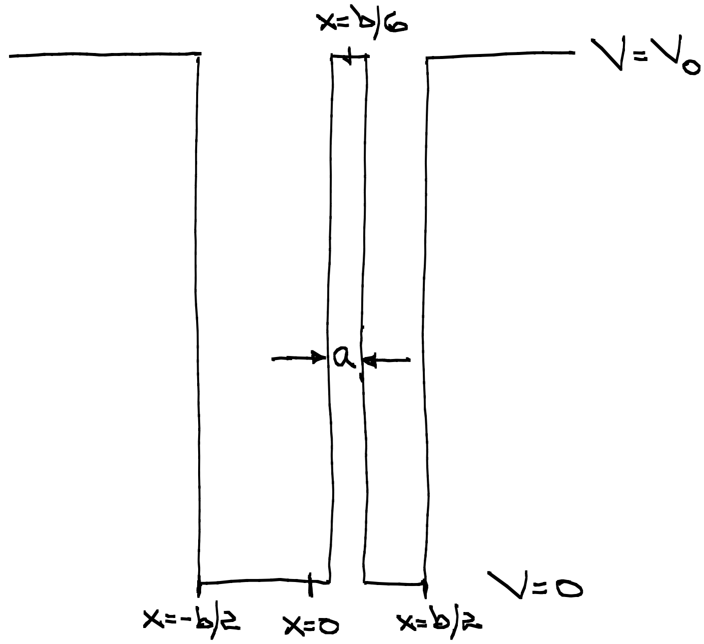
Phys 521

Homework #2

Solution Set

2.1



(a) The phase-space area available within the potential is

$$\underbrace{2\sqrt{2mV_0}}_{\text{momentum}}\underbrace{(b-a)}_{\text{position}} .$$

Since this area is large compared to h, the number of bound states is approximately

$$\boxed{\frac{2\sqrt{2mV_0}\,(b-a)}{h} = \frac{\sqrt{2mV_0}\,(b-a)}{\pi\hbar} = \frac{k_0(b-a)}{\pi} \gg 1.}$$

For the lowest levels, the exponential decay or growth across the barrier is governed by a decay constant given approximately by $k_0 = \sqrt{2mV_0}/\hbar$.

(b) Assume

$$\frac{2mV_0}{\hbar^2}ab = \frac{\sqrt{2mV_0}\,a}{\hbar}\frac{\sqrt{2mV_0}}{\hbar}b \ll 1.$$

The barrier is very thin, so the change in φ' across the barrier is very nearly given by $(2mV_0/\hbar^2)\,a\,\varphi(x_0)$. Across the entire well of width b, the change in $\varphi(x)$ that this derivative change can produce is limited by

$$\frac{2mV_0}{\hbar^2}\, a\, \varphi(x_0)\, b \ll \varphi(x_0).$$

This means that the barrier has scarcely any impact on the wave function and can thus be neglected.

Since the well is deep, the lowest levels are nearly the same as the levels of an infinite square well of width b:

$$k_n = \frac{n\pi}{b}, \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mb^2}, \quad n = 1, 2, 3, \ldots$$

$$\varphi_n = \sqrt{\frac{2}{b}} \begin{cases} \cos k_n x, & n \text{ odd} \\ \sin k_n x, & n \text{ even} \end{cases}$$

The lowest five bound states are approximately given by $n = 1, 2, 3, 4, 5$.

Another way to see why we can neglect the thin barrier is to treat the outer walls as infinitely high and to calculate the matrix elements of the Hamiltonian in the "unperturbed" states φ_n:

$$\langle \varphi_n | H | \varphi_m \rangle = \int_{-b/2}^{b/2} dx\, \varphi_n^*(x)\, H\, \varphi_m(x)$$

$$= E_n \delta_{nm} + V_0 \underbrace{\int_{x_0 - a/2}^{x_0 + a/2} dx\, \varphi_n^*(x)\varphi_m(x)}_{\simeq\, \varphi_n^*(x_0)\varphi_m(x_0)\, a}$$

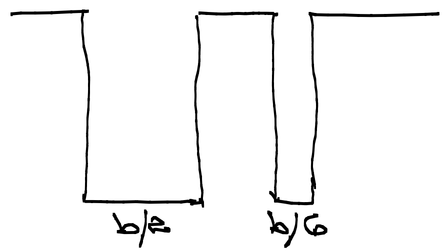
$$\simeq E_n \delta_{nm} + \frac{2V_0 a}{b} \times (\text{product of cosines and sines})$$

But $\dfrac{2V_0 a/b}{E_n} = \dfrac{1}{n^2\pi^2}\dfrac{2mV_0 ab}{\hbar^2} \ll 1$, so we can neglect the effect of the thin barrier.

(c)

$a = b/3$

$$\frac{\sqrt{2mV_0}\,a}{\hbar} \gg 1$$



In this situation, there is nearly complete exponential decay across the a-barrier (i.e., no tunneling), so effectively we have two isolated wells, a left well of width b/2 and a right well of width b/6. Both of these wells are deep, so their lowest bound states are very nearly those of infinitely deep wells.

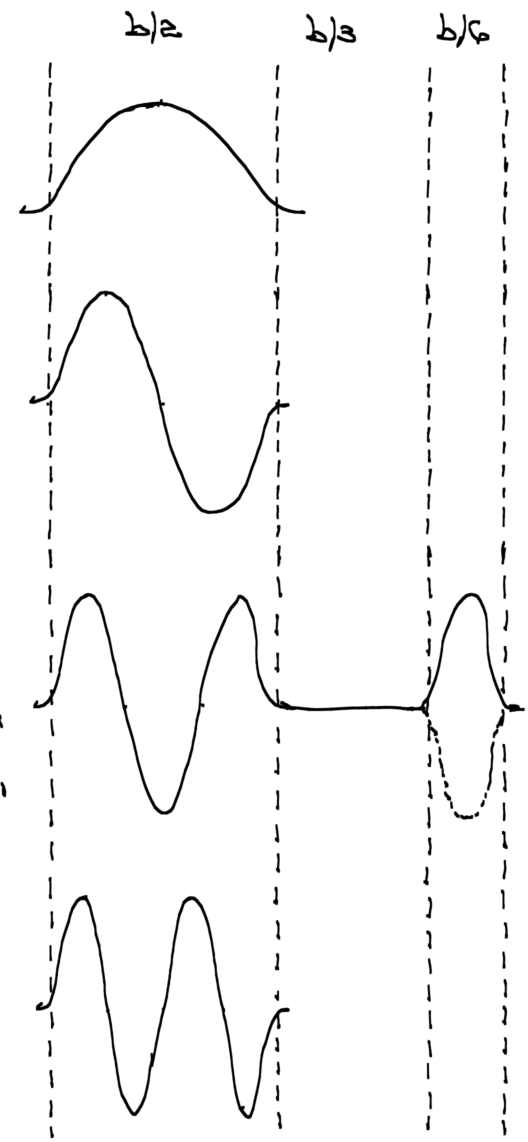
Left well

$$E_n = \frac{2n^2\pi^2\hbar^2}{m^2 b^2}$$

Right well

$$E_n = \frac{18n^2\pi^2\hbar^2}{m^2 b^2}$$

Lowest 5 bound states



① L, n=1 $\quad E = \dfrac{2\pi^2\hbar^2}{m^2 b^2}$

② L, n=2 $\quad E = \dfrac{8\pi^2\hbar^2}{m^2 b^2}$

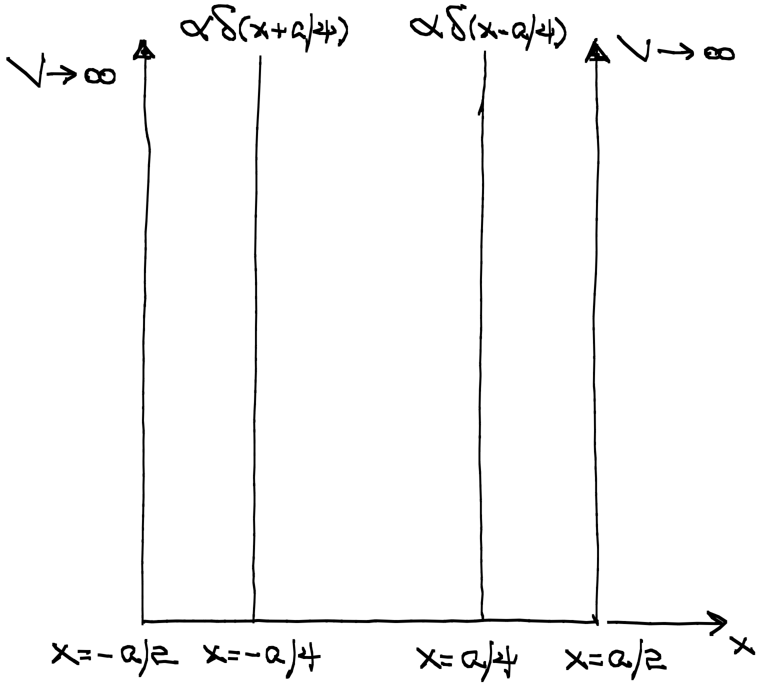
③ L, n=3 $\quad E = \dfrac{18\pi^2\hbar^2}{m^2 b^2}$

④ R, n=1 $\quad E = \dfrac{18\pi^2\hbar^2}{m^2 b^2}$

} degeneracy: the bound states are even and odd combinations of L3 and R1

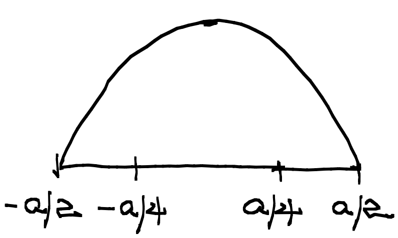
⑤ L, n=4 $\quad E = \dfrac{32\pi^2\hbar^2}{m^2 b^2}$

2.2.



(a)

$\alpha = 0$

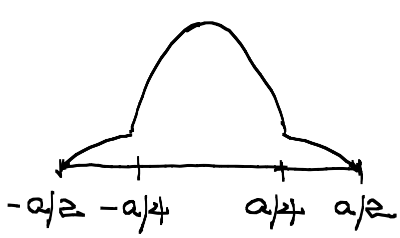


$\varphi(x) = A\cos kx, \; |x| < a/2$

$k = \pi/a$

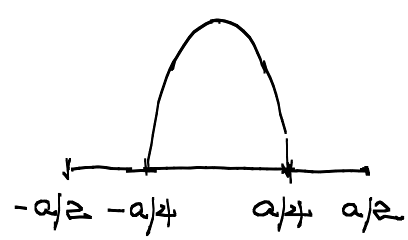
$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}$$

This is the ground-state wave function for a well of width a.



The barriers push down on the wave function. k and $E = \hbar^2 k^2/2m$ increase.

The wings $(a/4 < |x| < a/2)$ decrease in size because the wavelength can't fit in the wings.



$$\varphi(x) = \begin{cases} A\cos kx, & |x| < a/4 \\ 0, & a/4 < |x| < a/2 \end{cases}$$

$k = 2\pi/a$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{2\hbar^2 \pi^2}{ma^2}$$

For $|x| < a/4$, this is the ground-state wave function for a well of width a/2.

The wave function vanishes in the wings because the wavelength doesn't fit.

choose overall constant to make $A=1$

(b) Ground state: (even) $\varphi(x) = \begin{cases} A\cos kx, & |x| < a/4 \\ B\cos(k|x| - \phi), & a/4 < |x| \le a/2 \end{cases}$

We expect $\phi = 0$ and $B=1$ for $\alpha = 0$ and $B=0$ for $\alpha = \infty$.

$$E = \hbar^2 k^2/2m$$

BC's:
① $\varphi(a/2) = 0$
② $\varphi(-a/2) = 0$
③ $\varphi(a/4+\epsilon) = \varphi(a/4-\epsilon)$
④ $\varphi(-a/4-\epsilon) = \varphi(-a/4+\epsilon)$
⑤ $\varphi'(a/4+\epsilon) - \varphi'(a/4-\epsilon) = \frac{2m\alpha}{\hbar^2}\varphi(a/4)$
⑥ $\varphi'(-a/4-\epsilon) - \varphi'(-a/4+\epsilon) = -\frac{2m\alpha}{\hbar^2}\varphi(-a/4)$

Because $\varphi(x)$ is even, if we do ①, ③, and ⑤, we get ②, ④, and ⑥ automatically.

① $0 = \varphi(a/2) = B\cos(ka/2 - \phi) \implies ka/2 - \phi = \pi/2$ for ground state

③ $B\cos(ka/4 - \phi) = \varphi(a/4+\epsilon) = \varphi(a/4-\epsilon) = \cos(ka/4)$

⑤ $$\frac{2m\alpha}{\hbar^2}\cos(ka/4) = \frac{2m\alpha}{\hbar^2}\varphi(a/4)$$
$$= \varphi'(a/4+\epsilon) - \varphi'(a/4-\epsilon)$$
$$= -kB\sin(ka/4 - \phi) + k\sin(ka/4)$$

From ①, we have $ka/4 = \phi/2 + \pi/4$ and $ka/4 - \phi = -\phi/2 + \pi/4$. This suggests defining

$$\theta = \phi/2 + \pi/4 = ka/4, \quad \text{with } ka/4 - \phi = -\phi/2 + \pi/4 = \pi/2 - \theta$$

With these replacements, ③ and ⑤ become

Use
$\cos(\pi/2 - \theta) = \sin\theta$
$\sin(\pi/2 - \theta) = \cos\theta$

③ $B\sin\theta = \cos\theta \implies B = \cot\theta$

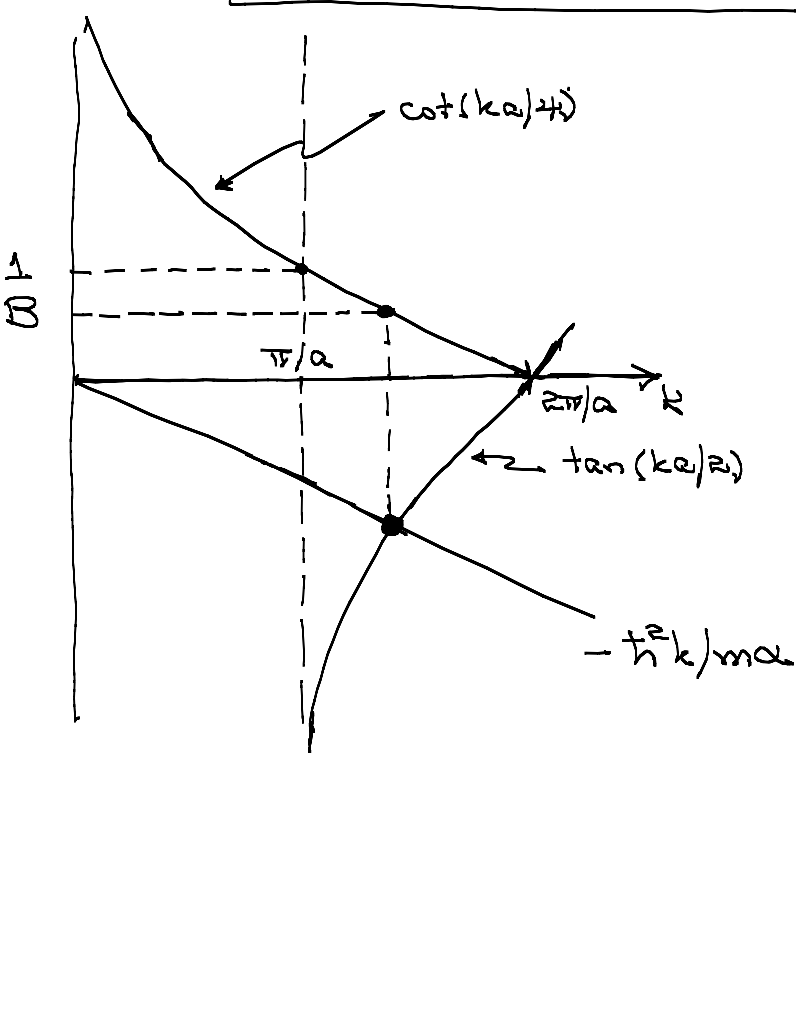
⑤ $\frac{2m\alpha}{\hbar^2}\cos\theta = -kB\cos\theta + k\sin\theta$

Plugging 3 into 5 to eliminate B from 5 gives

$$\frac{2m\alpha}{\hbar^2}\cos\theta = k\sin\theta - k\frac{\cos^2\theta}{\sin\theta}$$

$$\Rightarrow \frac{m\alpha}{\hbar^2}\underbrace{2\sin\theta\cos\theta}_{\sin 2\theta} = k\underbrace{(\sin^2\theta - \cos^2\theta)}_{-\cos 2\theta}$$

$$\Rightarrow \boxed{\begin{aligned} -\frac{\hbar^2 k}{m\alpha} &= \tan 2\theta = \tan(ka/2) \\ \phi &= ka/2 - \pi/2 \\ B &= \cot\theta = \cot(ka/4) \end{aligned}}$$



(c) One can see from the graph that when $\alpha = 0$, $k = \pi/a$ ($\phi = 0$) and $B = 1$. As α increases from zero, k increases (ϕ increases), and B decreases. As $\alpha \to \infty$, $k \to 2\pi/a$ ($\phi \to \pi/2$), and $B \to 0$.

2.3 $K_{I} \cdot 3$

$V(x) = -\alpha\,\delta(x)\,, \qquad E > 0$

(a) $x < 0; \quad \varphi(x) = A_1 e^{ikx} + A_1' e^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

$x > 0, \quad \varphi(x) = A_2 e^{ikx} + A_2' e^{-ikx}$

BC's at $x = 0$:

(1) φ continuous $\Rightarrow$ $\boxed{\varphi(0) = A_1 + A_1' = A_2 + A_2'}$

(2)

$$\underbrace{-\frac{2m\alpha}{\hbar^2}\varphi(0)}_{= -\frac{2m\alpha}{\hbar^2}(A_1 + A_1')} = \underbrace{\left.\frac{d\varphi}{dx}\right|_{x=+\epsilon}}_{= ik(A_2 - A_2')} - \underbrace{\left.\frac{d\varphi}{dx}\right|_{x=-\epsilon}}_{= ik(A_1 - A_1')}$$

$$\boxed{2i\frac{m\alpha}{\hbar^2 k}(A_1 + A_1') = A_2 - A_2' - A_1 + A_1'}$$

Solving for A_2 and A_2' in terms of A_1 and A_1', we get the M-matrix for a δ-potential.

$$\boxed{\begin{pmatrix} A_2 \\ A_2' \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} \qquad M = \begin{pmatrix} 1 + i\frac{m\alpha}{\hbar^2 k} & i\frac{m\alpha}{\hbar^2 k} \\ -i\frac{m\alpha}{\hbar^2 k} & 1 - i\frac{m\alpha}{\hbar^2 k} \end{pmatrix}}$$

We want the solution where no wave comes from the right, i.e., $A_2' = 0$. In this case,

$$A_2 = \left(1 + i\frac{m\alpha}{\hbar^2 k}\right) A_1 + i\frac{m\alpha}{\hbar^2 k} A_1'$$

$$0 = -i\frac{m\alpha}{\hbar^2 k} A_1 + \left(1 - i\frac{m\alpha}{\hbar^2 k}\right) A_1'$$

$$\boxed{\begin{aligned} A_1' &= i\,\frac{m\alpha/\hbar^2 k}{1 - i m\alpha/\hbar^2 k}\, A_1 \\ A_2 &= \frac{1}{1 - i m\alpha/\hbar^2 k}\, A_1 \end{aligned}}$$

We can choose $A_1 = 1$, and then let

$$\boxed{\begin{aligned} A_1' = A &= i\,\frac{m\alpha/\hbar^2 k}{1 - i m\alpha/\hbar^2 k} \\ A_2 = B &= \frac{1}{1 - i m\alpha/\hbar^2 k} \\ x<0,\quad \varphi(x) &= e^{ikx} + A e^{-ikx} \\ x>0,\quad \varphi(x) &= B e^{ikx} \end{aligned}}$$

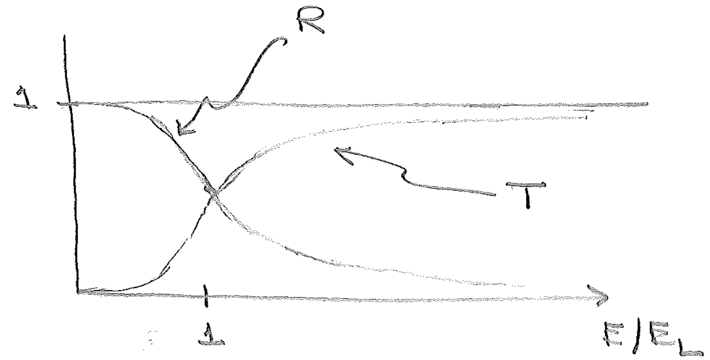
(b) $E_L \equiv \dfrac{m\alpha^2}{2\hbar^2}$

$$E/E_L = \frac{\hbar^2 k^2}{\cancel{2}m}\,\frac{\cancel{2}\hbar^2}{m\alpha^2} = \left(\frac{\hbar^2 k}{m\alpha}\right)^2, \quad \sqrt{E/E_L} = \frac{\hbar^2 k}{m\alpha}$$

$$A = i\,\frac{\sqrt{E_L/E}}{1 - i\sqrt{E_L/E}}, \quad B = \frac{1}{1 - i\sqrt{E_L/E}}$$

$$\begin{pmatrix}\text{reflection}\\ \text{coefficient}\end{pmatrix} = R = |A|^2 = \frac{E_L/E}{1+E_L/E} = \frac{1}{1+E/E_L}$$

$$\begin{pmatrix}\text{transmission}\\ \text{coefficient}\end{pmatrix} = T = |B|^2 = \frac{1}{1+E_L/E} = \frac{E/E_L}{1+E/E_L}$$



As $E \to \infty$, the particle doesn't notice the δ potential and is completely transmitted. The divergence in T when $E = -E_L$ is a sign of the bound state at that energy.

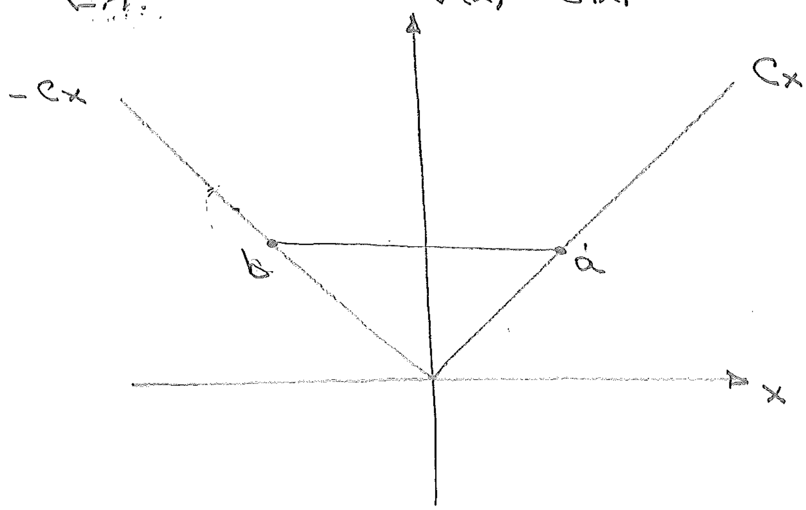
Comment: What if $\alpha < 0$, so that the δ-potential is a barrier rather than a well? Nothing except some phase changes. We can write

$$\sqrt{E/E_L} = -\frac{\hbar^2 k}{m\alpha}$$

$$A = -i\,\frac{\sqrt{E_L/E}}{1+i\sqrt{E_L/E}} \implies R = |A|^2 = \frac{1}{1+E/E_L}$$

$$B = \frac{1}{1+i\sqrt{E_L/E}} \implies T = |B|^2 = \frac{E/E_L}{1+E/E_L}$$

2A.



The WKB condition for bound states is that

$$\oint p\, dx = \left(n+\tfrac{1}{2}\right) h ,$$

where the integral is over a closed classical orbit. Rewriting this in terms of wave number $k = p/\hbar$,

$$\oint k\, dx = 2\pi\left(n+\tfrac{1}{2}\right) ,$$

which means that the phase accumulated over a closed orbit is a multiple of 2π plus an additional π. Since the integral over a periodic orbit goes from turning point b to turning point a and then back to b, we

Can rewrite the WKB condition as

$$\pi\left(n+\frac{1}{2}\right) = \int_b^a k\, dx = \frac{\sqrt{2m}}{\hbar}\int_b^a dx\,\sqrt{E-V(x)}\,.$$

In the case $V(x) = C|x|$, the turning points are determined by

$$E - C|x| = 0 \implies |x| = \frac{E}{C} \implies a = -b = \frac{E}{C}\,.$$

Thus we have

$$\pi\left(n+\frac{1}{2}\right) = \frac{\sqrt{2m}}{\hbar}\int_{-E/C}^{E/C} dx\,\sqrt{E-C|x|}$$

$$= \frac{2\sqrt{2m}}{\hbar}\int_0^{E/C} dx\,\sqrt{E-Cx}$$

$$= \frac{2\sqrt{2mE}}{\hbar}\int_0^{E/C} dx\,\sqrt{1-Cx/E}$$

$u = 1 - Cx/E$

$du = -\frac{C}{E}dx$

$$= \frac{2\sqrt{2mE}}{\hbar}\,\frac{E}{C}\,\underbrace{\int_0^1 du\cdot u^{1/2}}_{\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}}$$

$$= \frac{4\sqrt{2m}}{3\hbar C}E^{3/2}$$

$$E_n^{3/2} = \frac{3\pi\hbar C}{4\sqrt{2m}}\left(n+\frac{1}{2}\right) \implies \boxed{E_n = \left(\frac{3\pi\hbar C}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right)^{2/3}}$$

2.5. $K_I \cdot 4$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x)$$

(c) $$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} - \alpha\,\delta(x)\,\varphi(x) = E\,\varphi(x)$$

The position and momentum space wave functions are related by

$$\varphi(x) = \int_{-\infty}^{\infty}\frac{dp}{\sqrt{2\pi\hbar}}\,\bar{\varphi}(p)\,e^{ipx/\hbar} \iff \bar{\varphi}(p) = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi\hbar}}\,\varphi(x)\,e^{-ipx/\hbar}$$

$$\frac{d^2\varphi}{dx^2} = \int\frac{dp}{\sqrt{2\pi\hbar}}\underbrace{\left(\left(\frac{ip}{\hbar}\right)^2\bar{\varphi}(p)\right)}_{-\frac{p^2}{\hbar^2}\bar{\varphi}(p) = \left(\text{FT of } d^2\varphi/dx^2\right)} e^{ipx/\hbar}$$

$$\left(\begin{array}{c}\text{FT of}\\ \delta(x)\varphi(x)\end{array}\right) = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi\hbar}}\,\delta(x)\,\varphi(x)\,e^{-ipx/\hbar} = \frac{\varphi(0)}{\sqrt{2\pi\hbar}}$$

The equation for $\bar{\varphi}(p)$ is

$$\frac{p^2}{2m}\bar{\varphi}(p) - \frac{\alpha\,\varphi(0)}{\sqrt{2\pi\hbar}} = E\,\bar{\varphi}(p)$$

$$\left(\frac{p^2}{2m} - E\right)\bar{\varphi}(p) = \frac{\alpha\,\varphi(0)}{\sqrt{2\pi\hbar}}$$

$$\boxed{\bar{\varphi}(p) = \frac{\alpha\varphi(0)/\sqrt{2\pi\hbar}}{p^2/2m - E}}$$

②

because the bound state is even

(b) Bound state: $E<0$, $\varphi(0)\neq 0$

$$\bar{\varphi}(p) = \frac{\alpha\varphi(0)}{\sqrt{2\pi\hbar}} \frac{1}{p^2/2m - E} = \frac{1}{\sqrt{2\pi\hbar}} \frac{2m\alpha}{\hbar^2}\varphi(0)\frac{1}{(p/\hbar)^2+\rho^2} \qquad \rho^2 = -\frac{2mE}{\hbar^2} \qquad E = -\frac{\hbar^2\rho^2}{2m}$$

FT to find $\varphi(x)$

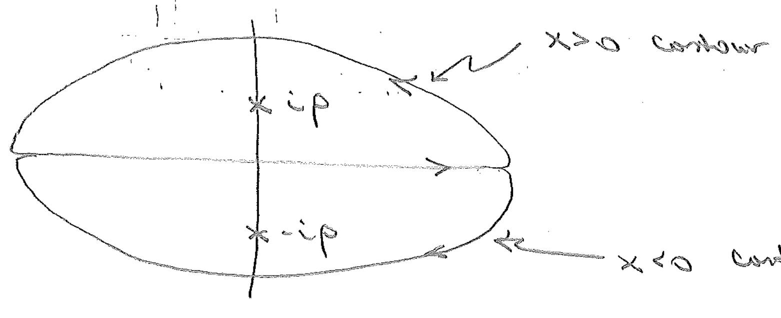
$$\varphi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\,\bar{\varphi}(p)\,e^{ipx/\hbar}$$

$$= \frac{2m\alpha}{\hbar^2}\varphi(0)\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\frac{e^{ipx/\hbar}}{(p/\hbar)^2+\rho^2}$$

$k = p/\hbar$
$$= \frac{2m\alpha}{\hbar^2}\varphi(0)\underbrace{\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k^2+\rho^2}} \qquad \rho^2 = -\frac{2mE}{\hbar^2}$$

Do this integral by contour integration

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{(k-i\rho)(k+i\rho)} = \left\{\begin{matrix} \dfrac{e^{-\rho x}}{2\rho}, & x>0 \\ \dfrac{e^{\rho x}}{2\rho}, & x<0 \end{matrix}\right\} = \frac{e^{-\rho|x|}}{2\rho}$$



$$\varphi(x) = \frac{m\alpha}{\rho\hbar^2}\varphi(0)\,e^{-\rho|x|}$$

Normalization $\Rightarrow \varphi(0) = \sqrt{\rho}$

Consistency at $x=0 \Rightarrow$ $\boxed{\rho = \frac{m\alpha}{\hbar^2} \Leftrightarrow E = -\frac{m\alpha^2}{2\hbar^2}}$

$$\boxed{\varphi(x) = \sqrt{\rho}\,e^{-\rho x} \Leftrightarrow \bar{\varphi}(p) = \sqrt{\frac{2}{\pi\rho\hbar}}\frac{\rho^2}{(p/\hbar)^2+\rho^2}}$$

(c) $E_k = \frac{\langle p^2 \rangle}{2m} \cdot \frac{1}{2m}\int_{-\infty}^{\infty} dp\; p^2 |\bar{\varphi}(p)|^2 = \frac{\hbar^2 p^2}{2m}$

↑ calculated in $K_I.2(c)$

Plug in the F.T of $\bar{\varphi}(p)$:

$$E_k = \frac{1}{2m}\int_{-\infty}^{\infty} dp\; p^2 \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, \varphi(x)\, e^{-ipx/\hbar} \int_{-\infty}^{\infty} dx'\, \varphi^*(x')\, e^{+ipx'/\hbar}$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx'\, \varphi^*(x') \int_{-\infty}^{\infty} dx\, \varphi(x)$$

$$\times \int_{-\infty}^{\infty} \frac{d(p/\hbar)}{2\pi} (p/\hbar)^2 e^{-i(x-x')(p/\hbar)}$$

Replace $p/\hbar$ by $\frac{1}{i}\frac{\partial}{\partial x}$, which comes out of the p-integral and gives a δ-fcn. After integration by parts, the δ-fcn. collapses the x and x' integrals to just one integral.

$$= -\frac{\partial^2}{\partial x^2} \underbrace{\int_{-\infty}^{\infty} \frac{d(p/\hbar)}{2\pi} e^{-i(x-x')p/\hbar}}_{\delta(x-x')}$$

$$= -\frac{\partial^2}{\partial x^2}\delta(x-x')$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx'\, \varphi^*(x') \int_{-\infty}^{\infty} dx\, \varphi(x) \frac{\partial^2}{\partial x^2}\delta(x-x')$$

Integrate by parts twice and and discard boundary terms on the grounds that $\varphi(\pm\infty)=0$ and $\varphi'(\pm\infty)=0$

$$= \int_{-\infty}^{\infty} dx\, \frac{\partial^2 \varphi}{\partial x^2}\, \delta(x-x')$$

$$= \frac{\partial^2 \varphi}{\partial x'^2}$$

$$\therefore \boxed{E_k = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\, \varphi^*(x) \frac{d^2\varphi}{dx^2}}$$

By the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} = \left(E + \alpha\,\delta(x)\right)\varphi,$$

$$E = -\frac{\hbar^2 p^2}{2m}$$

$$p = \frac{m\alpha}{\hbar^2}$$

So for the bound state

$$E_k = E\underbrace{\int_{-\infty}^{\infty} dx\,|\varphi(x)|^2}_{1} + \alpha\underbrace{\int_{-\infty}^{\infty} dx\,\delta(x)\,|\varphi(x)|^2}_{|\varphi(0)|^2}$$

$$E_k = \underset{-\frac{\hbar^2 p^2}{2m}}{\underset{\updownarrow}{E}} + \underbrace{\alpha|\varphi(0)|^2}_{\alpha p = \frac{\hbar^2 p^2}{m}}$$

$$\left(E = \underset{\substack{\uparrow \\ \text{kinetic energy}}}{E_k} \underbrace{- \alpha|\varphi(0)|^2}_{\text{potential energy}} \right)$$

$$\Longrightarrow E_k = \frac{\hbar^2 p^2}{2m}$$

$$\boxed{\begin{aligned} E_k &= \begin{pmatrix}\text{kinetic}\\ \text{energy}\end{pmatrix} = \frac{\hbar^2 p^2}{2m} \\ \begin{pmatrix}\text{potential}\\ \text{energy}\end{pmatrix} &= -\alpha|\varphi(0)|^2 = -\alpha p = -\frac{\hbar^2 p^2}{m} \\ E &= \begin{pmatrix}\text{kinetic}\\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{potential}\\ \text{energy}\end{pmatrix} = -\frac{\hbar^2 p^2}{2m} \end{aligned}}$$

(d) Continuum states: $E > 0$ (Extra credit: 10 points)

(C-T says you can't get these continuum stationary states from $\bar{\varphi}(p)$, but he is wrong.)

$$\bar{\varphi}(p) = \frac{\alpha\varphi(0)}{\sqrt{2\pi\hbar}} \frac{1}{p^2/2m - E} = \frac{2m\alpha}{\hbar^2}\varphi(0)\frac{1}{\sqrt{2\pi\hbar}}\frac{1}{(p/\hbar)^2 - 2mE/\hbar^2}$$

F.T. to get $\varphi(x)$:

$$\varphi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \bar{\varphi}(p)\, e^{ipx/\hbar}$$

$$= \frac{2m\alpha}{\hbar^2}\varphi(0)\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\frac{e^{ipx/\hbar}}{(p/\hbar)^2 - 2mE/\hbar^2}$$

$$= \frac{2m\alpha}{\hbar^2}\varphi(0)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k^2 - k_0^2}, \quad k_0^2 = \frac{2mE}{\hbar^2} > 0$$

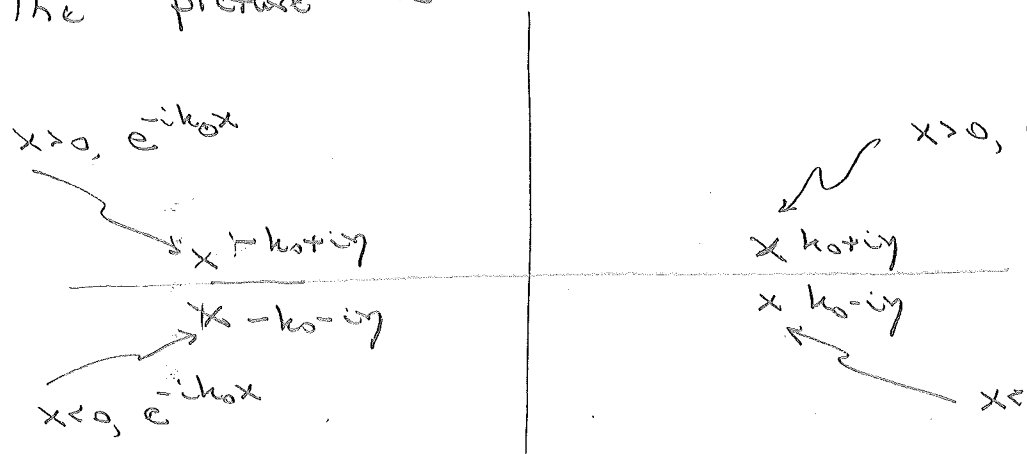
$$= \frac{2m\alpha}{\hbar^2}\varphi(0)\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\underbrace{\frac{1}{(k-k_0)(k+k_0)}}$$

$$\frac{1}{2k_0}\left(\frac{1}{k-k_0} - \frac{1}{k+k_0}\right)$$

There are two poles on the real axis, one at $k = k_0$ and one at $k = -k_0$. How those poles are moved slightly off the real axis determines which solution we get; i.e., it corresponds to the boundary conditions at $x = \pm\infty$. The pole at

$k = k_0$ gives e^{ik_0x} terms, i.e., waves travelling in the $+x$ direction. The pole at $k = -k_0$ gives e^{-ik_0x} terms, i.e., waves travelling in the $-x$ direction. Moving a pole above the real axis contributes to $x > 0$; moving it below contributes to $x < 0$.

The picture is



Each pole corresponds to one of the four waves in the problem.

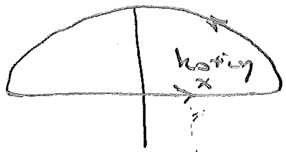
Suppose we want the solution where a wave propagating from $x = -\infty$ is partially reflected and partially transmitted. (i.e., no wave propagating from $x = +\infty$). Thus we do not want the pole at $k = -k_0 + i\eta$. So we write

$$\frac{1}{k-k_0} - \frac{1}{k+k_0} = \lim_{\eta\to 0}\left(\frac{\mu}{k-k_0-i\eta} + \frac{1-\mu}{k-k_0+i\eta} - \frac{1}{k+k_0+i\eta}\right),$$

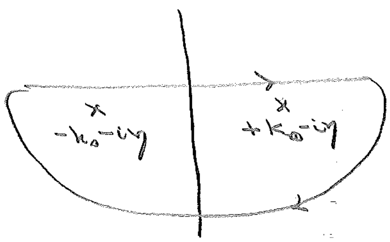
where μ is a constant to be determined.

$$\varphi(x) = \frac{m\alpha}{\hbar^2 k_0}\varphi(0)\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left(\frac{\mu}{\underset{k_0+i\eta}{k-k_0-i\eta}} + \frac{1-\mu}{\underset{k_0-i\eta}{k-k_0+i\eta}} - \frac{1}{\underset{-k_0-i\eta}{k+k_0+i\eta}}\right)$$

$x>0$: $$\varphi(x) = \frac{m\alpha}{\hbar^2 k_0}\varphi(0)\, i\mu\, e^{ik_0x} = i\frac{m\alpha}{\hbar^2 k_0}\varphi(0)\mu\, e^{ik_0x}$$



$x<0$: $$\varphi(x) = \frac{m\alpha}{\hbar^2 k_0}\varphi(0)(-i)\left((1-\mu)e^{ik_0x} - e^{-ik_0x}\right)$$



$$= i\frac{m\alpha}{\hbar^2 k_0}\varphi(0)\left((\mu-1)e^{ik_0x} + e^{-ik_0x}\right)$$

Consistency: $$\varphi(0) = \underbrace{i\mu\frac{m\alpha}{\hbar^2 k_0}}_{1}\varphi(0) \Longrightarrow \mu = -i\frac{\hbar^2 k_0}{m\alpha}$$

If we choose $i\frac{m\alpha}{\hbar^2 k_0}\varphi(0)(\mu-1) = 1$, then the stationary state becomes.

$$x<0, \quad \varphi(x) = e^{ik_0x} + \underbrace{\frac{1}{\mu-1}}e^{-ik_0x},$$

$$= \frac{\mu^{-1}}{1-\mu^{-1}} = i\frac{m\alpha/\hbar^2 k_0}{1 - i m\alpha/\hbar^2 k_0}$$

$$x>0, \quad \varphi(x) = \underbrace{\frac{\mu}{\mu-1}}e^{ik_0x},$$

$$\frac{1}{1-\mu^{-1}} = \frac{1}{1 - i m\alpha/\hbar^2 k_0}$$

Which is identical to the solution found in K_I.3.