Phys 521

Homework #5

Solution Set

5.1. C-T M_V-2

$$V(X,Y,Z) = \frac{1}{2}m\omega^2\left[\left(1+\frac{2\lambda}{3}\right)(X^2+Y^2)+\left(1-\frac{4\lambda}{3}\right)Z^2\right]$$

$$\omega \geq 0, \quad 0 \leq \lambda < \frac{3}{4}$$

(a) The Hamiltonian can be written as

$$H = \frac{P_x^2+P_y^2+P_z^2}{2m} + V(X,Y,Z)$$

$$= H_x + H_y + H_z$$

$$H_x = P_x^2/2m + \frac{1}{2}m\underbrace{\omega^2(1+2\lambda/3)}_{\omega_\perp^2}X^2$$

$$H_y = P_y^2/2m + \frac{1}{2}m\overbrace{\omega^2(1+2\lambda/3)}Y^2$$

$$H_z = P_z^2/2m + \frac{1}{2}m\underbrace{\omega^2(1-4\lambda/3)}_{\omega_z^2}Z^2$$

The x, y, & z degrees of freedom are separate harmonic oscillators for each of which we know the energy eigenstates and eigenvalues:

$$H_x|\varphi_{n_x}\rangle = \underbrace{\hbar\omega_\perp(n_x+\tfrac{1}{2})}_{E_{n_x}}|\varphi_{n_x}\rangle, \quad n_x = 0,1,2\ldots$$

$$H_y|\varphi_{n_y}\rangle = \underbrace{\hbar\omega_\perp(n_y+\tfrac{1}{2})}_{E_{n_y}}|\varphi_{n_y}\rangle, \quad n_y = 0,1,2,\ldots$$

$$H_z|\varphi_{n_z}\rangle = \hbar\omega_z\left(n_z+\tfrac{1}{2}\right)|\varphi_{n_z}\rangle, \quad n_z = 0, 1, 2, \ldots$$

The energy eigenstates of H are tensor products $|\varphi_{n_x n_y n_z}\rangle = |\varphi_{n_x}\rangle \otimes |\varphi_{n_y}\rangle \otimes |\varphi_{n_z}\rangle$:

$$H|\varphi_{n_x n_y n_z}\rangle = (H_x + H_y + H_z)|\varphi_{n_x n_y n_z}\rangle$$
$$= \underbrace{(E_{n_x} + E_{n_y} + E_{n_z})}_{E_{n_x n_y n_z}}|\varphi_{n_x n_y n_z}\rangle$$

$$\boxed{\begin{aligned}&\text{Energy eigenstates: } |\varphi_{n_x n_y n_z}\rangle = |\varphi_{n_x}\rangle \otimes |\varphi_{n_y}\rangle \otimes |\varphi_{n_z}\rangle\\ &\text{Energy eigenvalues: } E_{n_x n_y n_z} = E_{n_x} + E_{n_y} + E_{n_z}\\ &\qquad = \hbar\omega_\perp(n_x + n_y + 1)\\ &\qquad\quad + \hbar\omega_z\left(n_z + \tfrac{1}{2}\right)\end{aligned}}$$

(b) ① Ground state: $n_x = n_y = n_z = 0$

$$E_{000} = \hbar\left(\omega_\perp + \tfrac{1}{2}\omega_z\right) = \hbar\omega\left(\sqrt{1+2\lambda/3} + \tfrac{1}{2}\sqrt{1-4\lambda/3}\right)$$

$|\varphi_{000}\rangle = |\varphi_0\rangle \otimes |\varphi_0\rangle \otimes |\varphi_0\rangle \longleftarrow$ even parity

② First excited state: Notice that $\omega_\perp \geq \omega_z$

I. $\omega_\perp = \omega_z$ $(\lambda = 0)$

$$\left.\begin{array}{ccc} n_x & n_y & n_z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\} \quad E = E_{000} + \hbar\omega \qquad \text{odd parity}$$

II. $\omega_\perp > \omega_z$ $(\lambda > 0)$

n_x	n_y	n_z		
0	0	1	$E = E_{000} + \hbar\omega_z$ $= E_{000} + \hbar\omega\sqrt{1 - 4\lambda/3}$	odd parity

③ Second excited state

I. $\omega_\perp = \omega_z$ $(\lambda = 0)$

n_x	n_y	n_z
2	0	0
0	2	0
0	0	2
1	1	0
1	0	1
0	1	1

$E = E_{000} + 2\hbar\omega$ even parity

II. $2\omega_z > \omega_\perp$ $(0 < \lambda < \frac{1}{2})$

n_x	n_y	n_z
1	0	0
0	1	0

$$E = E_{000} + \hbar\omega_\perp$$
$$= E_{000} + \hbar\omega\sqrt{1 + 2\lambda/3}$$

odd parity

III. $2\omega_z = \omega_\perp$ $\left(\lambda = \frac{1}{2}\right)$ $\qquad \omega_\perp = 2\omega_z = \frac{2\omega}{\sqrt{3}}$

n_x	n_y	n_z		
1	0	0	$E = E_{000} + 2\hbar\omega/\sqrt{3}$	odd parity
0	1	0		
0	0	2		even parity

IV. $2\omega_z < \omega_\perp$ $\left(\frac{1}{2} < \lambda < \frac{3}{4}\right)$

n_x	n_y	n_z		
0	0	2	$E = E_{000} + 2\hbar\omega_z$	even parity

$$= E_{000} + 2\hbar\omega\sqrt{1 - 4\lambda/3}$$

5.2. $a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + iP/m\omega\right)$ $\qquad X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$

$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - iP/m\omega\right)$ $\qquad P/m\omega = -i\sqrt{\frac{\hbar}{2m\omega}}\left(a - a^\dagger\right)$

$H = \hbar\omega\, a^\dagger a$

MUS: $(a\cosh r + a^\dagger \sinh r)|\psi_r\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(e^r X + i e^{-r} P/m\omega\right)|\psi_r\rangle = 0$

(a) $(e^r X + i e^{-r} P/m\omega)|\psi_r\rangle = 0$

$0 = \langle\psi_r|(e^r X + i e^{-r} P/m\omega)|\psi_r\rangle \implies \boxed{\begin{array}{c}\langle X\rangle = 0 \\ \langle P\rangle = 0\end{array}}$

$0 = \langle\psi_r|(e^r X + i e^{-r} P/m\omega)^2|\psi_r\rangle$

$= \langle\psi_r|\left(e^{2r}X^2 - e^{-2r}P^2/m^2\omega^2 + i(XP + PX)/m\omega\right)|\psi_r\rangle$

$\implies 0 = e^{2r}\langle X^2\rangle - e^{-2r}\frac{\langle P^2\rangle}{m^2\omega^2}$

$$\boxed{0 = \langle XP + PX\rangle}$$

$0 = \langle\psi_r|(e^r X - i e^{-r} P/m\omega)(e^r X + i e^{-r} P/m\omega)|\psi_r\rangle$

$= \langle\psi_r|\left(e^{2r}X^2 + e^{-2r}P^2/m^2\omega^2 + i\underbrace{(XP - PX)}_{i\hbar}/m\omega\right)|\psi_r\rangle$

$\implies e^{2r}\langle X^2\rangle + e^{-2r}\frac{\langle P^2\rangle}{m^2\omega^2} = \frac{\hbar}{m\omega}$

$$\implies \boxed{\langle X^2\rangle = \frac{\hbar}{2m\omega}e^{-2r}, \qquad \langle P^2\rangle = \frac{\hbar m\omega}{2}e^{2r}}$$

(b) $S(r,\theta) = \exp\left(\frac{1}{2}r\left(a^2 e^{-2i\theta} - (a^\dagger)^2 e^{+2i\theta}\right)\right)$

$$S(r,\theta)\, a\, S^\dagger(r,\theta) = e^{\overbrace{\frac{1}{2}r(a^2e^{-2i\theta} - (a^\dagger)^2 e^{+2i\theta})}^{B}}\ \underset{\uparrow \atop A}{a}\ e^{\overbrace{-\frac{1}{2}r(a^2e^{-2i\theta} - (a^\dagger)^2 e^{+2i\theta})}^{-B}}$$

$B^\dagger = -B$

$$= a + [B,a] + \frac{1}{2!}[B,[B,a]] + \frac{1}{3!}[B,[B,[B,A]]] + \dots$$

$$[B,a] = -\frac{1}{2}r\underbrace{[(a^\dagger)^2, a]}\, e^{+2i\theta} = r a^\dagger e^{+2i\theta}$$

$= a^\dagger a^\dagger a - a a^\dagger a^\dagger$

$= a^\dagger(a a^\dagger - 1) - a a^\dagger a^\dagger$

$= a^\dagger a a^\dagger - a^\dagger - a a^\dagger a^\dagger$

$= (a a^\dagger - 1) a^\dagger - a^\dagger - a a^\dagger a^\dagger$

$= -2a^\dagger$

OR $[a,(a^\dagger)^2] = 2a^\dagger [a, a^\dagger] = 2a^\dagger$

$Ba - aB = r a^\dagger e^{+2i\theta}$

$a^\dagger B^\dagger - B^\dagger a^\dagger = r a e^{-2i\theta}$

$\quad\ \ \shortparallel$

$-a^\dagger B + B a^\dagger = [B, a^\dagger]$

$\therefore\ [B, a^\dagger] = r a e^{-2i\theta}$

$[B,a] = r a^\dagger e^{+2i\theta}$

$[B,[B,a]] = r^2 a$

$$S_R S^\dagger = a + r a^\dagger e^{+2i\theta} + \frac{1}{2!} r^2 a + \frac{1}{3!} r^3 a^\dagger e^{+2i\theta} + \dots$$

$$= a \underbrace{\sum_{k \text{ even}} \frac{r^k}{k!}}_{\frac{e^r + e^{-r}}{2} = \cosh r} + a^\dagger e^{+2i\theta} \underbrace{\sum_{k \text{ odd}} \frac{r^k}{k!}}_{\frac{e^r - e^{-r}}{2} = \sinh r}$$

$$\boxed{S_R S^\dagger = a \cosh r + a^\dagger e^{+2i\theta} \sinh r}$$

(c) $|\phi_{r,\theta}\rangle = S(r,\theta)|\varphi_0\rangle$

$$(a\cosh r + a^\dagger \sinh r)|\phi_{r,0}\rangle = (a\cosh r + a^\dagger \sinh r) S(r,0)|\varphi_0\rangle$$

$$= S(r,0) \underbrace{S^\dagger(r,0)(a\cosh r + a^\dagger \sinh r) S(r,0)}_{a} |\varphi_0\rangle$$

$$= S(r,0) \underbrace{a|\varphi_0\rangle}_{0}$$

$$= 0$$

$|\phi_{r,0}\rangle$ is an eigenstate of $a\cosh r + a^\dagger \sinh r$, so we can take $|\psi_r\rangle = |\phi_{r,0}\rangle$

(d) $|\psi(0)\rangle = |\phi_{r,0}\rangle = S(r,0)|\varphi_0\rangle$

$$|\psi(t)\rangle = \underbrace{U(t,0)}_{e^{-\frac{i}{\hbar}Ht} = e^{-i\omega t a^\dagger a}}|\psi(0)\rangle = U(t,0)\,S(r,0)\,|\varphi_0\rangle$$

$$|\psi(t)\rangle = \underbrace{U(t,0)\;S(r,0)\;U^\dagger(t,0)}_{S(r,-\omega t)}\;\underbrace{U(t,0)\,|\varphi_0\rangle}_{|\varphi_0\rangle}$$

Use $U(t,0)\,a\,U^\dagger(t,0) = e^{-i\omega t a^\dagger a}\,a\,e^{i\omega t a^\dagger a} = a\,e^{+i\omega t}$

$U(t,0)\,a^\dagger\,U^\dagger(t,0) = e^{-i\omega t a^\dagger a}\,a^\dagger\,e^{+i\omega t a^\dagger a} = a^\dagger e^{-i\omega t}$

$$U(t,0)\,S(r,0)\,U^\dagger(t,0) = \exp\left(\tfrac{1}{2}r\left(a^2 e^{2i\omega t} - (a^\dagger)^2 e^{-2i\omega t}\right)\right)$$

$$= S(r,-\omega t)$$

$$\boxed{|\psi(t)\rangle = S(r,-\omega t)\,|\varphi_0\rangle = |\phi_{r,-\omega t}\rangle \quad \begin{array}{l} r(t) = r \\ \theta(t) = -\omega t \end{array}}$$

$$|\psi(t=\pi/2\omega)\rangle = \underbrace{S(r,-\pi/2)}_{\downarrow}\,|\varphi_0\rangle = S(-r,0)\,|\varphi_0\rangle = |\phi_{-r,0}\rangle$$

$$= \exp\left(\tfrac{1}{2}r\left(a^2 e^{i\pi} - (a^\dagger)^2 e^{-i\pi}\right)\right)$$

$$= \exp\left(-\tfrac{1}{2}r\left(a^2 - (a^\dagger)^2\right)\right)$$

$$= S(-r,0)$$

$$\boxed{|\psi^{\mu}(t=\pi/2\omega)\rangle = |\phi_{-r,0}\rangle = |\psi^{\mu}_{-r}\rangle \text{ is a MUS.}}$$

$$\boxed{t=\frac{\pi}{2\omega}: \quad \langle X^2\rangle = \frac{\hbar}{2m\omega}e^{+2r}, \quad \langle P^2\rangle = \frac{\hbar m\omega}{2}e^{-2r}, \quad \langle XP+PX\rangle = 0}$$

Wave function:

$$0 = \langle x | (e^{r} X + i e^{-r} P/m\omega) | \psi_r \rangle$$

$$= e^{r} x \langle x | \psi_r \rangle + \frac{\hbar}{m\omega} e^{-r} \frac{d}{dx} \langle x | \psi_r \rangle$$

$$\frac{d}{dx} \psi_r(x) = -\frac{m\omega}{\hbar} e^{2r} x \, \psi_r(x)$$

$$\frac{d \ln \psi_r(x)}{dx} = -\frac{m\omega}{\hbar} e^{2r} x$$

$$\Longrightarrow \ln \psi_r(x) = -\frac{m\omega}{2\hbar} e^{2r} x^2 + (\text{const})$$

$$\psi_r(x) = C \exp\left(-\frac{m\omega}{2\hbar} e^{2r} x^2\right)$$

$$= C \exp\left(-\frac{x^2}{2(\hbar/m\omega) e^{-2r}}\right)$$

$$|\psi_r(x)|^2 = |C|^2 \exp\left(-\frac{x^2}{2(\hbar/2m\omega) e^{-2r}}\right)$$

$$\sigma^2 = (\Delta X)^2 = \langle X^2 \rangle$$

$$\Longrightarrow |C|^2 = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} = \left(\frac{1}{2\pi(\hbar/2m\omega) e^{-2r}}\right)^{1/2} = \left(\frac{m\omega e^{2r}}{\pi\hbar}\right)^{1/2}$$

$$\boxed{\psi_r(x) = e^{i\delta_r} \left(\frac{m\omega e^{2r}}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} e^{2r} x^2\right)}$$

This phase can be determined from $\psi_r(0) = \langle x=0 | \psi_r \rangle = \langle x=0 | S(r,0) | \varphi_0 \rangle$, but it is a tedious thing to do, so we won't bother.

5.3. Classical force on a harmonic oscillator.

$$\hat{H} = \hat{H}_0 + i\hbar\left[\hat{a}^{\dagger} f(t) - \hat{a} f^*(t)\right]$$

$$i\hbar \frac{d\hat{U}(t,0)}{dt} = \hat{H}(t)\,\hat{U}(t,0), \qquad \hat{U}(0,0) = \hat{1}.$$

B-C-H identity:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\, e^{\hat{B}}\, e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

if $\hat{A}$ and $\hat{B}$ commute with $[\hat{A},\hat{B}]$.

(a)

$$D(\hat{a},\alpha(t)) = \exp\left(\alpha(t)\hat{a}^{\dagger} - \alpha^*(t)\hat{a}\right)$$

$$= e^{-\frac{1}{2}|\alpha|^2}\, e^{\alpha \hat{a}^{\dagger}}\, e^{-\alpha^* \hat{a}}$$

$\hat{A} = \alpha \hat{a}^{\dagger}$

$\hat{B} = -\alpha^* \hat{a}$

$[\hat{A},\hat{B}] = -|\alpha|^2 [\hat{a}^{\dagger},\hat{a}]$

$= |\alpha|^2$

$$\frac{d\,D(\hat{a},\alpha(t))}{dt} = -\frac{1}{2}(\dot{\alpha}\alpha^* + \alpha\dot{\alpha}^*)D(\hat{a},\alpha)$$

$$+ \dot{\alpha}\hat{a}^{\dagger} D(\hat{a},\alpha) - \dot{\alpha}^* \underbrace{D(\hat{a},\alpha)\,\hat{a}}$$

Use $D^{\dagger}(\hat{a},\alpha)\,\hat{a}\,D(\hat{a},\alpha) = \hat{a} + \alpha$

or $D(\hat{a},\alpha)\,\hat{a}\,D^{\dagger}(\hat{a},\alpha) = \hat{a} - \alpha$ $= D(\hat{a},\alpha)\,\hat{a}\,D^{\dagger}(\hat{a},\alpha)\,D(\hat{a},\alpha)$

$= (\hat{a} - \alpha)\,D(\hat{a},\alpha)$

$$\boxed{\frac{d\,D(\hat{a},\alpha(t))}{dt} = \left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^{\dagger} - \dot{\alpha}^*\hat{a})\right] D(\hat{a},\alpha)}$$

(b) $$\hat{U}(t,0) = e^{-i\delta(t)}\,\hat{U}_0(t,0)\,D(\hat{a},\alpha(t))$$

$$\hat{U}_0(t,0) = e^{-i\omega t\,\hat{a}^{\dagger}\hat{a}}$$

$$\frac{d\hat{U}(t,0)}{dt} = \left(-i\dot{\delta}\,\hat{U}(t,0) + \frac{1}{i\hbar}\hat{H}_0\right)\hat{U}(t,0)$$

$$+\; e^{-i\delta}\,\hat{U}_0(t,0)\,\underbrace{\frac{dD(\hat{a},\alpha(t))}{dt}}$$

$$= \hat{U}_0(t,0)\left(-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\right)D(\hat{a},\alpha(t))$$

$$= \hat{U}_0(t,0)\left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\right]\hat{U}_0^\dagger(t,0)$$

Use $\hat{U}_0^\dagger\hat{a}\hat{U}_0 = \hat{a}e^{-i\omega t}$

or $\hat{U}_0\hat{a}\hat{U}_0^\dagger = \hat{a}e^{+i\omega t}$

$$\times\; \hat{U}_0(t,0)\,D(\hat{a},\alpha(t))$$

$$= \left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + \dot{\alpha}e^{-i\omega t}\hat{a}^\dagger - \dot{\alpha}^* e^{+i\omega t}\right]$$

$$\times\; \hat{U}_0(t,0)\,D(\hat{a},\alpha(t))$$

Schrödinger equation

$$\frac{d\hat{U}(t,0)}{dt} = \frac{1}{i\hbar}\hat{H}(t)\,\hat{U}(t,0)$$

$$= \frac{1}{i\hbar}\hat{H}_0\,\hat{U}(t,0) + \left[\hat{a}^\dagger f(t) - \hat{a}f^*(t)\right]\hat{U}(t,0)$$

$$= \frac{1}{i\hbar}\hat{H}_0\,\hat{U}(t,0)$$

$$+\left[-i\dot{\delta} - \frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + \dot{\alpha}e^{-i\omega t}\hat{a}^\dagger - \dot{\alpha}^* e^{i\omega t}\right]\hat{U}(t,0)$$

$$\therefore\;\boxed{\begin{array}{lcl} \dot{\delta} = \frac{i}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) & & \delta(t) = \frac{i}{2}\int_0^t dt'\,(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) \\ & \Rightarrow & \\ \dot{\alpha} = f(t)\,e^{+i\omega t} & & \alpha(t) = \int_0^t dt'\,f(t')\,e^{i\omega t'} \end{array}}$$

(c) $|\psi(0)\rangle = |\varphi_0\rangle$

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle = \hat{U}(t,0)|\varphi_0\rangle$$

$$\hat{U}_0(t,0)\, D(\hat{a},\alpha(t))\, \hat{U}_0^\dagger(t,0) = D(\hat{a}e^{+i\omega t},\alpha(t))$$

$$= D(\hat{a},\alpha(t)e^{-i\omega t})$$

So $\hat{U}(t,0) = e^{-i\delta(t)}\, \hat{U}_0(t,0)\, D(\hat{a},\alpha(t))$

$$= e^{-i\delta(t)}\, D(\hat{a},\alpha(t)e^{-i\omega t})\, \hat{U}_0(t,0)$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\delta(t)}\, \underbrace{D(\hat{a},\alpha(t)e^{-i\omega t})\, \underbrace{\hat{U}_0(t,0)|\varphi_0\rangle}_{|\varphi_0\rangle}}_{|\alpha(t)e^{-i\omega t}\rangle \text{ coherent state}}$$

$$\boxed{|\psi(t)\rangle = e^{-i\delta(t)}\,|\alpha(t)e^{-i\omega t}\rangle}$$

$$\langle\hat{a}\rangle_t = \alpha(t)e^{-i\omega t}$$

$$\langle\hat{a}^\dagger\rangle_t = \alpha^*(t)e^{+i\omega t}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle_t = |\alpha|^2$$

$$\langle\hat{a}^2\rangle_t = \alpha^2(t)e^{-2i\omega t}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + i\hat{p}/m\omega)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - i\hat{p}/m\omega)$$

$$\langle\hat{x}\rangle_t = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\langle\hat{a}\rangle) = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha(t)e^{-i\omega t})$$

$$\langle\hat{p}\rangle_t = \sqrt{2\hbar m\omega}\,\mathrm{Im}(\langle\hat{a}\rangle) = \sqrt{2\hbar m\omega}\,\mathrm{Im}(\alpha(t)e^{-i\omega t})$$

$$\boxed{(\Delta x)^2_t = \frac{\hbar}{2m\omega}, \quad (\Delta p)^2_t = \frac{\hbar m\omega}{2}}$$ ← CS variances same as vacuum

5.4. Classical force on a harmonic oscillator. II.

$$\hat{H} = \hat{H}_0 + i\hbar\left[\hat{a}^\dagger f(t) - \hat{a} f^*(t)\right]$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\hat{p}/m\omega\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\hat{p}/m\omega\right)$$

(a) HP eqns. of motion

$$i\hbar\frac{d\hat{x}}{dt} = [\hat{x}, \hat{H}] = \underbrace{[\hat{x}, \hat{H}_0]}_{i\hbar\frac{\hat{p}}{m}} + i\hbar\left(f(t)\underbrace{[\hat{x}, \hat{a}^\dagger]}_{= -\frac{i}{\sqrt{2\hbar m\omega}}i\hbar = \sqrt{\frac{\hbar}{2m\omega}}} - f^*(t)\underbrace{[\hat{x}, \hat{a}]}_{= \frac{i}{\sqrt{2\hbar m\omega}}i\hbar = -\sqrt{\frac{\hbar}{2m\omega}}}\right)$$

$$\boxed{\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m} + \sqrt{\frac{\hbar}{2m\omega}}\left(f(t) + f^*(t)\right) = \frac{\hat{p}}{m} + \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(f(t))}$$

$$i\hbar\frac{d\hat{p}}{dt} = [\hat{p}, \hat{H}] = \underbrace{[\hat{p}, \hat{H}_0]}_{-i\hbar m\omega^2\hat{x}} + i\hbar\left(f(t)\underbrace{[\hat{p}, \hat{a}^\dagger]}_{= -\sqrt{\frac{m\omega}{2\hbar}}i\hbar = -i\sqrt{\frac{\hbar m\omega}{2}}} - f^*(t)\underbrace{[\hat{p}, \hat{a}]}_{= -i\sqrt{\frac{\hbar m\omega}{2}}}\right)$$

$$\boxed{\begin{aligned}\frac{d\hat{p}}{dt} &= -m\omega^2\hat{x} + \sqrt{\frac{\hbar m\omega}{2}}\left(-if(t) + if^*(t)\right)\\ &= -m\omega^2\hat{x} + \sqrt{2\hbar m\omega}\,\mathrm{Im}(f(t))\end{aligned}}$$

$F(t) = \sqrt{2\hbar m\omega}\,\mathrm{Im}(f(t))$ is an ordinary force. The real part of $f(t)$ changes $\hat{x}$ directly, instead of $\hat{p}$, thus making this a generalized force.

$$i\hbar \frac{d\hat{a}}{dt} = [\hat{a}, \hat{H}] = \underbrace{[\hat{a}, \hat{H}_0]}_{\hbar\omega\hat{a}} + i\hbar f(t) \underbrace{[\hat{a}, \hat{a}^\dagger]}_{1}$$

$$\boxed{\frac{d\hat{a}}{dt} = -i\omega\hat{a} + f(t)} \quad \boxed{\frac{d\hat{a}^\dagger}{dt} = i\omega\hat{a}^\dagger + f^*(t)}$$

(b) Solution for $\hat{a}(t)$:

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} + \underbrace{\int_0^t dt'\, e^{-i\omega(t-t')} f(t')}_{e^{-i\omega t}\alpha(t)}$$

$$\boxed{\begin{aligned}\hat{a}(t) &= e^{-i\omega t}\left(\hat{a}(0) + \alpha(t)\right)\\ \hat{a}^\dagger(t) &= e^{+i\omega t}\left(\hat{a}^\dagger(0) + \alpha^*(t)\right)\end{aligned}}$$

$\delta(t)$ does not appear.

(c) $|\psi_H(t)\rangle = |\psi_H(0)\rangle = |\varphi_0\rangle$

$$\langle\hat{a}\rangle_t = \langle\psi_H(t)|\hat{a}(t)|\psi_H(t)\rangle = \langle\varphi_0|\hat{a}(t)|\varphi_0\rangle$$

$$= e^{-i\omega t}\Big(\underbrace{\langle\varphi_0|\hat{a}(0)|\varphi_0\rangle}_{0} + \underbrace{\langle\varphi_0|\alpha(t)|\varphi_0\rangle}_{\alpha(t)}\Big)$$

$$\boxed{\begin{aligned}\langle\hat{a}\rangle_t &= \alpha(t)\,e^{-i\omega t} = \int_0^t dt'\, e^{-i\omega(t-t')} f(t')\\ \langle\hat{a}^\dagger\rangle_t &= \langle\hat{a}\rangle_t^* = \alpha^*(t)\,e^{+i\omega t}\end{aligned}}$$

$$\langle \hat{a}^\dagger \hat{a} \rangle_t = \langle \psi_H(t) | \hat{a}_H^\dagger(t)\, \hat{a}_H(t) | \psi_H(t) \rangle$$

$$= \langle \varphi_0 | \underbrace{[\hat{a}^\dagger(0) + \alpha^*(t)][\hat{a}(0) + \alpha(t)]}_{\hat{a}^\dagger(0)\hat{a}(0) + \hat{a}^\dagger(0)\alpha(t) + \alpha^*(t)\hat{a}(0) + |\alpha(t)|^2} | \varphi_0 \rangle$$

$$\boxed{\langle \hat{a}^\dagger \hat{a} \rangle_t = |\alpha(t)|^2}$$

$$\langle \hat{a}^2 \rangle_t = \langle \psi_H(t) | \hat{a}_H^2(t) | \psi_H(t) \rangle$$

$$= \langle \varphi_0 | \left(\hat{a}(0) + \alpha(t)\right)^2 e^{-2i\omega t} | \varphi_0 \rangle$$

$$\boxed{\begin{aligned} \langle \hat{a}^2 \rangle_t &= \alpha^2(t)\, e^{-2i\omega t} \\ \langle (\hat{a}^\dagger)^2 \rangle_t &= \langle \hat{a}^2 \rangle_t^* = \alpha^{*2}(t)\, e^{2i\omega t} \end{aligned}}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a} - \hat{a}^\dagger)$$

$$\langle \hat{x} \rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a} \rangle_t + \langle \hat{a} \rangle_t^*\right)$$

$$\boxed{\begin{aligned} \langle \hat{x} \rangle_t &= \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\left(\alpha(t)\, e^{-i\omega t}\right) \\ \langle \hat{p} \rangle_t &= \sqrt{2\hbar m\omega}\, \mathrm{Im}\left(\alpha(t)\, e^{-i\omega t}\right) \end{aligned}}$$

$$\langle \hat{x}^2 \rangle_t = \frac{\hbar}{2m\omega}\langle (\hat{a} + \hat{a}^\dagger)^2 \rangle_t = \frac{\hbar}{2m\omega}\underbrace{\langle \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle_t}$$

$$= \langle \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} \rangle_t + 1$$

$$= \alpha^2 e^{-2i\omega t} + \alpha^{*2} e^{2i\omega t} + 2|\alpha|^2 + 1$$

$$= \left(\alpha e^{-i\omega t} + \alpha^* e^{+i\omega t}\right)^2 + 1$$

$$= 4\left[\mathrm{Re}\left(\alpha(t)\, e^{-i\omega t}\right)\right]^2 + 1$$

$$\boxed{\langle \hat{x}^2 \rangle_t = \frac{2\hbar}{m\omega}\left[\mathrm{Re}\left(\alpha(t)e^{-i\omega t}\right)\right]^2 + \frac{\hbar}{2m\omega} = \langle \hat{x} \rangle_t^2 + \frac{\hbar}{2m\omega} \qquad (\Delta x)_t^2 = \langle \hat{x}^2 \rangle_t - \langle \hat{x} \rangle_t^2 = \frac{\hbar}{2m\omega}}$$

$$\boxed{\langle \hat{p}^2 \rangle_t = 2\hbar m\omega\left[\mathrm{Im}\left(\alpha(t)e^{-i\omega t}\right)\right]^2 + \frac{\hbar m\omega}{2} = \langle \hat{p} \rangle_t^2 + \frac{\hbar m\omega}{2} \qquad (\Delta p)_t^2 = \langle \hat{p}^2 \rangle_t - \langle \hat{p} \rangle_t^2 = \frac{\hbar m\omega}{2}}$$