Phys 521

Midterm #2

Solution Set

1.

$$\hat{A} = |u_1\rangle\langle u_1| - |u_3\rangle\langle u_3| \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\hat{B} = |u_1\rangle\langle u_3| + |u_2\rangle\langle u_2| + |u_3\rangle\langle u_1| \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\hat{C} = \frac{1}{2}|u_1\rangle\langle u_3| - \frac{1}{2}|u_2\rangle\langle u_2| \;\; \frac{1}{2}|u_3\rangle\langle u_1| \longleftrightarrow \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{2}|u_1\rangle + \frac{1}{\sqrt{2}}|u_3\rangle + \frac{1}{2}|u_3\rangle \longleftrightarrow \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

(a) Eigenvectors and eigenvalues of $\hat{A}$, $\hat{B}$, and $\hat{C}$:

	Eigenvector	Eigenvalue
$\hat{A}$	$\vert u_1\rangle$	$+1$
	$\vert u_2\rangle$	0
	$\vert u_3\rangle$	-1
$\hat{B}$	$\vert v_1\rangle = \frac{1}{\sqrt{2}}(\vert u_1\rangle + \vert u_3\rangle)$	$+1$
	$\vert v_3\rangle = \frac{1}{\sqrt{2}}(\vert u_1\rangle - \vert u_3\rangle)$	-1
	$\vert v_2\rangle = \vert u_2\rangle$	$+1$

Use

$$|u_1\rangle\langle u_3| - |u_3\rangle\langle u_1| = |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|$$

$$\hat{B} = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| - |v_3\rangle\langle v_3|$$

Notice that

$$|u_1\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle + |v_3\rangle)$$

$$|u_3\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle - |v_3\rangle)$$

	Eigenvector	Eigenvalue
$\hat{C}$	$\vert v_1\rangle = \frac{1}{\sqrt{2}}(\vert u_1\rangle + \vert u_3\rangle)$	$1/2$
	$\vert v_3\rangle = \frac{1}{\sqrt{2}}(\vert u_1\rangle - \vert u_3\rangle)$	$-1/2$
	$\vert v_2\rangle = \vert u_2\rangle$	$-1/2$

$$\hat{C} = \frac{1}{2}|v_1\rangle\langle v_1| - \frac{1}{2}|v_2\rangle\langle v_2| - \frac{1}{2}|v_3\rangle\langle v_3|$$

$\hat{A}$ is nondegenerate, so it is a CSCO. $\hat{B}$ and $\hat{C}$ commute, and each has degeneracies, but together they have unique eigenvalues for each eigenvector, so $\hat{B}$ and $\hat{C}$ make up a CSCO.

(b) $|\psi(0)\rangle = \frac{1}{2}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|u_3\rangle = \frac{1}{\sqrt{2}}|v_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle$

	Possible results	Probability
$\hat{A}$	$+1$	$\lvert\langle u_1\vert\psi(0)\rangle\rvert^2 = \frac{1}{4}$
	0	$\lvert\langle u_2\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}$
	-1	$\lvert\langle u_3\vert\psi(0)\rangle\rvert^2 = \frac{1}{4}$
$\hat{B}$	1	$\lvert\langle v_1\vert\psi(0)\rangle\rvert^2 + \lvert\langle v_2\vert\psi(0)\rangle\rvert^2 = 1$
	-1	$\lvert\langle v_3\vert\psi(0)\rvert^2 = 0$
$\hat{C}$	$+\frac{1}{2}$	$\lvert\langle v_1\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}$
	$-\frac{1}{2}$	$\lvert\langle v_2\vert\psi(0)\rangle\rvert^2 + \lvert\langle v_3\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}$

(c) $\hat{H} = \hbar\omega\hat{A}$,

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-i\omega t\hat{A}} = e^{-i\omega t}|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + e^{i\omega t}|u_3\rangle\langle u_3|$$

$$\Longrightarrow |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \frac{1}{2}e^{-i\omega t}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}e^{i\omega t}|u_3\rangle$$

$$= \frac{1}{\sqrt{2}}\cos\omega t\,|v_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle - \frac{i}{\sqrt{2}}\sin\omega t\,|v_3\rangle$$

use $|u_1\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle + |v_3\rangle)$

$|u_3\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle - |v_3\rangle)$

	Possible results	Probability
$\hat{A}$	$+1$	$\lvert\langle u_1\vert\psi(0)\rangle\rvert^2 = \frac{1}{4}$
	0	$\lvert\langle u_2\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}$
	-1	$\lvert\langle u_3\vert\psi(0)\rangle\rvert^2 = \frac{1}{4}$
$\hat{B}$	1	$\lvert\langle v_1\vert\psi(0)\rangle\rvert^2 + \lvert\langle v_2\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}(1+\cos^2\omega t)$
	-1	$\lvert\langle v_3\vert\psi(0)\rvert^2 = \frac{1}{2}\sin^2\omega t$
$\hat{C}$	$+\frac{1}{2}$	$\lvert\langle v_1\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}\cos^2\omega t$
	$-\frac{1}{2}$	$\lvert\langle v_2\vert\psi(0)\rangle\rvert^2 + \lvert\langle v_3\vert\psi(0)\rangle\rvert^2 = \frac{1}{2}(1+\sin^2\omega t)$

(d) $\hat{H} = \hbar\omega\hat{C}$, $\quad \hat{C} = \frac{1}{2}|v_1\rangle\langle v_1| - \frac{1}{2}|v_2\rangle\langle v_2| - \frac{1}{2}|v_3\rangle\langle v_3|$

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-i\omega t\hat{C}} = e^{-i\omega t/2}|v_1\rangle\langle v_1| + e^{i\omega t/2}|v_2\rangle\langle v_2| + e^{i\omega t/2}|v_3\rangle\langle v_3|$$

$$\Longrightarrow |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t/2}|v_1\rangle + \frac{1}{\sqrt{2}}e^{i\omega t/2}|v_2\rangle$$

$$= \frac{1}{2}e^{-i\omega t/2}|u_1\rangle + \frac{1}{\sqrt{2}}e^{i\omega t/2}|u_2\rangle + \frac{1}{2}e^{-i\omega t/2}|u_3\rangle$$

	Possible results	Probability
$\hat{A}$	$+1$	$\|\langle u_1\|\psi(0)\rangle\|^2 = \frac{1}{4}$
	0	$\|\langle u_2\|\psi(0)\rangle\|^2 = \frac{1}{2}$
	-1	$\|\langle u_3\|\psi(0)\rangle\|^2 = \frac{1}{4}$
$\hat{B}$	1	$\|\langle v_1\|\psi(0)\rangle\|^2 + \|\langle v_2\|\psi(0)\rangle\|^2 = 1$
	-1	$\|\langle v_3\|\psi(0)\|^2 = 0$
$\hat{C}$	$+\frac{1}{2}$	$\|\langle v_1\|\psi(0)\rangle\|^2 = \frac{1}{2}$
	$-\frac{1}{2}$	$\|\langle v_2\|\psi(0)\rangle\|^2 + \|\langle v_3\|\psi(0)\rangle\|^2 = \frac{1}{2}$

2. $\psi(x,0) = \sqrt{2}\, a^{3/2}\, x\, e^{-a|x|}, \quad a>0$

Check normalization

$$\int dx\, |\psi(x,0)|^2 = 2a^3 \int_{-\infty}^{\infty} dx\, x^2 e^{-2a|x|}$$

$$= 4a^3 \int_0^{\infty} dx\, x^2 e^{-2ax}$$

$u = 2ax$
$du = 2a\,dx$

$$= \frac{1}{2}\underbrace{\int_0^{\infty} du\, u^2 e^{-u}}_{2!}$$

$$= 1$$

(a) $\bar\psi(p,0) = \langle p|\psi(0)\rangle$

$$= \int_{-\infty}^{\infty} dx\, \underbrace{\langle p|x\rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}}\langle x|\psi(0)\rangle$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, \psi(x,0)\, e^{-ipx/\hbar}$$

$$= \sqrt{\frac{a^3}{\pi\hbar}} \int_{-\infty}^{\infty} dx\, x e^{-a|x|} e^{-ipx/\hbar}$$

$$= \sqrt{\frac{a^3}{\pi\hbar}}\, -\frac{\hbar}{i}\frac{d}{dp} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-a|x|} e^{-ipx/\hbar}}$$

$$= \underbrace{\int_0^{\infty} dx\, e^{-(a+ip/\hbar)x}}_{=\frac{1}{a+ip/\hbar}} + \underbrace{\int_{-\infty}^{0} dx\, e^{(a-ip/\hbar)x}}$$

$$= \int_0^{\infty} dx\, e^{-(a-ip/\hbar)x}$$

$$= \frac{1}{a-ip/\hbar}$$

$$= \frac{1}{a+ip/\hbar} + \frac{1}{a-ip/\hbar}$$

$$= \frac{2a}{a^2+p^2/\hbar^2}$$

$$= \frac{2}{a}\frac{1}{(p/\hbar a)^2+1}$$

$$= 2i\sqrt{\frac{\hbar a}{\pi}}\, \underbrace{\frac{d}{dp}\frac{1}{(p/\hbar a)^2+1}}_{-\frac{2p/(\hbar a)^2}{[(p/\hbar a)^2+1]}}$$

$$\boxed{\tilde{\psi}(p,0) = -4i\sqrt{\frac{1}{\pi\hbar a}}\,\frac{p/\hbar a}{[(p/\hbar a)^2+1]^2}}$$

Check normalization

$$\int_{-\infty}^{\infty} dp\,|\tilde{\psi}(p,0)|^2 = \frac{16}{\pi}\int_{-\infty}^{\infty}\frac{dp}{\hbar a}\,\frac{(p/\hbar a)^2}{[(p/\hbar a)^2+1]^4} = 1$$

(b) $$\langle\hat{x}\rangle_0 = \int_{-\infty}^{\infty} dx\,\underset{\text{odd}}{\underset{\uparrow}{x}}\,\underset{\text{even}}{\underset{\uparrow}{|\psi(x,0)|^2}} = \boxed{0 = \langle\hat{x}\rangle_0}$$

$$\langle\hat{p}\rangle_0 = \int_{-\infty}^{\infty} dp\,\underset{\text{odd}}{\underset{\uparrow}{p}}\,\underset{\text{even}}{\underset{\uparrow}{|\tilde{\psi}(p,0)|^2}} = \boxed{0 = \langle\hat{p}\rangle_0}$$

$$\langle\hat{x}^2\rangle_0 = \int_{-\infty}^{\infty} dx\,x^2\,|\psi(x,0)|^2 = 2a^3\int_{-\infty}^{\infty} dx\,x^4 e^{-2a|x|}$$

$$= 4a^3\int_{-\infty}^{\infty} dx\,x^4 e^{-2ax}$$

$$= \frac{1}{8a^2}\underbrace{\int_{-\infty}^{\infty} du\,u^4 e^{-u}}_{4! = 24}$$

$$\boxed{\langle\hat{x}^2\rangle_0 = \frac{3}{a^2}}$$

$$\langle\hat{p}^2\rangle_0 = \int_{-\infty}^{\infty} dp\,p^2\,|\tilde{\psi}(p,0)|^2 = \frac{16}{\pi}(\hbar a)^2\underbrace{\int\frac{dp}{\hbar a}\,\frac{(p/\hbar a)^4}{[(p/\hbar a)^2+1]^2}}_{\frac{\pi}{16}} = \boxed{(\hbar a)^2 = \langle\hat{p}^2\rangle_0}$$

$$\boxed{\begin{aligned}\Delta x &= \left[\langle\hat{x}^2\rangle_0 - \langle\hat{x}\rangle_0^2\right]^{1/2} = \frac{\sqrt{3}}{a}\\ \Delta p &= \left[\langle\hat{p}^2\rangle_0 - \langle\hat{p}\rangle_0^2\right]^{1/2} = \hbar a\end{aligned}}$$

$$\Delta x\,\Delta p = \sqrt{3}\,\hbar$$

$$\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle_0 = 2\,\mathrm{Re}\left(\langle\hat{x}\hat{p}\rangle_0\right) = 2\,\mathrm{Re}\left(\langle\hat{p}\hat{x}\rangle_0\right)$$

$$\langle\hat{x}\hat{p}\rangle_0 = \langle\hat{p}\hat{x}\rangle_0^*$$

$$\langle \hat{p}\hat{x} \rangle_0 = \int dp\, \bar{\psi}(p,0)\, p \left(-\frac{\hbar}{i}\frac{d}{dp} \right) \psi(p,0)$$

$$= i\hbar \underbrace{\frac{16}{\pi \hbar a} \int_{-\infty}^{\infty} dp\, p \frac{p/\hbar a}{[(p/\hbar a)^2 + 1]^2} \frac{d}{dp} \left(\frac{p/\hbar a}{[(p/\hbar a)^2 + 1]^2} \right)}_{\text{real, so has to equal } -1/2 \text{ to get the commutator right}}$$

$$\Longrightarrow \boxed{\langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle_0 = 0}$$

$$\frac{d}{dp} \left(\frac{p/\hbar a}{[(p/\hbar a)^2 + 1]^2} \right) = \frac{1}{\hbar a} \left(\frac{1}{[(p/\hbar a)^2 + 1]^2} - 4 \frac{(p/\hbar a)^2}{[(p/\hbar a)^2 + 1]^3} \right)$$

Let's do it

$$= i\hbar \frac{16}{\pi} \int_{-\infty}^{\infty} \frac{dp}{\hbar a} \frac{p}{\hbar a} \frac{p/\hbar a}{[(p/\hbar a)^2 + 1]^2} \left(\frac{1}{[(p/\hbar a)^2 + 1]^2} - 4 \frac{(p/\hbar a)^2}{[(p/\hbar a)^2 + 1]^3} \right)$$

$$= i\hbar \frac{16}{\pi} \left(\underbrace{\int_{-\infty}^{\infty} du \frac{u^2}{(u^2+1)^4}}_{\frac{\pi}{16}} - 4 \underbrace{\int_{-\infty}^{\infty} du \frac{u^4}{(u^2+1)^5}}_{\frac{3\pi}{128}} \right)$$

$$= -\frac{1}{2} i\hbar$$

You can use the same sort of argument in the position domain to conclude that $\langle \hat{x}\hat{p} \rangle_0$ is pure imaginary, but if you want to do the integrals, you will need to worry about derivatives at $x=0$.