Phys 503

Lectures 9-11

Kinematics of rigid-body motion

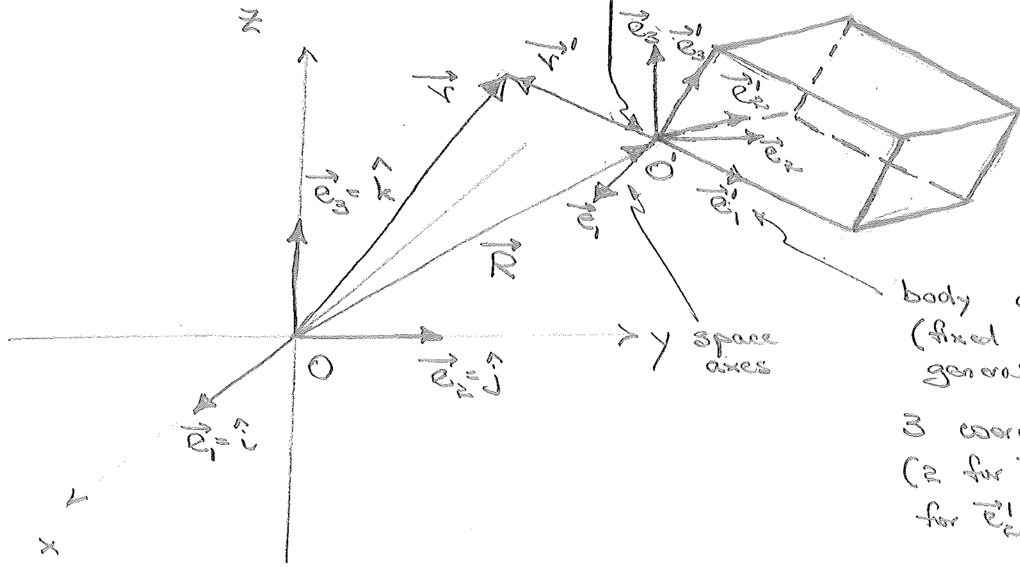
Rigid body: What is it?

Mathematics:

Physics: speed of sound $\rightarrow \infty$

Number of coordinates

Point fixed in body (generally accelerating) – convenient choices: (1) CM or (2) point fixed in inertial frame – 3 coordinates

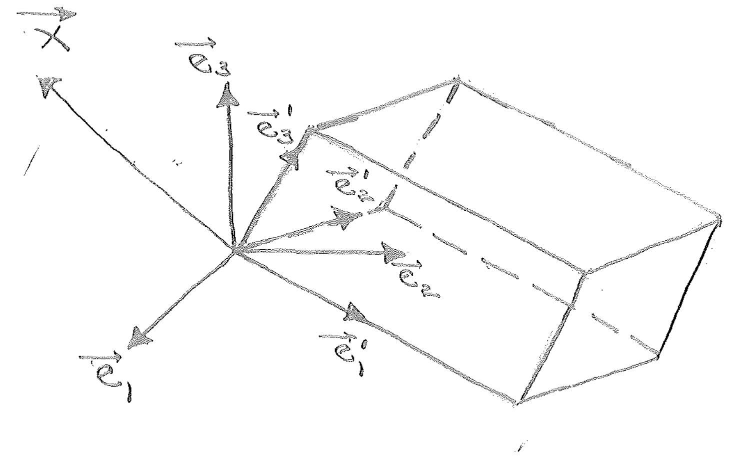


body axes (fixed in body; generally rotating)

3 coordinates (2 for $\vec{e}_1'$, 1 for $\vec{e}_2'$, 0 for $\vec{e}_3'$)

Inertial frame w/ right-handed, orthonormal axes (O in uniform motion – not accelerating – and axes not rotating)

Kinematics vs. dynamics



Relation between 2 sets of orthonormal axes:

$$\vec{e}_j' = A_{jk}\,\vec{e}_k$$

$$A_{jk} = \vec{e}_j' \cdot \vec{e}_k = \begin{pmatrix} k\text{-component} \\ \text{of } \vec{e}_j' \end{pmatrix} = \begin{pmatrix} \text{direction cosine} \\ \text{between } \vec{e}_j' \text{ and } \vec{e}_k \end{pmatrix}$$

Orthonormality $\Rightarrow$ $\delta_{jk} = \vec{e}_j' \cdot \vec{e}_k' = A_{jl} A_{km} \underbrace{\vec{e}_l \cdot \vec{e}_m}_{\delta_{lm}} = A_{jl} A_{kl}$

$$\boxed{\delta_{jk} = A_{jl} A_{kl}}$$

What about right handedness of primed basis vectors?

Matrix notation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

A_{jk} is "matrix element" in row j and column k

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{pmatrix} = A \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

Matrix multiplication

Successive transformations (group property):

$$\vec{e}_j^{\,\prime\prime} = B_{jk}\vec{e}_k^{\,\prime} = \underbrace{B_{jk}A_{k\ell}}_{\substack{(BA)_{j\ell} \\ \uparrow \\ \text{matrix product}}}\vec{e}_\ell \iff \begin{pmatrix}\vec{e}_1^{\,\prime\prime}\\ \vec{e}_2^{\,\prime\prime}\\ \vec{e}_3^{\,\prime\prime}\end{pmatrix} = B\begin{pmatrix}\vec{e}_1^{\,\prime}\\ \vec{e}_2^{\,\prime}\\ \vec{e}_3^{\,\prime}\end{pmatrix} = BA\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}$$

Orthonormality: $\delta_{jk} = A_{j\ell}\underset{\substack{\uparrow \\ \text{transposed matrix}}}{\tilde{A}_{\ell k}} = (A\tilde{A})_{jk}$

$$\underline{1} = A\tilde{A} \iff \tilde{A} = A^{-1}$$

A is an orthogonal matrix or orthogonal transformation

rows of A are components of orthonormal vectors (jth row is k-components of $\vec{e}_j^{\,\prime}$)

$$\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} = A^{-1}\begin{pmatrix}\vec{e}_1^{\,\prime}\\ \vec{e}_2^{\,\prime}\\ \vec{e}_3^{\,\prime}\end{pmatrix} = \tilde{A}\begin{pmatrix}\vec{e}_1^{\,\prime}\\ \vec{e}_2^{\,\prime}\\ \vec{e}_3^{\,\prime}\end{pmatrix} \iff \vec{e}_j = \tilde{A}_{jk}\vec{e}_k^{\,\prime} = \vec{e}_k^{\,\prime}A_{kj}$$

$$\iff (\vec{e}_1\ \vec{e}_2\ \vec{e}_3) = (\vec{e}_1^{\,\prime}\ \vec{e}_2^{\,\prime}\ \vec{e}_3^{\,\prime})\,A$$

(another matrix notation)

A different route: $\vec{e}_j = (A^{-1})_{jk}\vec{e}_k^{\,\prime}$, where $(A^{-1})_{jk} = \vec{e}_j \cdot \vec{e}_k^{\,\prime} = A_{kj}$, (i.e., $A^{-1} = \tilde{A}$. Hence A is an orthogonal matrix. Notice that $\delta_{jk} = (A^{-1})_{j\ell}(A^{-1})_{\ell k} = A_{\ell j}A_{\ell k}$, i.e., columns of A are components of orthonormal vectors. $\boxed{\text{jth column is k-components of } \vec{e}_j}$.

Summarize:

$$\vec{e}_j^{\,\prime} = A_{jk}\vec{e}_k$$
$$\vec{e}_j = \vec{e}_k^{\,\prime}A_{kj}$$

<u>No</u> <u>confusion</u> <u>about</u> <u>notation</u>!

How does a vector transform?

Here's where the confusion comes in. Must understand, not memorize.

<u>Passive</u> <u>transformation</u>: $\vec{x} = x_j \vec{e}_j = x'_j \vec{e}'_j$ is a fixed vector, but its components change.

$$x'_j \vec{e}'_j = x_k \vec{e}_k = A_{jk} x_k \vec{e}'_j \longleftarrow$$ Write primed components in two ways

$$\Longrightarrow \quad x'_j = A_{jk} x_k \quad \Longleftrightarrow \quad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Equivalent derivation

$$\underset{x'_j}{\underset{\parallel}{\vec{x}\cdot\vec{e}'_j}} = A_{jk} \underset{x_k}{\underset{\parallel}{\vec{x}\cdot\vec{e}_k}}$$

Shorthand (Textbook)

$$\vec{x}' = A \vec{x}$$

operator on the components of a fixed vector

For example, $\vec{x} = \vec{e}_1$

$$\begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\vec{e}_1 = \vec{e}'_j A_{j1}$$

DANGER!! There is no <u>other</u> vector $\vec{x}'$.

<u>Active</u> <u>transformation</u>: Vector $\vec{x} = x_j \vec{e}_j = x'_j \vec{e}'_j$ rotates <u>with</u> <u>the</u> <u>axes</u> to vector $\vec{y} = y_j \vec{e}_j = y'_j \vec{e}'_j$, i.e., $\boxed{y'_j = x_j}$.

What if called this $\vec{x}'$?

$$\vec{y} = y'_j \vec{e}'_j = x_j \underbrace{A_{jk} \vec{e}_k}_{} = y_k \vec{e}_k \longleftarrow$$ Write components in unprimed basis in 2 ways.

$$y_k = x_j A_{jk} = \tilde{A}_{kj} x_j \Longleftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \tilde{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Equivalent derivation: $\underset{y_k}{\underset{\parallel}{\vec{y}\cdot\vec{e}_k}} = A_{jk} \underset{y'_j = x_j}{\underset{\parallel}{\vec{y}\cdot\vec{e}'_j}}$

or $(y_1 \; y_2 \; y_3) = (x_1 \; x_2 \; x_3) A$

Shorthand
(Textbook)

$$\vec{y} = \tilde{A}\,\vec{x}$$

$$\tilde{A} = A^{-1}$$

$\tilde{A}$: operator that rotates vectors

$\vec{x}'$?

For example,

$\vec{x} = \vec{e}_1$ and $\vec{y} = \vec{e}_1'$

$$\vec{e}_1' = \tilde{A}\,\vec{e}_1$$

$$\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} = \tilde{A} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Review

Proper vs. improper transformations

Right-handed basis: $\vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3 = +1$, $\vec{e}_j \cdot \vec{e}_k \times \vec{e}_l = \epsilon_{jkl}$

$$\vec{e}_1' \cdot \vec{e}_2' \times \vec{e}_3' = A_{1j} A_{2k} A_{3l} \underbrace{\vec{e}_j \cdot \vec{e}_k \times \vec{e}_l}_{\epsilon_{jkl}}$$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$= \det A = \begin{cases} +1, & \text{proper orthogonal transformation} \\ -1, & \text{improper orthogonal transformation} \end{cases}$$

$$1 = A\tilde{A} \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$$

Review:

$$\vec{e}_j{}' = A_{jk}\vec{e}_k \iff \vec{e}_j = \tilde{A}_{jk}\vec{e}_k{}' = A_{kj}\vec{e}_k{}'$$

$$A_{jl}A_{kl} = \delta_{jk} \iff A\tilde{A} = 1$$

Passive transformation:

$$x_j' = \vec{x}\cdot\vec{e}_j{}' = A_{jk}\,\vec{x}\cdot\vec{e}_k = A_{jk}x_k$$

$$\iff \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

What's the inverse?

Shorthand: $\vec{x}\,' = A\vec{x}$ ← matrix acts on 3-vector of components

Active transformation:

$$y_j = \vec{y}\cdot\vec{e}_j = A_{kj}\,\vec{y}\cdot\vec{e}_k{}' = A_{kj}x_k$$

$$y_k' = x_k$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \tilde{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

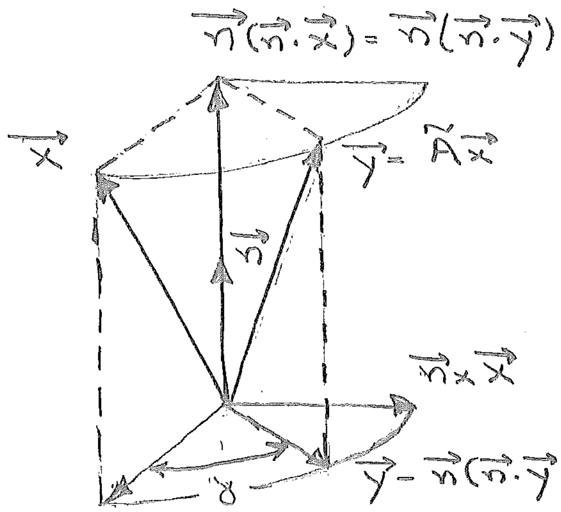
Shorthand: $\vec{y} = \tilde{A}\vec{x}$ ← matrix acts on 3-vector of components or directly on vectors

Notice the way the group property works

$$\tilde{A}_2\tilde{A}_1\vec{x} = \widetilde{(A_1A_2)}\,\vec{x}$$

Textbook uses $\vec{y} = A\vec{x}$, preferring to remember that rotation is in opposite direction from passive transformation.

General rotation



Right-handed rotation by γ about $\vec{n}$

[Textbook uses $\Phi = -\gamma$ and $\vec{y} = A\vec{x}$.]

$$\vec{y} - \vec{n}(\vec{n}\cdot\vec{y}) = -\vec{n}\times(\vec{n}\times\vec{x})\cos\gamma + \vec{n}\times\vec{x}\sin\gamma$$

$$\vec{x} - \vec{n}(\vec{n}\cdot\vec{x}) = -\vec{n}\times(\vec{n}\times\vec{x})$$

$$\vec{y} = \vec{n}(\vec{n}\cdot\vec{x}) - \vec{n}\times(\vec{n}\times\vec{x})\cos\gamma + \vec{n}\times\vec{x}\sin\gamma$$

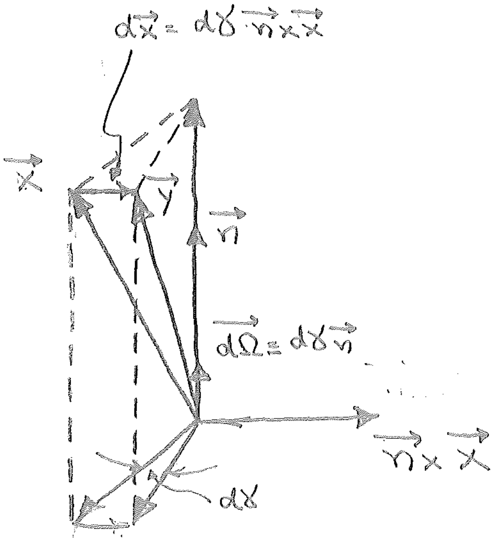
$$= \vec{x}\cos\gamma + (1-\cos\gamma)\vec{n}(\vec{n}\cdot\vec{x}) + \vec{n}\times\vec{x}\sin\gamma = \tilde{A}\vec{x}$$

$$y_j = \underbrace{\left[\cos\gamma\,\delta_{jk} + (1-\cos\gamma)n_j n_k + \sin\gamma\,\epsilon_{jlk} n_l\right]}_{\tilde{A}_{jk}} x_k$$

$$\boxed{A_{jk} = \cos\gamma\,\delta_{jk} + (1-\cos\gamma)n_j n_k + \sin\gamma\,\epsilon_{jkl} n_l}$$

Infinitesimal rotations $\longrightarrow$ aiming toward angular velocity

Finite rotations do not commute, but infinitesimal rotations do. Finite can be built up from infinitesimal; commutators of the infinitesimal generators express non-commutativity of finite rotations.



$|\vec{x}| \sin\theta\, d\gamma = |\vec{n} \times \vec{x}|\, d\gamma$

$$\vec{y} = \vec{x} + d\vec{x}$$

specialization of finite formula to

$$\boxed{d\vec{x} = d\gamma\, \vec{n} \times \vec{x} = d\vec{\Omega} \times \vec{x} = -\vec{x} \times d\vec{\Omega}}$$

$$y_j = x_j + dx_j = \underbrace{(\delta_{jk} - \epsilon_{jkl}\, d\Omega_l)}_{\tilde{A}_{jk} = A_{kj}} x_k$$

$$\Longrightarrow \boxed{\begin{aligned} A_{jk} &= \delta_{jk} + \epsilon_{jkl}\, d\Omega_l \\ &= \delta_{jk} + d\gamma\, \epsilon_{jkl} n_l \\ &\equiv \delta_{jk} + \epsilon_{jk} \end{aligned}}$$

$$A = 1 + \epsilon \qquad \epsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \longleftarrow \text{2-tensor}$$

This is the product of separate rotations about the 3 coordinate axes, which shows that infinitesimal rotations commute.

$$\boxed{\text{Angular velocity } \vec{\omega} = \frac{d\vec{\Omega}}{dt} \neq \frac{d}{dt}\vec{\Omega}}$$

① Example of 90° rotations about ⊥ axes.

② Contrast translations, which commute. Can use vector positions; differentiate & get velocity and acceleration.

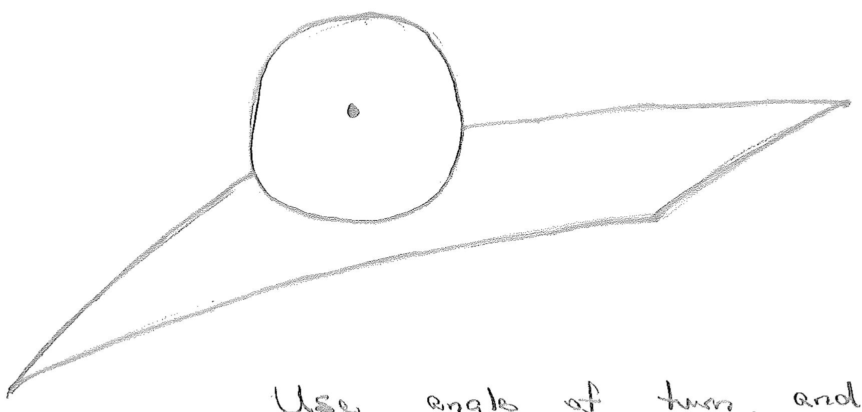
③ Can't use vectors to describe orientation. Angular coordinates are not components of vectors, but their derivatives can be used to construct an angular-velocity vector.

How to find $\vec{\omega}$ in terms of angular coördinates.
First step in any rigid body problem.

1. Adopt body principal axes and Euler angles as generalized coördinates. Write $\vec{\omega}$ in terms of Euler angles and their derivatives.

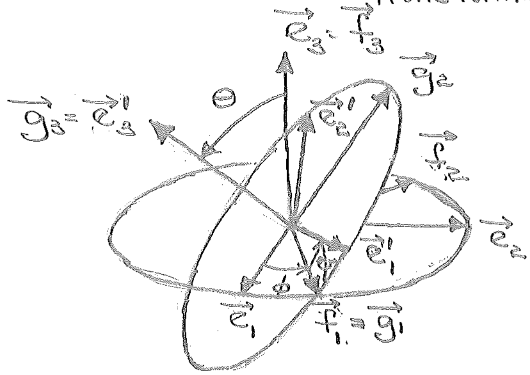
But, often you don't want to use Euler angles as coördinates. For example, when a coin rolls without slipping on a curved surface; one wants coördinates matched to rolling constraint.

2. Adopt body principal axes $\vec{e}_j'$. Write small changes $d\vec{e}_j' = \vec{d\Omega} \times \vec{e}_j'$ in terms of chosen coördinates, and read off $\vec{d\Omega}$. Then $\vec{\omega} = \vec{d\Omega}/dt$.



Use angle of turn, and 2 coördinates for normal vector.

<u>Euler</u> <u>angles</u>: A particular choice of 3 coordinates to specify the orientation of body axes in terms of space axes — i.e., to specify orthogonal transformation A. (assumes <u>proper</u> orthogonal transformation)



Use Levitron as example.

$$\begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

$$\begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}}_{C} \begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_3 \end{pmatrix}$$

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{pmatrix} = \underbrace{BCD}_{A} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

explicit form in text

3.4. Goldstein 4.19.

$\vec{\omega}$ in terms of Euler angles:

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$$

infinitesimal rotations add

$$\vec{\omega}_\phi = \dot{\phi}\vec{e}_3, \quad \vec{\omega}_\theta = \dot{\theta}\vec{f}_1, \quad \vec{\omega}_\psi = \dot{\psi}\vec{e}_3{}'$$

$$\vec{\omega}_\phi = \dot{\phi}\vec{e}_3 = \dot{\phi}\vec{f}_3 = \dot{\phi}(\sin\theta\,\vec{g}_2 + \cos\theta\,\vec{g}_3)$$

$$= \dot{\phi}\left[\sin\theta(\sin\psi\,\vec{e}_1{}' + \cos\psi\,\vec{e}_2{}') + \cos\theta\,\vec{e}_3{}'\right]$$

$$= \dot{\phi}\sin\theta\sin\psi\,\vec{e}_1{}' + \dot{\phi}\sin\theta\cos\psi\,\vec{e}_2{}' + \dot{\phi}\cos\theta\,\vec{e}_3{}'$$

$$\vec{\omega}_\theta = \dot{\theta}\vec{f}_1 = \dot{\theta}\cos\phi\,\vec{e}_1 + \dot{\theta}\sin\phi\,\vec{e}_2$$

$$= \dot{\theta}\vec{g}_1 = \dot{\theta}\cos\psi\,\vec{e}_1{}' - \dot{\theta}\sin\psi\,\vec{e}_2{}'$$

$$\vec{\omega}_\psi = \dot{\psi}\vec{e}_3{}' = \dot{\psi}\vec{g}_3 = \dot{\psi}(-\sin\theta\,\vec{f}_2 + \cos\theta\,\vec{f}_3)$$

$$= \dot{\psi}\left[-\sin\theta(-\sin\phi\,\vec{e}_1 + \cos\phi\,\vec{e}_2) + \cos\theta\,\vec{e}_3\right]$$

$$= \dot{\psi}\sin\theta\sin\phi\,\vec{e}_1 - \dot{\psi}\sin\theta\cos\phi\,\vec{e}_2 + \dot{\psi}\cos\theta\,\vec{e}_3$$

$$\boxed{\begin{aligned}\vec{\omega} &= (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\vec{e}_1 + (\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)\vec{e}_2 + (\dot{\phi} + \dot{\psi}\cos\theta)\vec{e}_3 \\ &= (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\vec{e}_1{}' + (-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\vec{e}_2{}' + (\dot{\psi} + \dot{\phi}\cos\theta)\vec{e}_3{}'\end{aligned}}$$

Note primed & unprimed are interchanged under $\theta \leftrightarrow -\theta$, $\phi \leftrightarrow -\psi$, and $\frac{d}{dt} \leftrightarrow -\frac{d}{dt}$.

Time derivative of a vector

Draw picture

$$\vec{G}(t) = \underbrace{G_j(t)}_{\text{"inertial" coordinates}}\ \underbrace{\vec{e}_j}_{\text{space axes}} = \underbrace{G_j'(t)}_{\text{"noninertial" coordinates}}\ \underbrace{\vec{e}_j{}'(t)}_{\text{rotating body axes}}$$

$$\frac{d\vec{G}}{dt} = \frac{dG_j(t)}{dt}\vec{e}_j \equiv \left(\frac{d\vec{G}}{dt}\right)_{\text{space}}$$

$$= \underbrace{\frac{dG_j'(t)}{dt}\vec{e}_j{}'(t)}_{\equiv \left(\frac{d\vec{G}}{dt}\right)_{\text{body or rotating}}} + G_j'(t)\frac{d\vec{e}_j{}'}{dt}$$

(1) $\vec{e}_j{}'(t+dt) = A_{jk}\,\vec{e}_k{}'(t) = (\delta_{jk} + \epsilon_{jk\ell}\, d\Omega_\ell')\,\vec{e}_k{}'(t)$

infinitesimal rotation by $d\gamma$ about $\vec{n} = n_j'\vec{e}_j{}'(t)$;
infinitesimal rotation vector is $\vec{d\Omega} = d\gamma\,\vec{n} = d\Omega_j'\,\vec{e}_j{}'(t)$

$$\therefore\ d\vec{e}_j{}' = \vec{e}_j{}'(t+dt) - \vec{e}_j{}'(t) = \epsilon_{jk\ell}\, d\Omega_\ell'\,\vec{e}_k{}'(t)$$

$$\Longleftrightarrow \boxed{d\vec{e}_j{}' = \vec{e}_k{}'\,\epsilon_{k\ell j}\, d\Omega_\ell' = \vec{d\Omega}\times\vec{e}_j{}'} \longleftarrow$$

Can write down directly from form of inf rotation

$$\frac{d\vec e_j'}{dt} = \epsilon_{jkl}\,\omega_l'\,\vec e_k'(t)\,, \qquad \omega_l'(t) \equiv \frac{d\Omega_l'}{dt} \longleftarrow$$ components in body frame

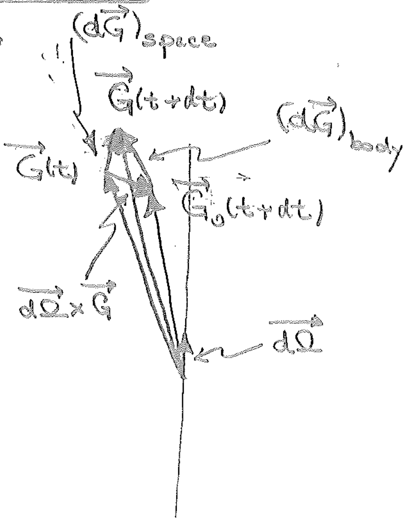
$$\boxed{\frac{d\vec e_j'}{dt} = \vec\omega \times \vec e_j'} \qquad \omega_j(t) = \frac{d\Omega_j}{dt} \longleftarrow$$ $\overrightarrow{d\Omega}$ is an infinitesimal vector — not the differential of a vector

$$\boxed{\frac{\overrightarrow{d\Omega}}{dt} = \vec\omega = \omega_j'(t)\,\vec e_j'(t) = \omega_j(t)\,\vec e_j = \begin{pmatrix}\text{instantaneous}\\ \text{angular}\\ \text{velocity}\end{pmatrix},}$$

↑ not $\frac{d}{dt}\vec\Omega$

$$G_j'(t)\,\frac{d\vec e_j'}{dt} = \vec e_k'(t)\,\epsilon_{klj}\,\omega_l'\,G_j' = \vec\omega \times \vec G$$

$$\boxed{\left(\frac{d\vec G}{dt}\right)_{space} = \left(\frac{d\vec G}{dt}\right)_{body} + \vec\omega \times \vec G}$$



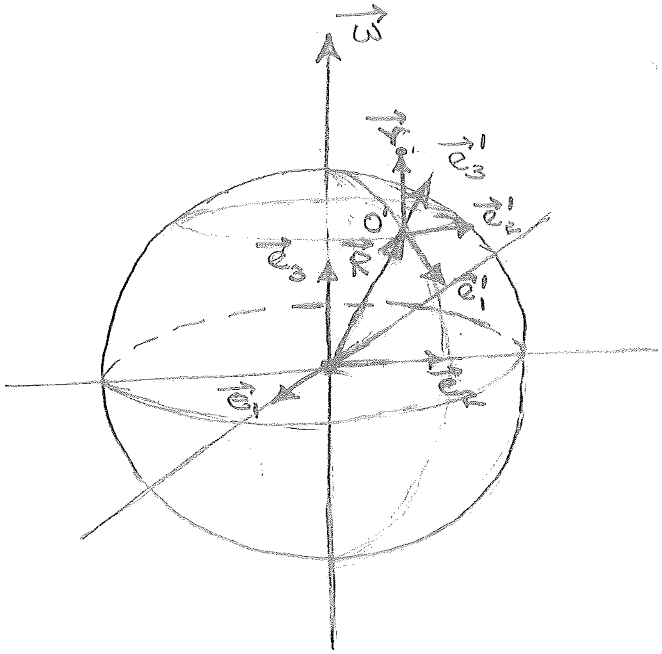
(2) $$(d\vec G)_{space} = \vec G(t+dt) - \vec G(t)$$

$$= \underbrace{\left[\vec G(t+dt) - \vec G_0(t+dt)\right]}_{(d\vec G)_{body}} + \underbrace{\left[\vec G_0(t+dt) - \vec G(t)\right]}_{\overrightarrow{d\Omega}\times\vec G}$$

$\vec G_0$ is fixed in body frame

$$\left(\frac{d\vec G}{dt}\right)_{space} = \left(\frac{d\vec G}{dt}\right)_{body} + \vec\omega \times \vec G$$

Centrifugal and Coriolis forces



$$\vec{R} = X_j \vec{e}_j = X'_j \vec{e}'_j = R\,\vec{e}'_3$$

$$\vec{r} = x_j \vec{e}_j = x'_j \vec{e}'_j$$

$$\vec{W} = \left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{space} = (\dot{X}_j + \dot{x}_j)\,\vec{e}_j$$

$$= \left(\frac{d(\vec{R}+\vec{r})}{dt}\right)_{rot} + \vec{\omega}\times(\vec{R}+\vec{r})$$

$$\shortparallel$$

$$(\cancelto{0}{\dot{X}'_j} + \dot{x}'_j)\,\vec{e}'_j = \dot{x}'_j\,\vec{e}'_j = \vec{V}_r$$

$$= \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times(\vec{R}+\vec{r})$$

$$\shortparallel \quad \vec{V}_r$$

Traditional approach:

$$\vec{a} = \left(\frac{d\vec{W}}{dt}\right)_{space} = \dot{W}_j\,\vec{e}_j = (\ddot{X}_j + \ddot{x}_j)\,\vec{e}_j$$

$$= \left(\frac{d\vec{W}}{dt}\right)_{rot} + \vec{\omega}\times\vec{W} \qquad \left(\frac{d\vec{\omega}}{dt}\right)_{space} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \cancelto{0}{\vec{\omega}\times\vec{\omega}}$$

$$= \underbrace{\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot}}_{\ddot{x}'_j\,\vec{e}'_j = \vec{a}_r} + 2\vec{\omega}\times\underbrace{\left(\frac{d\vec{r}}{dt}\right)_{rot}}_{\vec{V}_r = \dot{x}'_j\,\vec{e}'_j} + \vec{\omega}\times(\vec{\omega}\times\vec{R}) + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\boxed{\vec{a} = \frac{\vec{F}}{m} = \vec{a}_r + 2\vec{\omega}\times\vec{V}_r + \vec{\omega}\times(\vec{\omega}\times\vec{R}) + \vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

Lagrangian formulation:

$$T = \frac{1}{2} m \vec{W} \cdot \vec{W} = \frac{1}{2} m W_j' W_j'$$

$$W_j' = \dot{x}_j' + \epsilon_{jkl}\,\omega_k' (X_l' + x_l')$$

$$V = V(x_1', x_2', x_3')$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{x}_j'} = \frac{\partial T}{\partial \dot{x}_j'} = m W_k' \underbrace{\frac{\partial W_k'}{\partial \dot{x}_j'}}_{\delta_{jk}} = m W_j' = m\left(\dot{x}_j' + \epsilon_{jkl}\,\omega_k'(X_l' + x_l')\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j'}\right) = m\left(\ddot{x}_j' + \epsilon_{jkl}\,\omega_k' \dot{x}_l'\right)$$

$$\frac{\partial L}{\partial x_j} = \frac{\partial T}{\partial x_j} - \frac{\partial V}{\partial x_j}$$

$$\frac{\partial T}{\partial x_j} = m W_m' \frac{\partial W_m'}{\partial x_j} = m W_m' \left(\epsilon_{mkl}\,\omega_k' \delta_{lj}\right) = -m\,\epsilon_{jkl}\,\omega_k' W_l'$$

$$= -m\,\epsilon_{jkl}\,\omega_k' \dot{x}_l' - m\,\epsilon_{jkl}\,\omega_k'\,\epsilon_{lmn}\,\omega_m'(X_n' + x_n') - \frac{\partial V}{\partial x_j'}$$

$$m\ddot{x}_j' = -2m\,\epsilon_{jkl}\,\omega_k' \dot{x}_l' - m\,\epsilon_{jkl}\,\omega_k'\,\epsilon_{lmn}\,\omega_m'(X_n' + x_n') - \frac{\partial V}{\partial x_j'}$$

$$\boxed{m\vec{a}_r = -2m\vec{\omega} \times \vec{V}_r - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{F}}$$

Review of kinematics

Orthogonal transformations

Rotations: Passive and active transformations

Angular coördinates: Euler angles are one possibility

Angular velocity $\vec{\omega}$

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$$

instantaneous angular velocity