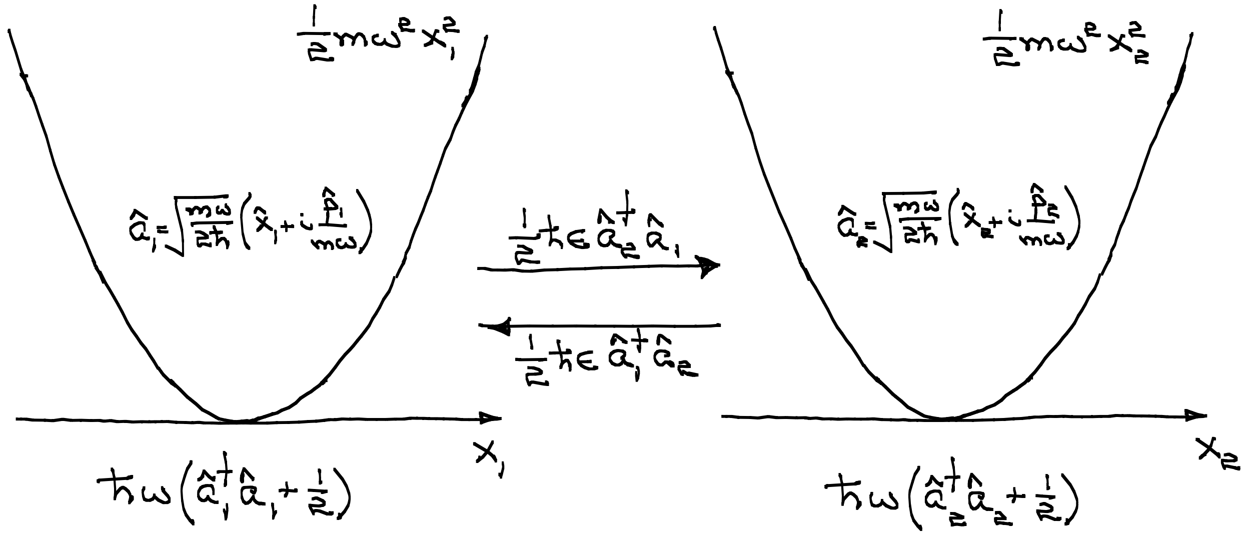
Phys 522

Midterm #3

Solution Set

1.



Bare Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_2^2 \qquad [\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk} \quad j,k=1,2$$

Bare creation and annihilation operators

$$\hat{a}_j = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_j + i\frac{\hat{p}_j}{m\omega}\right) \qquad j=1,2$$
$$\hat{a}_j^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_j - i\frac{\hat{p}_j}{m\omega}\right)$$

$$= \hbar\omega\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right)$$

Bare number states

$$\overline{|n_1, n_2\rangle} = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}\underbrace{\overline{|0,0\rangle}}$$

vacuum for the bare Hamiltonian: $\hat{a}_1\overline{|0,0\rangle} = 0 = \hat{a}_2\overline{|0,0\rangle}$

Total Hamiltonian

interaction strength

$$\hat{H} = \hat{H}_0 + \underbrace{\frac{1}{2}\hbar\epsilon\left(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2\right)}_{\text{tunneling interaction}}$$

$\hat{a}_2^\dagger\hat{a}_1$: one quantum tunnels from oscillator 1 to oscillator 2

$\hat{a}_1^\dagger\hat{a}_2$: one quantum tunnels from oscillator 2 to oscillator 1

Decoupling the total Hamiltonian: Sum and difference oscillators

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\hat{a}_2 \pm \hat{a}_1\right) \qquad \hat{a}_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_2^\dagger \pm \hat{a}_1^\dagger\right)$$

$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left(\hat{a}_+ - \hat{a}_-\right)$$
$$\hat{a}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}_+ + \hat{a}_-\right)$$

$$\Longrightarrow \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 = \hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_-$$
$$\hat{H}_0 = \hbar\omega\left(\hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_- + 1\right)$$

So $\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 = \frac{1}{2}\Big[\underbrace{(\hat{a}_+^\dagger + \hat{a}_-^\dagger)(\hat{a}_+ - \hat{a}_-) + (\hat{a}_+^\dagger - \hat{a}_-^\dagger)(\hat{a}_+ + \hat{a}_-)}_{\text{cross-terms cancel}}\Big]$

$$= \hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-$$

Plugging this into the total Hamiltonian, we get

$$\hat{H} = \hbar\overbrace{\left(\omega + \tfrac{1}{2}\epsilon\right)}^{\omega_+}\left(\hat{a}_+^\dagger \hat{a}_+ + \tfrac{1}{2}\right) + \hbar\overbrace{\left(\omega - \tfrac{1}{2}\epsilon\right)}^{\omega_-}\left(\hat{a}_-^\dagger \hat{a}_- + \tfrac{1}{2}\right)$$

$$= \hbar\omega_+ \hat{a}_+^\dagger \hat{a}_+ + \hbar\omega_- \hat{a}_-^\dagger \hat{a}_- + \hbar\omega$$

Energy eigenstates: $|n_+, n_-\rangle = \dfrac{(\hat{a}_+^\dagger)^{n_+}(\hat{a}_-^\dagger)^{n_-}}{\sqrt{n_+!\,n_-!}}|0,0\rangle$ vacuum for the total Hamiltonian $\hat{a}_+|0,0\rangle = 0 = \hat{a}_-|0,0\rangle$

Energy eigenvalue: $E = \hbar\omega_+ n_+ + \hbar\omega_- n_- + \hbar\omega$

$$= \hbar\omega(n_+ + n_- + 1) + \tfrac{1}{2}\hbar\epsilon(n_+ - n_-)$$

(a) The first thing we need to know is that both vacuums are the same, and this is obvious since $\hat{a}_1, \hat{a}_2, \hat{a}_+$, and $\hat{a}_-$ all annihilate both $\overline{|0,0\rangle}$ and $|0,0\rangle$. So we have

$$|0,0\rangle = \overline{|0,0\rangle}.$$

The lowest six eigenstates are those with no excitations (vacuum, 1 state), one excitation (2 states), or two excitations (3 states).

	Eigenvalue	Eigenstate
$n_+=0, n_-=0$	$\hbar\omega$	$\|0,0\rangle = \overline{\|0,0\rangle}$
$n_+=1, n_-=0$	$2\hbar\omega + \frac{1}{2}\hbar\epsilon$	$\|1,0\rangle = \hat{a}_+^\dagger\|0,0\rangle = \frac{1}{\sqrt{2}}(\hat{a}_2^\dagger + \hat{a}_1^\dagger)\overline{\|0,0\rangle} = \frac{1}{\sqrt{2}}(\overline{\|0,1\rangle} + \overline{\|1,0\rangle})$
$n_+=0, n_-=1$	$2\hbar\omega - \frac{1}{2}\hbar\epsilon$	$\|0,1\rangle = \hat{a}_-^\dagger\|0,0\rangle = \frac{1}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_1^\dagger)\overline{\|0,0\rangle} = \frac{1}{\sqrt{2}}(\overline{\|0,1\rangle} - \overline{\|1,0\rangle})$
$n_+=2, n_-=0$	$3\hbar\omega + \hbar\epsilon$	$\|2,0\rangle = \frac{\hat{a}_+^\dagger}{\sqrt{2}}\|1,0\rangle$

$$= \frac{1}{2\sqrt{2}}(\hat{a}_2^\dagger + \hat{a}_1^\dagger)(\overline{|0,1\rangle} + \overline{|1,0\rangle})$$

$$= \frac{1}{2\sqrt{2}}\left(\sqrt{2}\,\overline{|0,2\rangle} + \overline{|1,1\rangle} + \overline{|1,1\rangle} + \sqrt{2}\,\overline{|2,0\rangle}\right)$$

$$= \frac{1}{\sqrt{2}}\overline{|1,1\rangle} + \frac{1}{2}\left(\overline{|0,2\rangle} + \overline{|2,0\rangle}\right)$$

$n_+=0, n_-=2 \quad 3\hbar\omega - \hbar\epsilon \qquad |0,2\rangle = \frac{\hat{a}_-^\dagger}{\sqrt{2}}|0,1\rangle$

$$= \frac{1}{2\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_1^\dagger)(\overline{|0,1\rangle} - \overline{|1,0\rangle})$$

$$= \frac{1}{2\sqrt{2}}(\sqrt{2}\,\overline{|0,2\rangle} - \overline{|1,1\rangle} - \overline{|1,1\rangle} + \sqrt{2}\,\overline{|2,0\rangle})$$

$$= -\frac{1}{\sqrt{2}}\overline{|1,1\rangle} + \frac{1}{2}(\overline{|0,2\rangle} + \overline{|2,0\rangle})$$

$n_+=1, n_-=1 \quad 3\hbar\omega \qquad |1,1\rangle = \hat{a}_-^\dagger|1,0\rangle \quad (\text{or } \hat{a}_+^\dagger|0,1\rangle)$

$$= \frac{1}{2}(\hat{a}_2^\dagger - \hat{a}_1^\dagger)(\overline{|0,1\rangle} + \overline{|1,0\rangle})$$

$$= \frac{1}{2}(\sqrt{2}\,\overline{|0,2\rangle} + \overline{|1,1\rangle} - \overline{|1,1\rangle} - \sqrt{2}\,\overline{|2,0\rangle})$$

$$= \frac{1}{\sqrt{2}}(\overline{|0,2\rangle} - \overline{|2,0\rangle})$$

(b) HP equations of motion: $i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]$ for arbitrary $\hat{A}$

$$i\hbar\frac{d\hat{a}_\pm}{dt} = [\hat{a}_\pm, \hat{H}] = \hbar\omega_\pm \underbrace{[\hat{a}_\pm, \hat{a}_\pm^\dagger \hat{a}_\pm]}_{\hat{a}_\pm} = \hbar\omega_\pm \hat{a}_\pm \Rightarrow \boxed{\frac{d\hat{a}_\pm}{dt} = -i\omega_\pm \hat{a}_\pm}$$

$$i\hbar\frac{d\hat{a}_1}{dt} = [\hat{a}_1, \hat{H}] = \hbar\omega \underbrace{[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_1]}_{\hat{a}_1} + \frac{1}{2}\hbar\epsilon \underbrace{[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_2]}_{\hat{a}_2}$$

$$= \hbar\omega\hat{a}_1 + \frac{1}{2}\hbar\epsilon\hat{a}_2$$

$$\boxed{\frac{d\hat{a}_1}{dt} = -i\omega\hat{a}_1 - i\frac{\epsilon}{2}\hat{a}_2} \qquad \boxed{\frac{d\hat{a}_2}{dt} = -i\omega\hat{a}_2 - i\frac{\epsilon}{2}\hat{a}_1}$$

Since the Hamiltonian is invariant under interchange of 1 and 2, we don't need to derive this equation.

It is trivial to solve for $\hat{a}_\pm$, and all other quantities can be derived from $\hat{a}_\pm$. In writing the solutions, operators without a time argument are initial values at $t=0$; these initial values are the operators in the Schrödinger picture.

evolution operator $\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$

$$\hat{a}_\pm(t) = \hat{U}^\dagger(t)\,\hat{a}_\pm\,\hat{U}(t) = e^{-i\omega_\pm t}\hat{a}_\pm = e^{-i\omega t}e^{\mp i\epsilon t/2}\hat{a}_\pm$$

$$\hat{a}_{\substack{1\\2}}(t) = \frac{1}{\sqrt{2}}\left(\hat{a}_+(t) \mp \hat{a}_-(t)\right) = \frac{1}{\sqrt{2}}e^{-i\omega t}\left(e^{-i\epsilon t/2}\hat{a}_+ \mp e^{+i\epsilon t/2}\hat{a}_-\right)$$

$$\hat{a}_+ = \frac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_1) \qquad \hat{a}_- = \frac{1}{\sqrt{2}}(\hat{a}_2 - \hat{a}_1)$$

$$\boxed{\begin{aligned} \hat{a}_1(t) &= \hat{U}^\dagger(t)\,\hat{a}_1\,\hat{U}^\dagger(t) = e^{-i\omega t}\left(\cos(\epsilon t/2)\,\hat{a}_1 - i\sin(\epsilon t/2)\,\hat{a}_2\right) \\ \hat{a}_2(t) &= \hat{U}^\dagger(t)\,\hat{a}_2\,\hat{U}^\dagger(t) = e^{-i\omega t}\left(\cos(\epsilon t/2)\,\hat{a}_2 - i\sin(\epsilon t/2)\,\hat{a}_1\right) \end{aligned}}$$

The $1 \leftrightarrow 2$ exchange symmetry of the Hamiltonian means that one of these solutions determines the other.

(c) Initial coherent state with

$$\langle \hat{x}_2\rangle_0 = A, \quad \langle \hat{p}_2\rangle_0 = 0 \implies \langle \hat{a}_2\rangle_0 = \sqrt{\frac{m\omega}{2\hbar}}\,A$$

$$\langle \hat{x}_1\rangle_0 = 0, \quad \langle \hat{p}_1\rangle_0 = 0 \implies \langle \hat{a}_1\rangle_0 = 0$$

$$(\Delta x_1)_0 = (\Delta x_2)_0 = \sqrt{\frac{\hbar}{2m\omega}}\,, \quad (\Delta p_1)_0 = (\Delta p_2)_0 = \sqrt{\frac{\hbar m\omega}{2}}$$

We can read off $\langle \hat{a}_1\rangle_t$ and $\langle \hat{a}_2\rangle_t$ from the solutions for the Heisenberg operators:

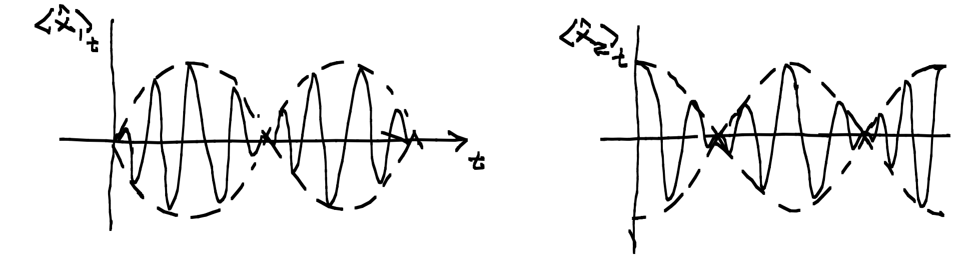
$$\langle \hat{a}_1\rangle_t = -i\sqrt{\frac{m\omega}{2\hbar}}\,A\,e^{-i\omega t}\sin(\epsilon t/2)$$

$$\langle \hat{a}_2\rangle_t = \sqrt{\frac{m\omega}{2\hbar}}\,A\,e^{-i\omega t}\cos(\epsilon t/2)$$

The position and momentum expectation values follow:

$$\boxed{\begin{aligned} \langle \hat{x}_1\rangle_t &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a}_1\rangle_t + \langle \hat{a}_1\rangle_t^*\right) = -A\sin\omega t\,\sin(\epsilon t/2) \\ \langle \hat{p}_1\rangle_t &= -i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle \hat{a}_1\rangle_t - \langle \hat{a}_1\rangle_t^*\right) = -m\omega A\cos\omega t\,\sin(\epsilon t/2) \\ \langle \hat{x}_2\rangle_t &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a}_2\rangle_t + \langle \hat{a}_2\rangle_t^*\right) = A\cos\omega t\,\cos(\epsilon t/2) \\ \langle \hat{p}_1\rangle_t &= -i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle \hat{a}_1\rangle_t - \langle \hat{a}_1\rangle_t^*\right) = -m\omega A\sin\omega t\,\cos(\epsilon t/2) \end{aligned}}$$

These solutions describe the familiar beats



For a coherent state, the uncertainties are constant

$$\boxed{(\Delta x_1)_t = (\Delta x_2)_t = \sqrt{\frac{\hbar}{2m\omega}} \quad , \quad (\Delta p_1)_t = (\Delta p_2)_t = \sqrt{\frac{\hbar m\omega}{2}}}$$

(d) The states

$$|+\rangle_{\vec{u}} = e^{-i\varphi/2}\cos(\theta/2)\,|+\rangle + e^{+i\varphi/2}\sin(\theta/2)\,|-\rangle$$

$$|-\rangle_{\vec{u}} = |+\rangle_{-\vec{u}}$$

are ± 1 eigenstates of $\hat{\vec{\sigma}}\cdot\vec{u} = \hat{\sigma}_{\vec{u}}$, i.e.,

$$\hat{\sigma}_{\vec{u}}|+\rangle_{\vec{u}} = |+\rangle_{\vec{u}} \qquad \Longleftrightarrow \qquad \sigma_{\vec{u}} = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|$$

$$\sigma_{\vec{u}}|-\rangle_{\vec{u}} = -|-\rangle_{\vec{u}}$$

Eigenvalue Eigenvector

$\hat{\sigma}_z$

$+1$ $\quad |+\rangle_{\vec{e}_z} = |+\rangle = |1,0\rangle = \frac{1}{\sqrt{2}}\left(\overline{|0,1\rangle} + \overline{|1,0\rangle}\right)$

-1 $\quad |+\rangle_{-\vec{e}_z} = |-\rangle = |0,1\rangle = \frac{1}{\sqrt{2}}\left(\overline{|0,1\rangle} - \overline{|1,0\rangle}\right)$

$\hat{\sigma}_x$

$+1$ $\quad |+\rangle_{\vec{e}_x} = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right) = \overline{|0,1\rangle}$

-1 $\quad |+\rangle_{-\vec{e}_x} = -\frac{i}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) = -\frac{i}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = -i\,\overline{|1,0\rangle}$

$\hat{\sigma}_y$

$+1$ $\quad |+\rangle_{\vec{e}_y} = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|+\rangle + e^{i\pi/4}|-\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|1,0\rangle + e^{i\pi/4}|0,1\rangle\right) = \frac{1}{\sqrt{2}}\left(\overline{|0,1\rangle} - i\,\overline{|1,0\rangle}\right)$$

-1 $\quad |+\rangle_{-\vec{e}_y} = \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|+\rangle + e^{-i\pi/4}|-\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|1,0\rangle + e^{-i\pi/4}|0,1\rangle\right) = \frac{1}{\sqrt{2}}\left(\overline{|0,1\rangle} + i\,\overline{|1,0\rangle}\right)$$

(e) To find the Hamiltonian in the 1-quantum subspace, one needs the matrix elements

$$\langle 1,0|\hat{H}|1,0\rangle = \hbar\omega_+ + \hbar\omega$$

$$\langle 0,1|\hat{H}|0,1\rangle = \hbar\omega_- + \hbar\omega$$

$$\langle 1,0|\hat{H}|0,1\rangle = 0 = \langle 0,1|\hat{H}|1,0\rangle$$

$$\Longrightarrow \hat{H} = \hbar\omega_+ \underbrace{|1,0\rangle\langle 1,0|}_{\equiv |+\rangle\langle +| = \frac{1}{2}(\hat{I}+\hat{\sigma}_z)} + \hbar\omega_- \underbrace{|0,1\rangle\langle 0,1|}_{= |-\rangle\langle -| = \frac{1}{2}(\hat{I}-\hat{\sigma}_z)} + \hbar\omega\overbrace{\left(|1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1|\right)}^{\hat{I}}$$

$$= \hbar\underbrace{\tfrac{1}{2}(\omega_+ + \omega_-)}_{\omega}\hat{I} + \hbar\underbrace{\tfrac{1}{2}(\omega_+ - \omega_-)}_{\frac{1}{2}\epsilon}\hat{\sigma}_z + \hbar\omega\hat{I}$$

$$\boxed{\hat{H} = 2\hbar\omega\hat{I} + \tfrac{1}{2}\hbar\epsilon\hat{\sigma}_z}$$

The $2\hbar\omega\hat{I}$ term, half zero-point energy and half the average energy of one quantum, could be omitted, since it only produces a phase.

(f) $|\psi(0)\rangle = \overline{|0,1\rangle} = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right) = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$

Method 1: $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-2i\omega t}e^{-i\hat{\sigma}_z\epsilon t/2}$ rotation by ϵt about z axis plus a phase

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\left(e^{-i\epsilon t/2}|+\rangle + e^{i\epsilon t/2}|-\rangle\right)$$

$$|\psi(t)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\left(e^{-i\epsilon t/2}|1,0\rangle + e^{+i\epsilon t/2}|0,1\rangle\right)$$

$$= e^{-2i\omega t}\left(\cos(\epsilon t/2)\overline{|0,1\rangle} - i\sin(\epsilon t/2)\overline{|1,0\rangle}\right)$$

Method 2: $\hat{U}(t) = e^{-2i\omega t}\left(\hat{I}\cos(\epsilon t/2) - i\hat{\sigma}_z\sin(\epsilon t/2)\right)$

$$|\psi(t)\rangle = \hat{U}(t)\overline{|0,1\rangle} = e^{-2i\omega t}\left(\cos(\epsilon t/2)\overline{|0,1\rangle} - i\sin(\epsilon t/2)\hat{\sigma}_z\overline{|0,1\rangle}\right)$$

$$= |1,0\rangle\underbrace{\langle 1,0\overline{|0,1\rangle}}_{1/\sqrt{2}} - |0,1\rangle\underbrace{\langle 1,0\overline{|0,1\rangle}}_{1/\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right)$$

$$= \overline{|1,0\rangle}$$

$$\boxed{\begin{aligned}|\psi(t)\rangle &= e^{-2i\omega t}\left(\cos(\epsilon t/2)\,\overline{|0,1\rangle} - i\sin(\epsilon t/2)\,\overline{|1,0\rangle}\right)\\ &= e^{-2i\omega t}\frac{1}{\sqrt{2}}\left(e^{-i\epsilon t/2}|1,0\rangle + e^{i\epsilon t/2}|0,1\rangle\right)\end{aligned}}$$

Method 3: $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{a}_+^\dagger\hat{a}_+\omega_+ t - i\hat{a}_-^\dagger\hat{a}_-\omega_- t}\,e^{-i\omega t}$

$$= e^{-i(\hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_-)\omega t}\,e^{-i(\hat{a}_+^\dagger\hat{a}_+ - \hat{a}_-^\dagger\hat{a}_-)\epsilon t/2}\,e^{-i\omega t}$$

$$|\psi(t)\rangle = \hat{U}(t)\,\overline{|0,1\rangle} = \frac{1}{\sqrt{2}}\left(\hat{U}(t)|1,0\rangle + \hat{U}(t)|0,1\rangle\right)$$

$$= e^{-2i\omega t}\frac{1}{\sqrt{2}}\left(e^{-i\epsilon t/2}|1,0\rangle + e^{+i\epsilon t/2}|0,1\rangle\right)$$