^2 E}{mk^2}} = \sqrt{1 + \left(\frac{2sE}{ZZ'e^2}\right)^2}$$

$r \to \infty$ at $\theta = \pi \Rightarrow \epsilon\cos(\underbrace{\pi - \Theta - \Psi}_{\Psi = \frac{1}{2}(\pi - \Theta)}) = 1$

$$\cos\Psi = \cos\left(\frac{1}{2}(\pi - \Theta)\right) = \sin\frac{\Theta}{2}$$

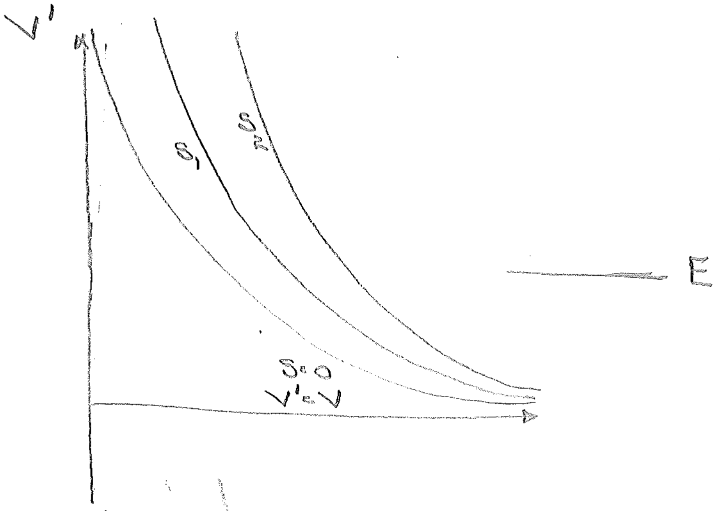
$$\boxed{\sin\frac{\Theta}{2} = \frac{1}{\epsilon} = \frac{1}{\sqrt{1+\left(\frac{2sE}{ZZ'e^2}\right)^2}}}$$

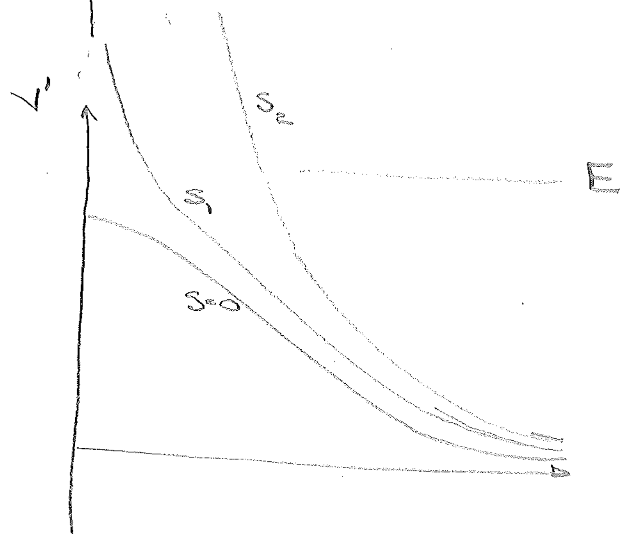
1-1 behavior:

$s=0$: $\Theta = \pi$

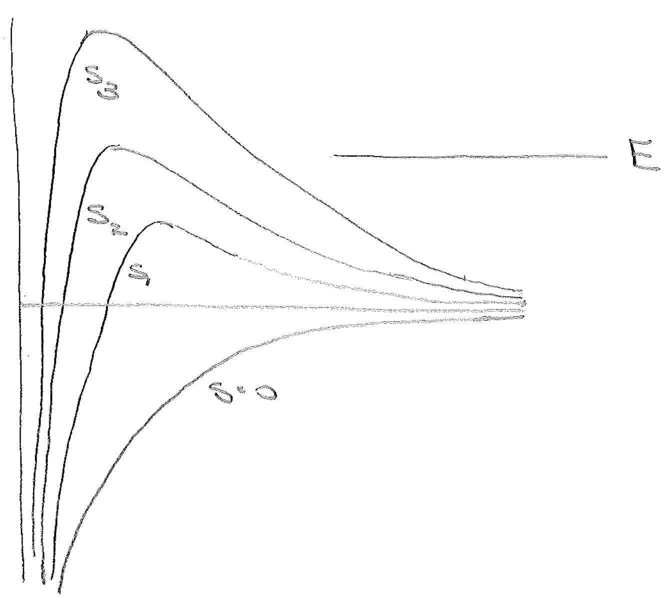
$s\to\infty$: $\Theta \to 0$.

Range of 1 behaviors:





max scattering angle



Differential cross section:

$$\sigma(\Theta,\Phi)\, d\Omega = \frac{(\text{number scattered into } d\Omega \text{ per s})}{I}$$

$\hookrightarrow \sin\Theta\, d\Theta\, d\Phi$

Central force: $\sigma(\Theta,\Phi) = \sigma(\Theta)$, $d\Omega = 2\pi \sin\Theta\, d\Theta$

$$\sigma(\Theta)\, 2\pi \sin\Theta\, |d\Theta| = \sum_i \frac{\cancel{I}\, 2\pi s_i\, |ds_i|}{\cancel{I}}$$

$\sigma(\Theta)$: Scattering angle; not total θ

$|d\Theta|$: Why?

$\sum_i$: all s_i's that give Θ

$$\boxed{\sigma(\Theta) = \sum_i \frac{s_i}{\sin\Theta} \left| \frac{ds_i}{d\Theta} \right|}$$

Hard part: inverting $\Theta(s)$ to get $s(\Theta)$

Total cross section: $\sigma_T = \int \sigma(\Omega)\, d\Omega = 2\pi \int_0^{\pi} d\Theta\, \sin\Theta\, \sigma(\Theta) = 2\pi \int_{\Theta(s) \neq 0} ds\, s$

Example: Coulomb scattering

$$\cot^2(\Theta/2) = \frac{1}{\sin^2(\Theta/2)} - 1 = \epsilon^2 - 1 = \left(\frac{2sE}{ZZ'e^2}\right)^2$$

$$\Longrightarrow s = \frac{ZZ'e^2}{2E} \cot\frac{\Theta}{2}$$

$$\frac{ds}{d\Theta} = \frac{ZZ'e^2}{2E} \frac{1}{2} \underbrace{\left[-1 - \frac{\cos^2(\Theta/2)}{\sin^2(\Theta/2)}\right]}_{-\frac{1}{\sin^2(\Theta/2)}} = -\frac{ZZ'e^2}{4E} \frac{1}{\sin^2(\Theta/2)}$$

$$\sigma(\Theta) = \frac{ZZ'e^2}{2E} \frac{\cot(\Theta/2)}{2\sin(\Theta/2)\cos(\Theta/2)} \frac{ZZ'e^2}{4E} \frac{1}{\sin^2(\Theta/2)}$$

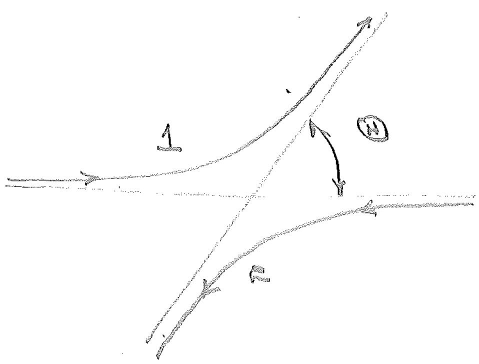
$$\boxed{\sigma(\Theta) = \left(\frac{ZZ'e^2}{4E}\right)^2 \frac{1}{\sin^4(\Theta/2)}}$$

$$\sigma_T \to \infty \quad \left(\begin{array}{l}\text{Blows up at } \Theta = 0, \\ \text{i.e., } s \to \infty\end{array}\right)$$

Relative coordinates are not inertial — neither CM nor lab

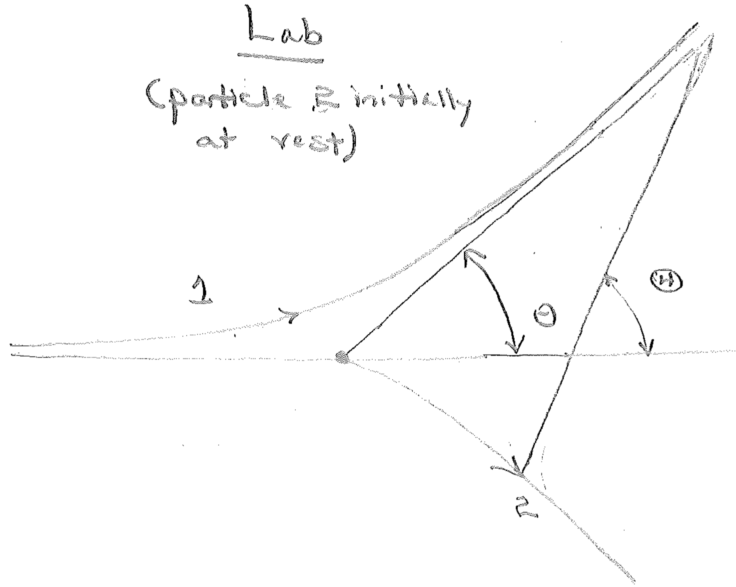
Laboratory description:

CM



Lab

(particle 2 initially at rest)

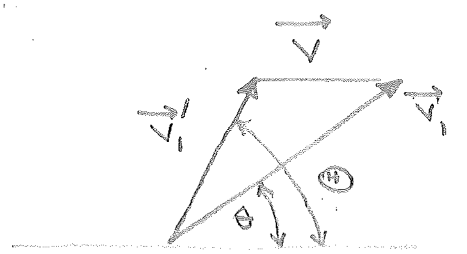


$\vec{R}, \vec{V}$ - CM in lab

$\vec{r}_j, \vec{v}_j$ - particles

before "0" after " "

lab " " CM "'"



CM momentum conservation: $m_1 v_{10}' = m_2 v_{20}'$ and $m_1 v_1' = m_2 v_2'$

$$\Rightarrow \frac{v_1'}{v_{10}'} = \frac{v_2'}{v_{20}'}$$

CM energy conservation: $v_1' = v_{10}'$.

CM: $(m_1 + m_2) V = m_1 v_{10}$

$$v_1 \sin\theta = v_1' \sin\Theta$$

$$v_1 \cos\theta - V = v_1' \cos\Theta$$

$$\Longrightarrow \quad \tan\theta = \frac{v_1' \sin\Theta}{V + v_1' \cos\Theta} = \frac{\sin\Theta}{\cos\Theta + V/v_1'}$$

Use $v_1' = v_{10}' = v_{10} - V$

$$\therefore \quad \frac{v_1'}{V} = \frac{v_{10}}{V} - 1 = \frac{m_1 + m_2}{m_1} - 1 = \frac{m_2}{m_1}$$

$$\boxed{\tan\theta = \frac{\sin\Theta}{\cos\Theta + m_1/m_2}}$$

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = \frac{c^2 + 2(m_1/m_2)c + (m_1/m_2)^2 + s^2}{(c + m_1/m_2)^2}$$

$$\Longrightarrow \quad \boxed{\cos\theta = \frac{\cos\Theta + m_1/m_2}{\sqrt{1 + 2(m_1/m_2)\cos\Theta + (m_1/m_2)^2}}}$$

$m_1 = m_2$

$$\cos\theta = \sqrt{\frac{1 + \cos\Theta}{2}} = \cos(\Theta/2)$$

$$\Longrightarrow \theta = \Theta/2$$

Relation of Cross Sections:

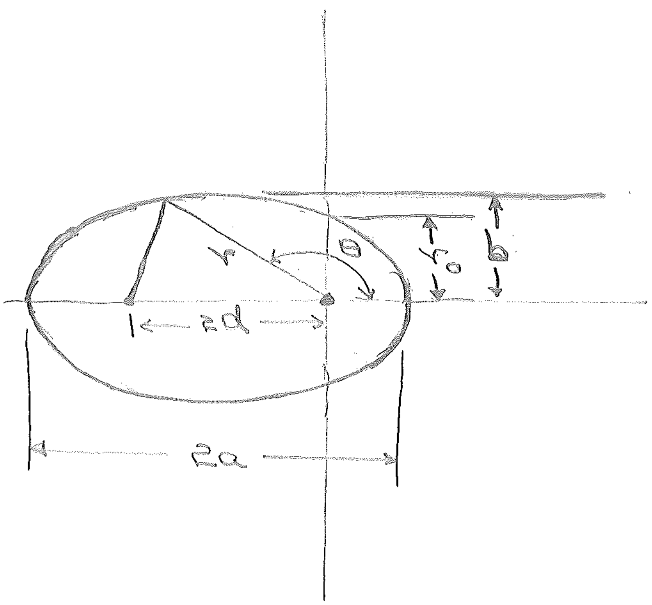
relative-coordinate I - same as in lab

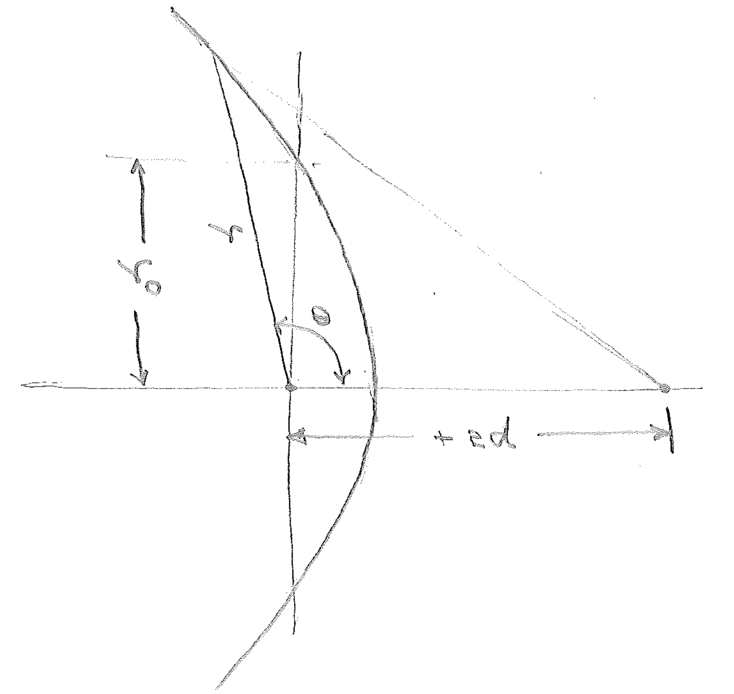
$$I\,\sigma(\Theta)\, 2\pi \sin\Theta\, |d\Theta| = I\,\sigma'(\theta)\, 2\pi \sin\theta\, |d\theta|$$

$$\Longrightarrow \sigma'(\theta) = \sigma(\Theta) \frac{\sin\Theta}{\sin\theta} \left| \frac{d\Theta}{d\theta} \right| = \sigma(\Theta) \left| \frac{d\cos\Theta}{d\cos\theta} \right|$$

$$= \sigma(\Theta) \frac{\left(1 + 2(m_1/m_2)\cos\Theta + (m_1/m_2)^2\right)^{3/2}}{1 + (m_1/m_2)\cos\Theta}$$

Conics





$$\sqrt{r^2+4d^2-2r(2d)\cos(\pi-\theta)} + r = 2a \qquad \sqrt{r^2+4d^2-2r(-2d)\cos\theta} - r = -2a$$

$$r^2+4d^2+4dr\cos\theta = (2a-r)^2 \qquad r^2+4d - 4dr\cos\theta = (2a-r)^2$$

← Change sign of d and a →

$$\cancel{r^2} + \cancel{4}d^2 + \cancel{4}dr\cos\theta = \cancel{r^2} - \cancel{4}ar + \cancel{4}a^2$$

$$r(d\cos\theta + a) = a^2 - d^2$$

$$\boxed{\frac{1}{r} = \frac{a}{a^2-d^2}\left(1 + \frac{d}{a}\cos\theta\right) = \frac{1}{r_0}(1 + e\cos\theta)}$$

$$\boxed{e \equiv \frac{d}{a}}$$

$$\theta = \frac{\pi}{2}: \quad r = \boxed{r_0 \equiv \frac{a^2-d^2}{a} = a(1-e^2) = d\,\frac{1-e^2}{e}}$$

Semimajor axis: $$\boxed{a = \frac{d}{e} = \frac{r_0}{1-e^2}}$$

Seminor axis: $b^2 + d^2 = a^2 \Rightarrow b^2 = a^2 - d^2 = a r_0$

$$\boxed{b = \sqrt{a r_0} = a\sqrt{1-e^2}}$$

$r_{min} = a - d = a(1-e)$	$r_{min} = r(\theta = 0) = \frac{r_0}{1+e} = a(1-e)$
$r_{max} = a + d = a(1+e)$	$r_{max} = r(\theta = \pi) = \frac{r_0}{1-e} = a(1+e)$

there is no $\theta = \pi$ for hyperbola orbits

Circular limit is easy

Parabola: $d, a \to \infty$, e, r_0 finite