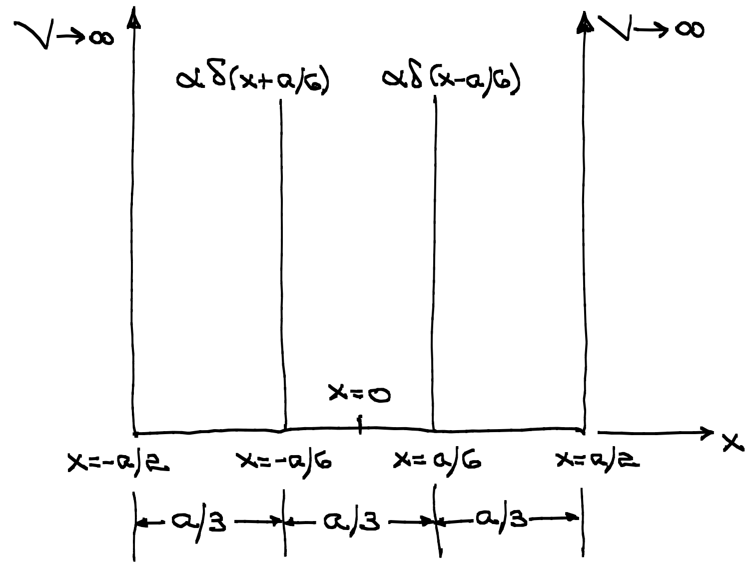
Phys 521

Midterm #1

Solution Set

1.1



(a)

$\alpha = 0$: The stationary states are those of an infinite well of width a.

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \begin{cases} \cos k_n x, & n = 1, 3, \ldots \\ \sin k_n x, & n = 2, 4, \ldots \end{cases} \qquad k_n = \frac{n\pi}{a}, \quad E_n = \frac{\hbar^2 k_n^2}{2m}$$

$\alpha \to \infty$: There are three infinite wells, each of width $a/3$. All the stationary states are three-fold degenerate. The stationary states are given by

$$\varphi_n^\alpha(x) = \sqrt{\frac{6}{a}} \begin{cases} \cos \bar{k}_n x_\alpha, & n = 1, 3, \ldots \\ \sin \bar{k}_n x_\alpha, & n = 2, 4, \ldots \end{cases} \qquad \bar{k}_n = \frac{3n\pi}{a}, \quad E_n = \frac{\hbar^2 \bar{k}_n^2}{2m}$$

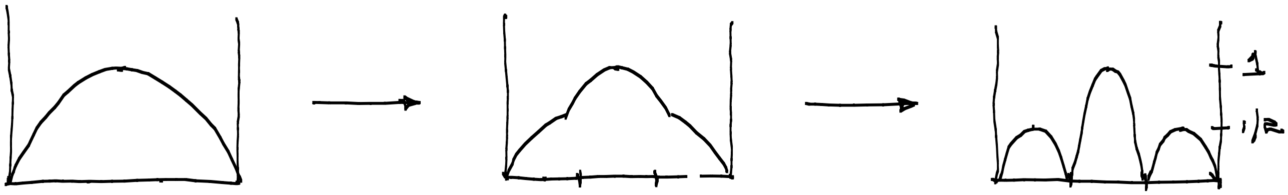
Left well ($\alpha = L$): $x_L = x + a/3$, $-a/2 < x < -a/6$
Middle well ($\alpha = M$): $x_M = x$, $-a/6 < x < a/6$
Right well ($\alpha = R$): $x_R = x - a/3$, $a/6 < x < a/2$

Any linear combination of the three degenerate stationary states is also a stationary state with the same energy.

Transformation of the lowest three stationary states. Since the δ-barriers are parity-symmetric, an even state stays even, and an odd state stays odd.

$\alpha = 0$ | typical α | $\alpha \to \infty$

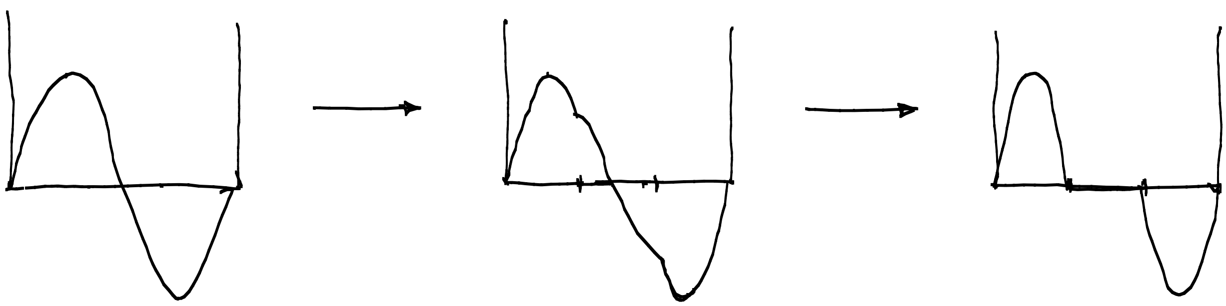
(1)



$k = \pi/a$ | $\pi/a < k < 3\pi/a$ | $k = 3\pi/a$

The relative size of the 3 bumps is determined by orthogonality to (3).

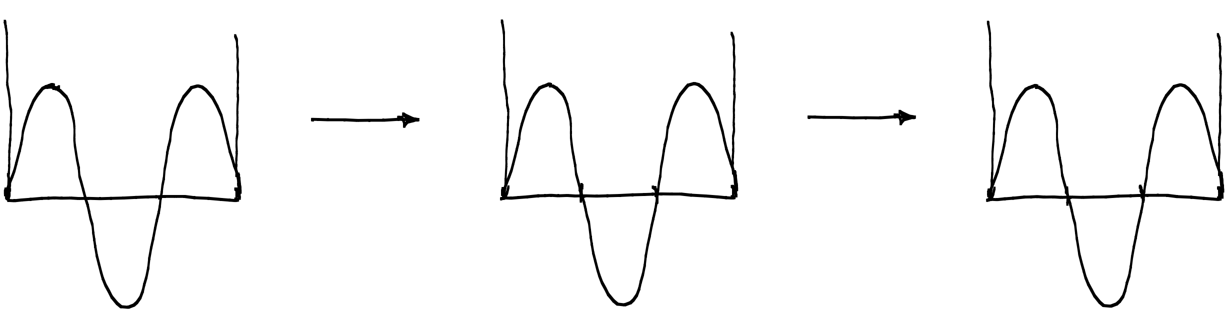
(2)



$k = 2\pi/a$ | $2\pi/a < k < 3\pi/a$ | $k = 3\pi/a$

An odd wave function with this wave number does not fit into the middle region, so the wave function vanishes there.

(3)



$k = 3\pi/a$ | $k = 3\pi/a$ | $k = 3\pi/a$

Since the δ-barriers are at zeroes of this state, the barriers have no effect.

Now focus on the ground state, which is even and has the form

$$\varphi(x) = \begin{cases} \cos kx, & |x| < a/6, \\ B\cos(k|x| - \theta), & a/6 < |x| < a/2, \\ 0, & |x| > a/2. \end{cases}$$

Because of the parity symmetry, the boundary conditions at negative x will automatically be satisfied if we enforce those at positive x.

BC's:

use the 1st zero to make k as small as possible and thus get the ground state

① $\varphi(a/2) = 0 \Rightarrow \cos(ka/2 - \theta) = 0 \Rightarrow ka/2 - \theta = \pi/2 \Rightarrow \theta = ka/2 - \pi/2$

② $\varphi(a/6 - \epsilon) = \varphi(a/6 + \epsilon) \Rightarrow \cos ka/6 = B\cos(ka/6 - \theta)$

③ $\varphi'(a/6 + \epsilon) - \varphi'(a/6 - \epsilon) = \frac{2m\alpha}{\hbar^2}\varphi(a/6)$

$$\Rightarrow -B\sin(ka/6 - \theta) + \sin ka/6 = \frac{2m\alpha}{k\hbar^2}\cos ka/6$$

(b) ①: $\frac{ka}{6} - \theta = \frac{\pi}{2} - \frac{ka}{3} \Rightarrow \cos\left(\frac{ka}{6} - \theta\right) = \sin\frac{ka}{3} = 2\sin\frac{ka}{6}\cos\frac{ka}{6}$

Plug this into ②:

$$\cos\frac{ka}{6} = 2B\sin\frac{ka}{6}\cos\frac{ka}{6} \Rightarrow \boxed{2B\sin\frac{ka}{6} = 1}$$

(c) ③: $-B\underbrace{\sin\left(\frac{\pi}{2} - \frac{ka}{3}\right)}_{\cos\frac{ka}{3}} + \sin\frac{ka}{6} = \frac{2m\alpha}{k\hbar^2}\cos\frac{ka}{6}$

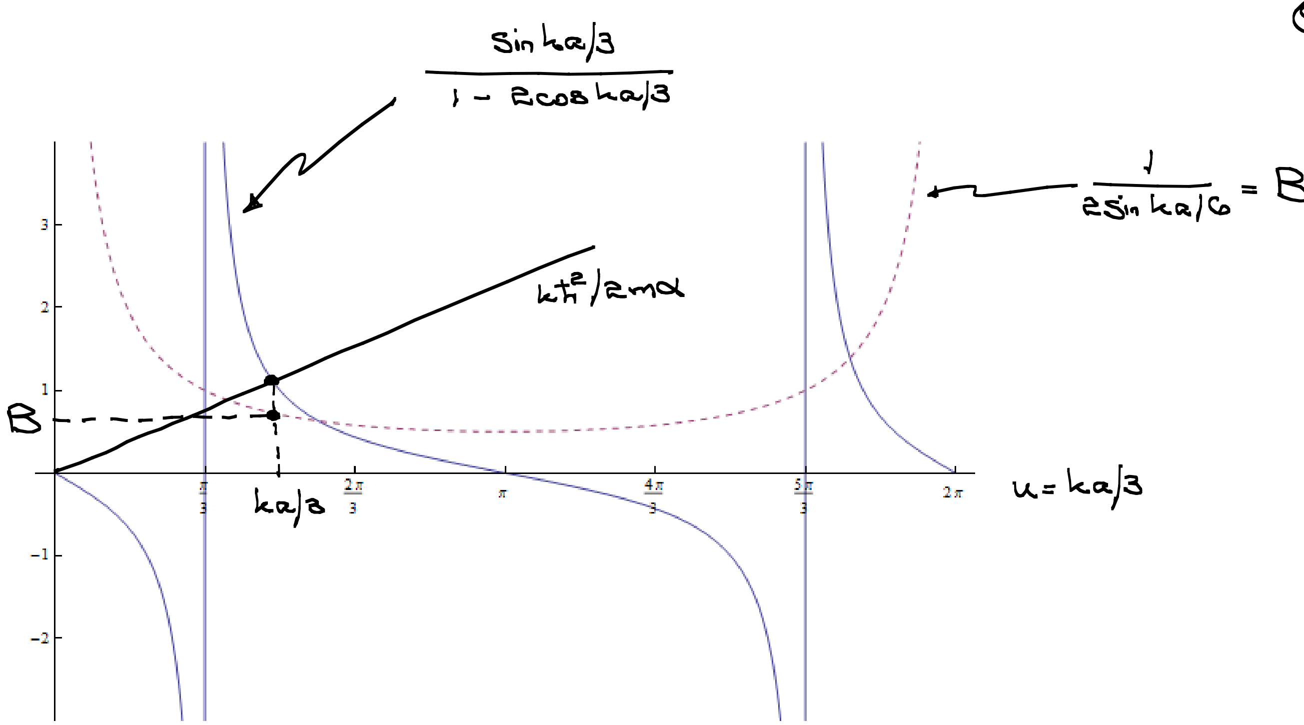
$$\Rightarrow \sin\frac{ka}{6} - \frac{2m\alpha}{k\hbar^2}\cos\frac{ka}{6} = B\cos\frac{ka}{3} = \frac{\cos ka/3}{2\sin ka/6}$$

$$\Rightarrow \underbrace{2\sin^2\frac{ka}{6}}_{1-\cos\frac{ka}{3}} - \frac{2m\alpha}{k\hbar^2}\underbrace{2\sin\frac{ka}{6}\cos\frac{ka}{6}}_{\sin\frac{ka}{3}} = \cos\frac{ka}{3}$$

$$\Rightarrow \frac{2m\alpha}{k\hbar^2}\sin\frac{ka}{3} = 1 - 2\cos\frac{ka}{3}$$

$$\Rightarrow \boxed{\frac{k\hbar^2}{2m\alpha} = \frac{\sin ka/3}{1 - 2\cos ka/3}}$$

(d)



The intersection of $k\hbar^2/2m\alpha$ with the solid line gives $ka/3$. The value of the dashed curve at this k gives B.

$$\alpha = 0 : \quad \frac{ka}{3} = \frac{\pi}{3} \implies k = \frac{\pi}{a} \qquad (\text{and } \theta = 0)$$

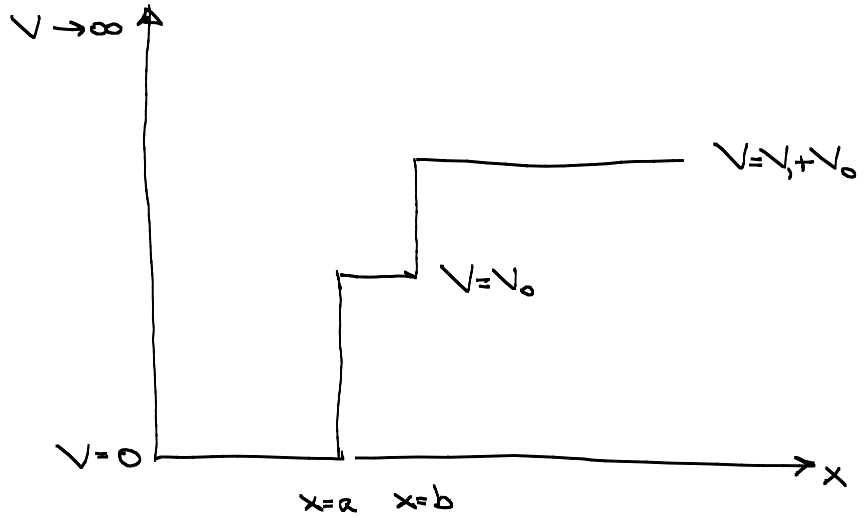
$$\implies B = 1$$

$$\alpha \to \infty : \quad \frac{ka}{3} = \pi \implies k = \frac{3\pi}{a} \qquad (\text{and } \theta = \pi)$$

$$\implies B = \frac{1}{2 \sin ka/6} = \frac{1}{2}$$

As α increases from 0 to ∞, k increases from $\frac{\pi}{a}$ to $\frac{3\pi}{a}$, and B decreases from 1 to $1/2$. All this is in accord with the drawings in part (a).

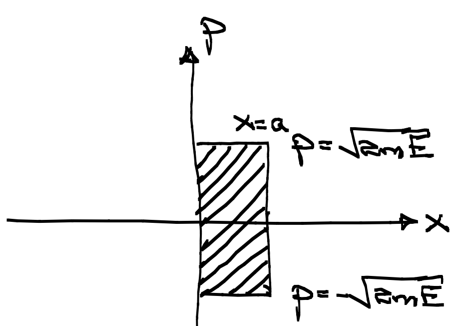
1.2



(a) There are two cases: $0 < E \leq V_0$ and $V_0 < E \leq V_0 + V_1$.

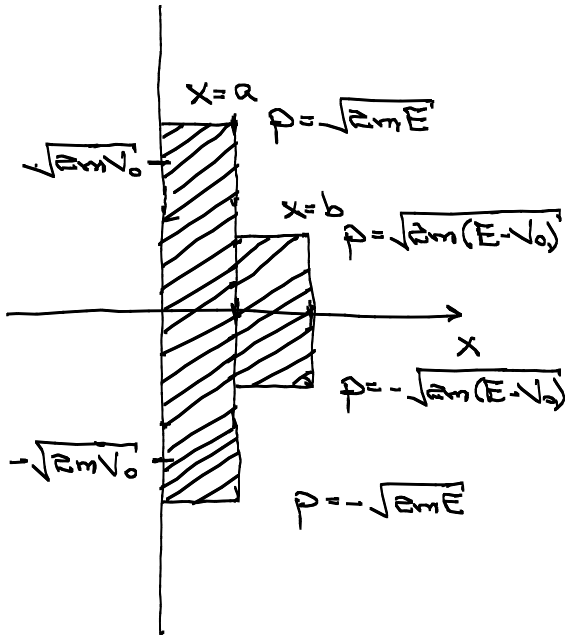
$0 < E \leq V_0$: The particle can move between $x=0$ and $x=a$, and it can have any momentum between $-\sqrt{2mE}$ and $+\sqrt{2mE}$, so the accessible phase-space area is $a \cdot 2\sqrt{2mE}$. The approximate number of bound states is

$$\boxed{\begin{pmatrix}\text{\# of bound states} \\ \text{with energy} < E \leq V_0\end{pmatrix} \simeq \frac{2a\sqrt{2mE}}{h}}$$



Accessible phase space

$V_0 < E \leq V_0 + V_1$: When the particle is in the region $0 \leq x \leq a$, it can have any momentum between $-\sqrt{2mE}$ and $+\sqrt{2mE}$. When the particle is in the region $a \leq x \leq b$, it can have any momentum between $-\sqrt{2m(E-V_0)}$ and $+\sqrt{2m(E-V_0)}$. Thus the accessible phase-space area is



Accessible phase space

$$a\,2\sqrt{2mE} + (b-a)\,2\sqrt{2m(E-V_0)}$$

and the approximate number of bound states is

$$\boxed{\left(\begin{array}{l}\#\text{ of bound states with}\\ \text{energy} < E,\text{ where } V_0 < E \le V_0 + V_1\end{array}\right) \simeq \frac{2a\sqrt{2mE}}{h} + \frac{2(b-a)\sqrt{2m(E-V_0)}}{h}}$$

(b) For $E \le V_0$, the nth bound state has energy

$$\frac{2a\sqrt{2mE_n}}{h} = n \implies \boxed{E_n = \frac{n^2h^2}{8a^2m^2} = \frac{n^2\pi^2\hbar^2}{2a^2m^2}\,,\quad n=1,\dots,N,\ E_n \le V_0}$$

To make things easy, we assume that $N \equiv \frac{2a\sqrt{2mV_0}}{h} \gg 1$ is a large integer that specifies the last bound state in the deeper well. For $V_0 < E \le V_0 + V_1$, the $(N+l)$th bound state has energy determined by

$$\frac{2a\sqrt{2mE_{N+l}}}{h} + \frac{2(b-a)\sqrt{2m(E_{N+l}-V_0)}}{h} = N+l$$

To solve for E_{N+l}, let $c = b-a$, $E_{N+l}/V_0 = x^2$, and write

$$N\cdot x + \frac{c}{a}N\sqrt{x^2-1} = N+l \implies \frac{c}{a}\sqrt{x^2-1} = 1 - x + \frac{l}{N}$$

$$\implies \frac{c^2}{a^2}(x^2-1) = 1 - 2x + x^2 + 2(1-x)\frac{l}{N} + \frac{l^2}{N^2}$$

$$\left(1-\frac{c^2}{a^2}\right)x^2 - 2\left(1+\frac{l}{N}\right)x + \frac{c^2}{a^2} + \left(1+\frac{l}{N}\right)^2 = 0$$

$$x = \frac{(1+\ell/N) \pm \sqrt{(1+\ell/N)^2 - (1-c^2/a^2)[c^2/a^2 + (1+\ell/N)^2]}}{1 - c^2/a^2}$$

For $\ell = 0$, we should have $x = 1$, so we need to use the lower sign:

$$\boxed{\frac{E_{N+\ell}}{V_0} = \frac{1+\ell/N - \sqrt{(1+\ell/N)^2 - (1-c^2/a^2)[c^2/a^2 + (1+\ell/N)^2]}}{1 - c^2/a^2}}$$

Another, equivalent way to get these estimates of the bound-state energies is to say that the integrated phase across the well is a multiple of π.

$$0 < E \le V_0 : \quad n\pi = \int_0^a dx\, \underset{\sqrt{2mE}/\hbar}{\underset{\uparrow}{k(x)}} = \frac{\sqrt{2mE}\,a}{\hbar} \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, \ldots, N$$

$$E_N = \frac{N^2\pi^2\hbar^2}{2ma^2} = V_0$$

$$V_0 < E \le V_0 + V_1 : \quad (N+\ell)\pi = \int_0^b dx\, k(x)$$

$$= \int_0^a dx \frac{\sqrt{2mE}}{\hbar} + \int_a^b dx \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

$$= \frac{\sqrt{2mE}}{\hbar} a + \frac{\sqrt{2m(E-V_0)}}{\hbar}(b-a)$$

$$N + \ell = \frac{2a\sqrt{2mE}}{h} + \frac{2(b-a)\sqrt{2m(E-V_0)}}{h}$$

Same as above