Phys 521

Homework #1

Solution Set

(ii) For angular momentum 0, the effective radial potential for central-force motion is $V(r)$ itself.

For radial uncertainty of about r, the radial momentum uncertainty must exceed $\frac{\hbar}{2r}$

So, $$E \gtrsim \frac{\left(\frac{\hbar}{2r}\right)^2}{2m} - \frac{\alpha}{r^s} = \frac{\hbar^2}{8mr^2} - \frac{\alpha}{r^s}$$

Ground-state energy ⇒ minimize E relative to r

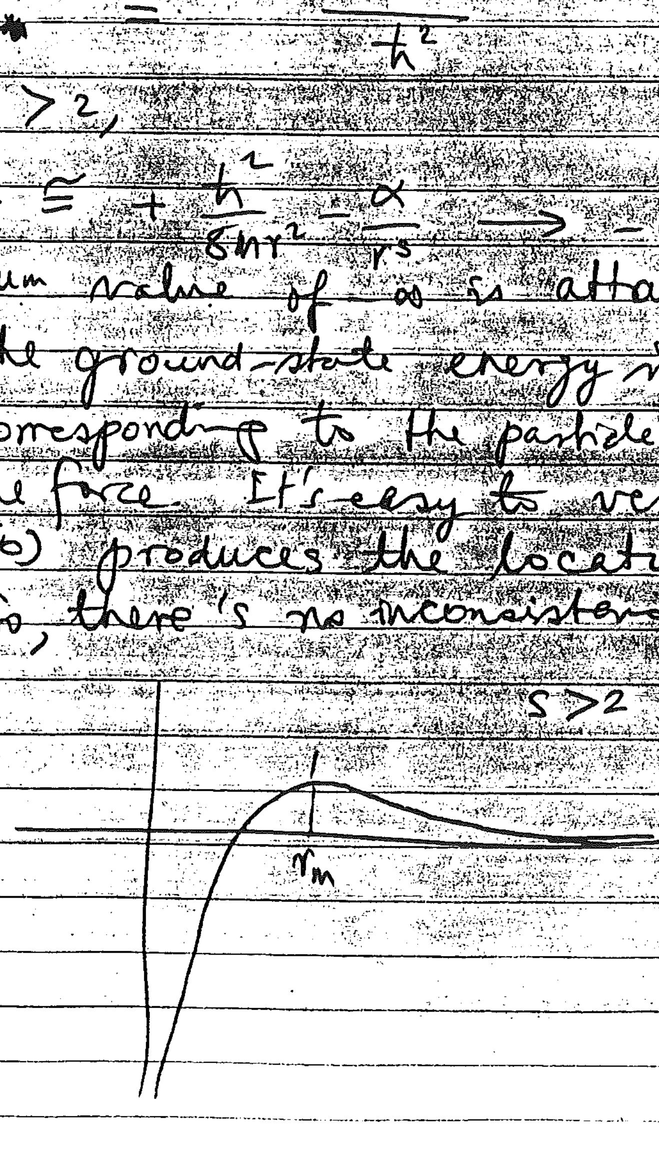
$$-\frac{2\hbar^2}{8mr_m^3} + \frac{\alpha s}{r_m^{s+1}} = 0$$

i.e. $$r_m^{s-2} = \frac{4m\alpha s}{\hbar^2} \qquad (*)$$

• $\underline{s>2}$: For $s>2$,

$$E \cong +\frac{\hbar^2}{8mr^2} - \frac{\alpha}{r^s} \longrightarrow -\infty \text{ as } r \to 0$$

So, the minimum value of $-\infty$ is attained when $r \to 0$. Thus, the ground-state energy in this case will be $-\infty$ corresponding to the particle sitting at the center of the force. It's easy to verify that for $s>2$, eq $(*)$ produces the location of E_{max}, not E_{min}. So, there's no inconsistency.



(2)

- $\underline{s<2}$: For $s<2$, (*) does give the correct location of E_{min}

$$E_{min} = \frac{\hbar^2}{8m r_m^2} - \frac{\alpha}{r_m^s}$$

$$= \frac{\hbar^2}{8m\left(\frac{4m\alpha s}{\hbar^2}\right)^{\frac{2}{s-2}}} - \frac{\alpha}{\left(\frac{4m\alpha s}{\hbar^2}\right)^{s/s-2}}$$

- $\underline{s=2}$: Then $E = \left(\frac{\hbar^2}{8m} - \alpha\right)\frac{1}{r^2}$

  * So, if $\alpha > \frac{\hbar^2}{8m}$, then $E = \frac{-ive}{r^2} \longrightarrow -\infty$ as $r \to 0$. Again, the minimum energy is $-\infty$ corresponding to particle at the center of force.
  * If $\alpha < \hbar^2/8m$, then the particle will recede to ∞; no binding takes place: $E_{min} = 0$

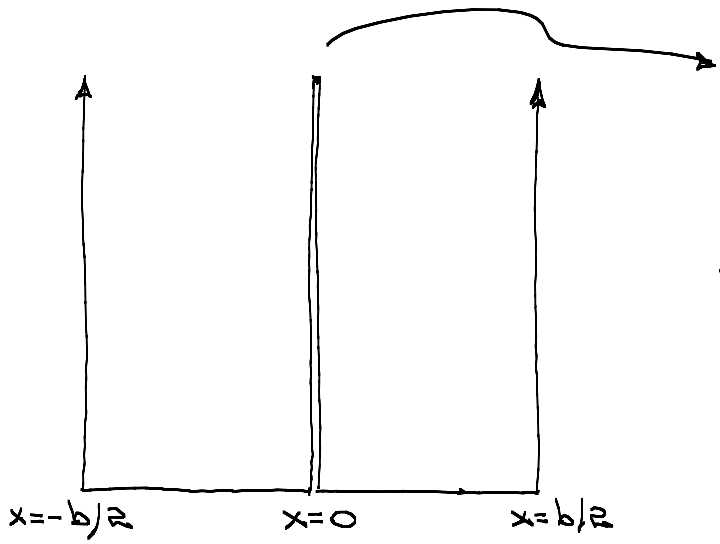
- Classically, for ang mom. $= 0$, as soon as s is +ive (attractive potential), the minimum energy state is $E_{min} = -\infty$ for $r=0$ (particle at force center). Q. Mechanically, only when $s \geq 2$ that such a singular state is possib. for $0 < s < 2$, one still has a finite energy ground state.

(3) (a) The 1^{st} cond'n is equivalent to the operator statement

$$(\hat{p} - \langle p\rangle)\psi = -\frac{1}{\lambda}(x - \langle x\rangle)\psi \quad , \quad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

~~[illegible]~~ =

1.2 $\quad V(x) = \begin{cases} \alpha\,\delta(x), & |x| < b/2 \\ \infty, & |x| > b/2 \end{cases}$



δ barrier can be viewed as a thin, high barrier of height V_0 and width a, with $V_0 a = \alpha$. Varying α corresponds to varying V_0, with a held fixed. If $\alpha < 0$, we have a well instead of a barrier.

Schrödinger equation: $-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi$

Boundary conditions:

① $\varphi(b/2) = 0$.

② $\varphi(-b/2) = 0$.

③ φ continuous at $x = 0$.

④ Integrate Schrödinger equation from $x = -\epsilon$ to $x = +\epsilon$.

$$-\frac{\hbar^2}{2m}\varphi'\Big|_{-\epsilon}^{+\epsilon} = \int_{-\epsilon}^{+\epsilon} dx\,\left(E - \alpha\,\delta(x)\right)\varphi(x) = -\alpha\,\varphi(0)$$

$$\Rightarrow \varphi'(+\epsilon) - \varphi'(-\epsilon) = \frac{2m\alpha}{\hbar^2}\varphi(0)$$

(a) Odd bound states

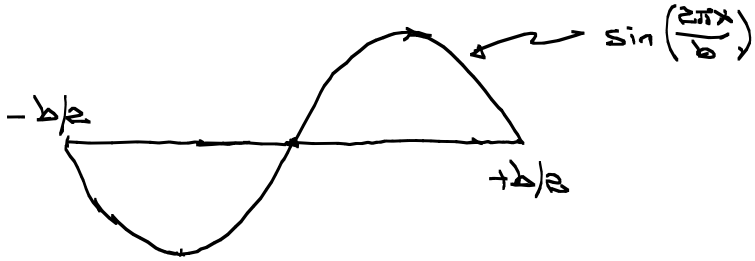
Since an odd wave function vanishes at the origin, it doesn't notice the δ barrier/well, so the problem reduces to an infinite square well with no δ barrier/well.

$$\varphi(x) = A\sin kx$$

BC's: ① and ② $\Leftrightarrow$ $0 = \varphi(b/2) = A\sin(kb/2) \Rightarrow kb/2 = n\pi/2$

③ and ④ are automatic since $\varphi(0) = 0$ $\qquad n = 2, 4, 6, \ldots$

$\boxed{\begin{array}{l}\underline{\text{Odd}} \text{ states: } k_n = n\pi/b, \quad n = 2, 4, 6, \ldots \\ \varphi_n(x) \propto \sin(k_n x) = \sin\left(\frac{n\pi x}{b}\right), \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mb^2} \\ n=2: \quad k = \frac{2\pi}{b}, \quad E = \frac{2\pi^2\hbar^2}{mb^2}, \quad \varphi(x) \propto \sin\left(\frac{2\pi x}{b}\right)\end{array}}$



<u>Even</u> bound states

(b) Barrier: $\alpha > 0$

It is clear that all bound states have positive energy. Thus, for $0 < |x| < b/2$, the general solution of the Schrödinger equation is a linear combination of $\cos kx$ and $\sin kx$, where $E = \hbar^2 k^2/2m$. The even states have wave functions

$$\varphi(x) = A\sin(k|x| - \phi) = \begin{cases} A\sin(kx - \phi), & 0 < x < b/2 \\ -A\sin(kx + \phi), & -b/2 < x < 0 \end{cases}$$

<u>BC's</u>: ① and ② $\Longleftrightarrow$ $\sin(kb/2 - \phi) = 0 \Rightarrow 0 = \sin(kb/2)\cos\phi - \cos(kb/2)\sin\phi$

or $\tan\phi = \tan(kb/2)$

③ automatic

④ $\varphi'(x) = \begin{cases} Ak\cos(kx - \phi), & 0 < x < b/2 \\ -Ak\cos(kx + \phi), & -b/2 < x < 0 \end{cases}$

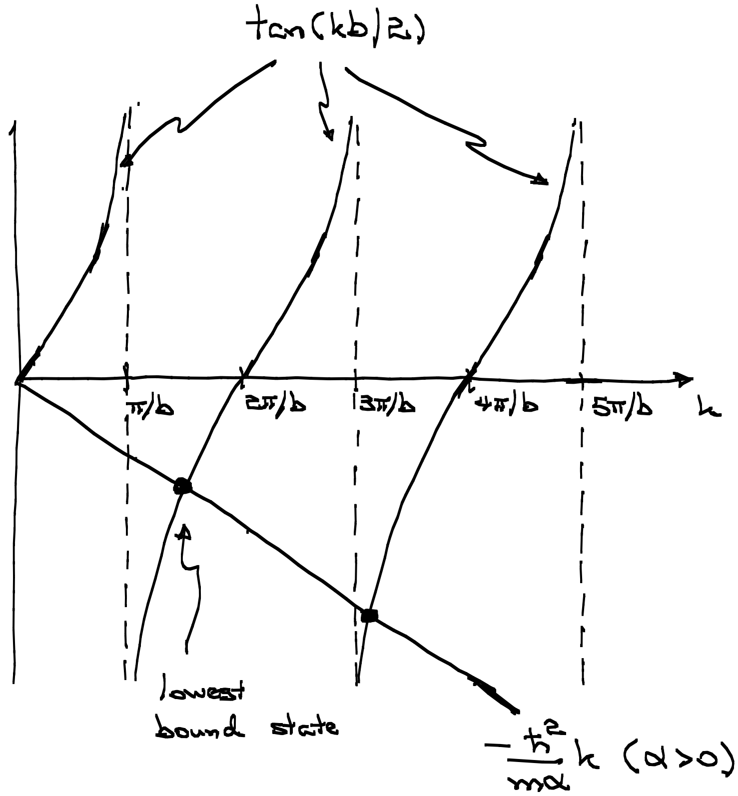
So

$$2Ak\cos\phi = \varphi'(+\epsilon) - \varphi'(-\epsilon) = \frac{2m\alpha}{\hbar^2}\varphi(0) = -\frac{2m\alpha}{\hbar^2}A\sin\phi$$

$$\Rightarrow \tan\phi = \frac{\sin\phi}{\cos\phi} = -\frac{\hbar^2 k}{m\alpha}$$

Combining ① and ④ gives

$$\boxed{\tan(kb/2) = \tan\phi = -\frac{\hbar^2 k}{m\alpha}, \quad E = \frac{\hbar^2 k^2}{2m}}$$

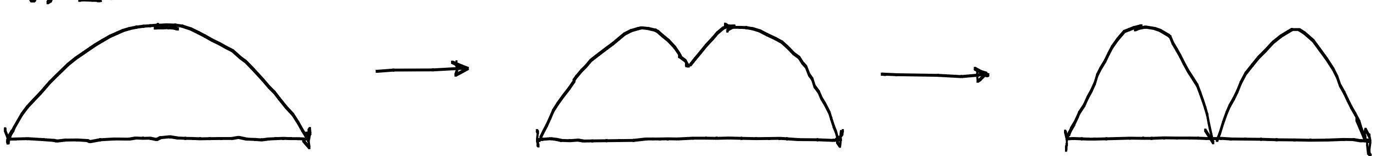


The even bound states have

$$\underset{\underset{\alpha = 0}{\uparrow}}{\frac{n\pi}{b}} < k_n < \underset{\underset{\alpha = \infty}{\uparrow}}{\frac{(n+1)\pi}{b}}, \quad n = 1, 3, 5, \ldots$$

When $\alpha = 0$, we have an infinite square well of width b and $k_n = n\pi/b$, $n = 1, 3, 5, \ldots$. When $\alpha = \infty$, we have two disconnected square wells of width $b/2$ and $k_n = (n+1)\pi/b$, $n = 1, 3, 5, \ldots$; the even and odd wave functions become degenerate and can be superposed to make energy eigenstates on the left and right sides of the barrier. We can illustrate what happens as α increases by considering $n = 1$.

$n = 1$:



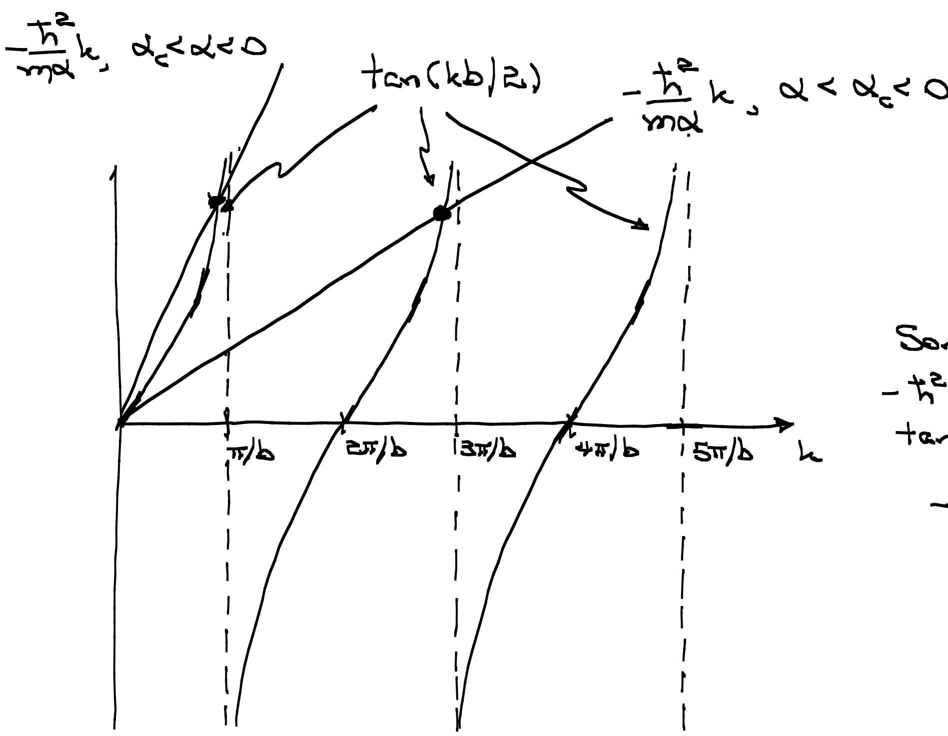
$\alpha = 0$ | typical $\alpha > 0$ | $\alpha = \infty$

The barrier pushes down on the wave function at $x = 0$ and eventually produces 2 separate humps (degenerate with $n = 2$).

(c) <u>Well</u>: $\alpha < 0$

Assuming $E>0$, we can appropriate the results of (b): the energy eigenvalues are determined by

$$\boxed{\tan(kb/2) = \tan\phi = -\frac{\hbar^2 k}{m\alpha}, \quad E = \frac{\hbar^2 k^2}{2m}}$$



Something happens when $-\hbar^2 k/m\alpha$ is tangent to $\tan(kb/2)$ at $x=0$, i.e., when $-\frac{\hbar^2}{m\alpha} = \frac{b}{2} \Leftrightarrow \alpha = -\frac{2\hbar^2}{mb} \equiv \alpha_c$.

The even bound states have

$$\underset{\alpha=-\infty}{\frac{(n-1)\pi}{b}} < k_n < \underset{\substack{\uparrow \\ \alpha=0}}{\frac{n\pi}{b}}, \quad n = 1, 3, 5, \ldots, \quad E_n = \frac{\hbar^2 k_n^2}{2m}$$

For $\alpha=0$, we have an infinite square well of width b and $k_n = n\pi/b$, $n = 1, 3, 5, \ldots$. For $\alpha = -\infty$, we have two disconnected square wells of width $b/2$ and $k_n = (n-1)\pi/b$, $n = 1, 3, 5, \ldots$.

This discussion is correct <u>except</u> for $n=1$. As α decreases from 0, k_1 decreases and goes to zero when $\alpha = \alpha_c$ (E_1 also goes to zero). This state appears to vanish, but what actually happens is that it becomes a state with negative energy.

We can find the negative-energy bound state by writing $E = -\hbar^2 p^2/2m$. Then $\varphi(x)$ is a linear combination of $\cosh px$ and $\sinh px$. Let's use

$$\varphi(x) = B \sinh(p|x| - \theta).$$

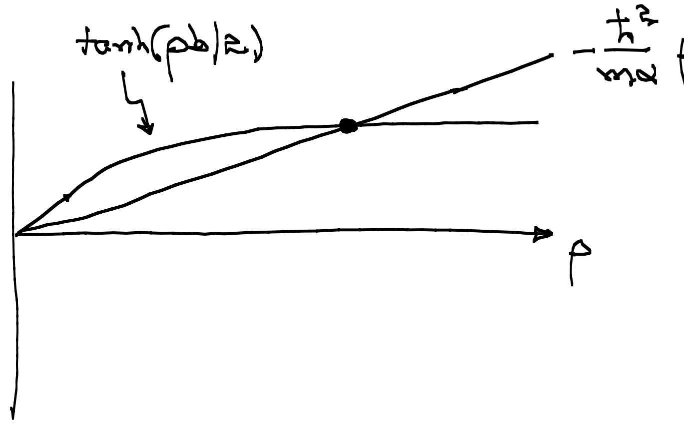
We can use our results from (b) with the substitutions

$$B = A/i, \quad p = ik, \quad \theta = i\phi.$$

This gives

Use $\tan(ix) = i \tanh x$

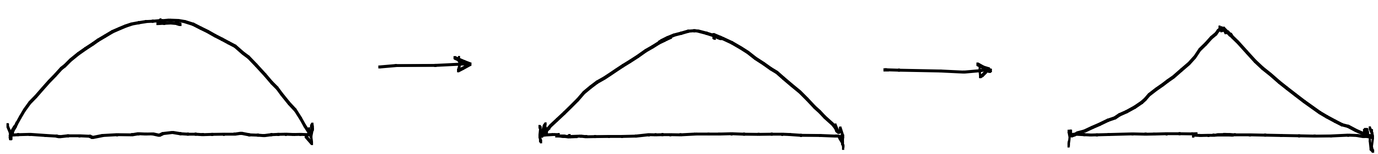
$$\tanh(pb/2) = \tanh\theta = -\frac{\hbar^2 p}{m\alpha}$$



There is a single negative-energy state if $\alpha < \alpha_c = -2\hbar^2/mb$. This is the $n=1$ state

We can see the effect of the well by considering how the wave functions for the first two even states change as α decreases from 0.

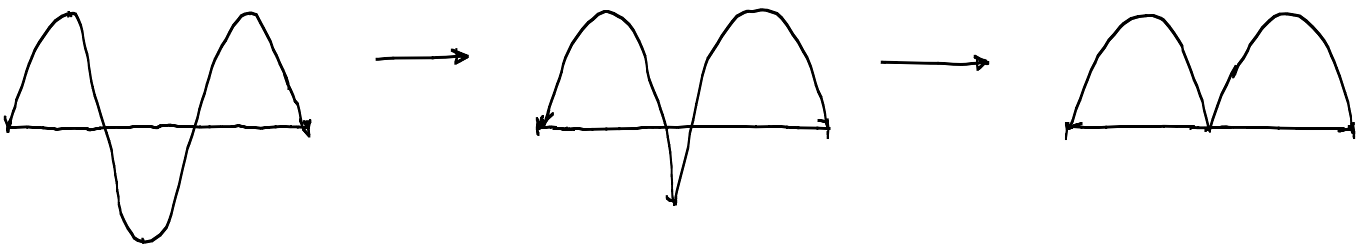
$n=1$:



$\alpha = 0$ typical $\alpha_c < \alpha < 0$ typical $\alpha < \alpha_c$

The well pulls up on the wave function at $x=0$ and eventually produces a wave function concentrated at $x=0$.

$n=3$:



$\alpha=0$ typical $\alpha<0$ $\alpha=-\infty$

The well pulls up on the wave function at $x=0$ and eventually produces two separate humps (degenerate with $n=2$).

1.3.

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \underbrace{\alpha\,\delta(x)}_{\downarrow\; V_0 a} \quad , \quad V(x) = -\alpha\,\delta(x)$$

Stationary states: $H\varphi(x) = E\varphi(x)$

(a) BC at $x=0$.

$$\underbrace{E\int_{-\epsilon}^{+\epsilon}\varphi(x')\,dx'}_{=\,0} = \int_{-\epsilon}^{+\epsilon} H\varphi(x')\,dx'$$

because φ isn't singular at $x=0$

$$= -\frac{\hbar^2}{2m}\underbrace{\int_{-\epsilon}^{+\epsilon} dx'\,\frac{d^2\varphi}{dx'^2}}_{\left.\frac{d\varphi}{dx}\right|_{-\epsilon}^{+\epsilon}} - \alpha\underbrace{\int_{-\epsilon}^{\epsilon} dx'\,\delta(x')\varphi(x')}_{\varphi(0)}$$

$$\therefore \boxed{\left.\frac{d\varphi}{dx}\right|_{x=+\epsilon} - \left.\frac{d\varphi}{dx}\right|_{x=-\epsilon} = -\frac{2m\alpha}{\hbar^2}\varphi(0)}$$

(b) $E<0$

$$x<0, \quad \varphi(x) = A_1 e^{\rho x} + \underbrace{A_1' e^{-\rho x}}_{A_1' = 0} = A_1 e^{\rho x}, \qquad \rho = \sqrt{-\frac{2mE}{\hbar^2}}$$

$$x>0, \quad \varphi(x) = \underbrace{A_2 e^{\rho x}}_{A_2 = 0} + A_2' e^{-\rho x} = A_2' e^{-\rho x}$$

Indeed, since the potential $V(x) = -\alpha\,\delta(x)$ is parity-symmetric, we know that $\varphi(x)$ can be chosen to be even or odd. If it is odd $(A_1 = -A_2')$, then it is discontinuous at $x=0$. So $\varphi(x)$ must be even $(A_1 = A_2' \equiv A)$:

$$\varphi(x) = A\, e^{-\rho|x|}.$$

The boundary condition gives us

$$-\frac{2m\alpha}{\hbar^2}\underbrace{\varphi(0)}_{A} = \underbrace{\left.\frac{d\varphi}{dx}\right|_{x=+\epsilon}}_{-A\rho} - \underbrace{\left.\frac{d\varphi}{dx}\right|_{x=-\epsilon}}_{A\rho}$$

$$\therefore\ -\frac{2m\alpha}{\hbar^2} = -2\rho \implies \boxed{\rho = \frac{m\alpha}{\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}}$$

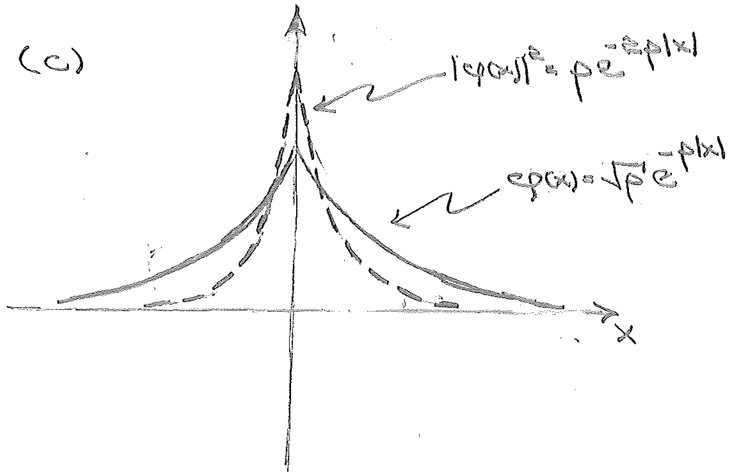
One bound state

Normalization: $$1 = \int_{-\infty}^{\infty} dx\, |\varphi(x)|^2 = 2A^2 \underbrace{\int_0^{\infty} dx\, e^{-2\rho x}}_{\frac{1}{2\rho}} = \frac{A^2}{\rho}$$

$$\implies A = \sqrt{\rho}$$

$$\boxed{\varphi(x) = \sqrt{\rho}\, e^{-\rho|x|}}$$

(c)



On <u>each</u> side of $x=0$, $|\varphi(x)|^2$ extends over a few decay lengths $1/2\rho$ of the exponential. A sensible estimate of Δx is

$$\boxed{\Delta x \sim 2(1/2\rho) = 1/\rho.}$$

Comment:

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\; x\,|\varphi(x)|^2 = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\; x^2\,|\varphi(x)|^2$$

$$\int_0^\infty dx\, x^n e^{-2\rho x} = \frac{n!}{(2\rho)^{n+1}}$$

$$= 2\rho \underbrace{\int_0^\infty dx\; x^2 e^{-2\rho x}}_{\frac{2}{(2\rho)^3}}$$

$$= \frac{1}{2\rho^2}$$

$$\langle(\Delta x)^2\rangle \equiv \langle (x - \langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\rho^2}$$

$$\langle(\Delta x)^2\rangle^{1/2} = \begin{pmatrix}\text{uncertainty}\\ \text{in } x\end{pmatrix} = \frac{1}{\sqrt{2}\,\rho} \longleftarrow \text{Exact answer}$$

(d) The position and momentum wave functions are related by

$$\varphi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, \bar{\varphi}(p)\, e^{ipx/\hbar} \iff \bar{\varphi}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, \varphi(x)\, e^{-ipx/\hbar}$$

$$\bar{\varphi}(p) = \sqrt{\frac{\rho}{2\pi\hbar}} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\rho|x|} e^{-ipx/\hbar}}$$

$$= \underbrace{\int_0^{\infty} dx\, e^{-(\rho + ip/\hbar)x}}_{\frac{1}{\rho + ip/\hbar}} + \underbrace{\int_{-\infty}^{0} dx\, e^{(\rho - ip/\hbar)x}}$$

$$= \int_0^{\infty} dx\, e^{-(\rho - ip/\hbar)x} = \frac{1}{\rho - ip/\hbar}$$

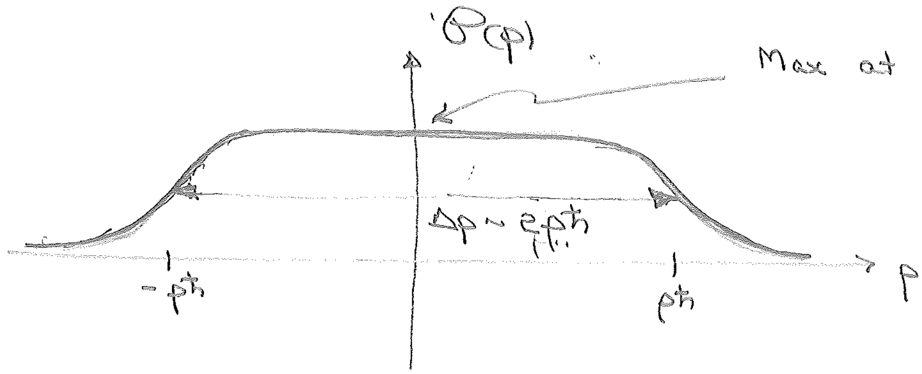
$$= \frac{1}{\rho + ip/\hbar} + \frac{1}{\rho - ip/\hbar}$$

$$= \frac{2\rho}{\rho^2 + (p/\hbar)^2}$$

$$= \frac{2\rho\hbar^2}{p^2 + (\rho\hbar)^2}$$

$$\bar{\varphi}(p) = \sqrt{\frac{2}{\pi(\rho\hbar)}} \frac{(\rho\hbar)^2}{p^2 + (\rho\hbar)^2} = \sqrt{\frac{2}{\pi(\rho\hbar)}} \frac{1}{1 + (p/\rho\hbar)^2}$$

$$\boxed{\mathcal{P}(p)\, dp = |\bar{\varphi}(p)|^2 dp = \frac{2}{\pi} \frac{dp}{\rho\hbar} \frac{1}{(1 + (p/\rho\hbar)^2)^2}}$$



$$\boxed{\begin{array}{c}\Delta p \sim 2\rho\hbar \\ \Delta x\, \Delta p \sim 2\hbar\end{array}}$$

Comment:

$$\langle p^n \rangle = \int_{-\infty}^{\infty} dp\, p^n\, \mathcal{P}(p)$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dp}{p\hbar} \frac{p^n}{(1 + (p/p\hbar)^2)^2}$$

$$\frac{p}{p\hbar} = \tan u$$

$$\frac{dp}{p\hbar} = \frac{du}{\cos^2 u}$$

$$= \frac{2}{\pi}(p\hbar)^n \int_{-\pi/2}^{+\pi/2} du\, \cos^2 u\, (\tan u)^n$$

Note that all odd moments are zero, and the even moments for $n \geq 4$ diverge

$n=0$: $$\int_{-\infty}^{\infty} dp\, \mathcal{P}(p) = \frac{2}{\pi} \underbrace{\int_{-\pi/2}^{+\pi/2} du\, \cos^2 u}_{\pi/2} = 1$$

$n=1$: $\langle p \rangle = 0$

$n=2$: $$\langle p^2 \rangle = \int_{-\infty}^{\infty} dp\, p^2\, \mathcal{P}(p) = \frac{2}{\pi}(p\hbar)^2 \underbrace{\int_{-\pi/2}^{+\pi/2} du\, \sin^2 u}_{\pi/2} = (p\hbar)^2$$

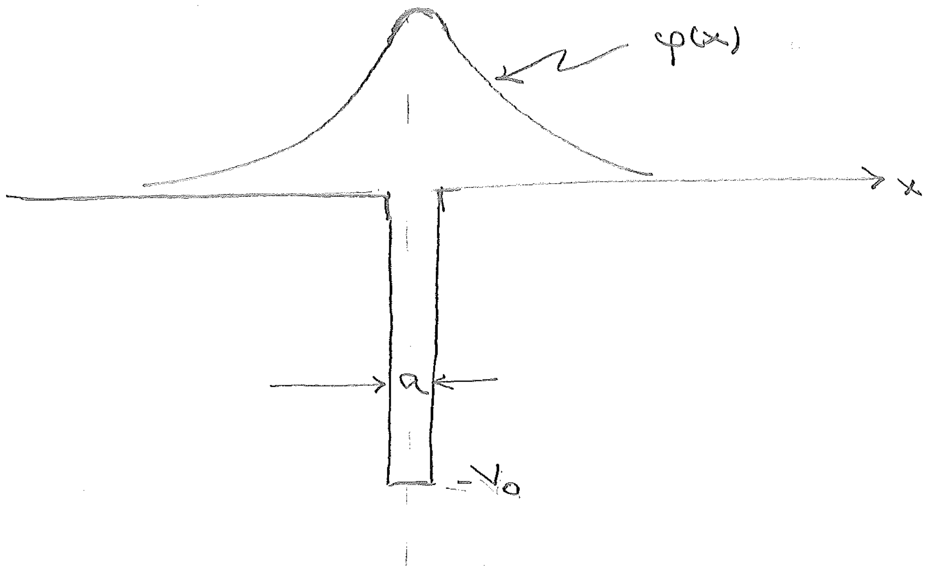
$$\langle (\Delta p)^2 \rangle = \langle (p - \langle p \rangle)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = (p\hbar)^2$$

$$\langle (\Delta p)^2 \rangle^{1/2} = p\hbar$$

Exact

$$\langle (\Delta x)^2 \rangle^{1/2} \langle (\Delta p)^2 \rangle^{1/2} = \frac{\hbar}{\sqrt{2}}$$

Comment: We can obtain the energy eigenvalue from our results for a square-well potential in the limit $V_0 \to \infty$, $a \to 0$, $V_0 a = \alpha =$ constant



The number of bound states is $1 + \mathrm{Int}\underbrace{\left(\frac{\sqrt{2mV_0}\,a}{\pi\hbar}\right)}_{\frac{\sqrt{2mV_0 a}\,\sqrt{a}}{\pi\hbar} \to 0} \to 1.$

The energy of this one bound state is determined by

$$\cos^2(ka/2) = 1 - \sin^2(ka/2) = (k/k_0)^2$$

$$k_0^2 = \frac{2mV_0}{\hbar^2} \to \infty$$

$$k^2 = \frac{2m(E+V_0)}{\hbar^2} \to \infty$$

$$\left(\frac{k}{k_0}\right)^2 = \frac{E+V_0}{V_0} = 1 + \frac{E}{V_0} \to 1$$

$$k_0^2 a^2 = \frac{2mV_0 a^2}{\hbar^2} = \frac{2mV_0 a}{\hbar^2}\, a \to 0$$

$$k^2a^2 = \left(\frac{k}{k_0}\right)^2 k_0^2 a^2 \longrightarrow 0$$

The equation for the eigenvalue becomes

$$\underbrace{1 - \left(\frac{ka}{2}\right)^2}_{\left(\frac{k}{k_0}\right)^2 (k_0 a/2)^2 = \frac{1}{4} k_0^2 a^2 = \frac{mV_0 a^2}{2\hbar^2}} = \left(\frac{k}{k_0}\right)^2 = 1 + \frac{E}{V_0}$$

$$\frac{E}{V_0} = -\frac{mV_0 a^2}{2\hbar^2}$$

$$\boxed{E = -\frac{m(V_0 a)^2}{2\hbar^2}}$$