Phys 521

Homework #6

Solution Set

6.1. $H_0 = \frac{1}{2}\hbar\omega\left(|+\rangle\langle+| - |-\rangle\langle-|\right) = \frac{1}{2}\hbar\omega\,\sigma_z \longleftrightarrow \frac{1}{2}\hbar\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\langle+|V|-\rangle = \langle-|V|+\rangle = \hbar\gamma$

$V = \hbar\gamma\,|+\rangle\langle-| + \hbar\gamma\,|-\rangle\langle+| = \hbar\gamma\,\sigma_x \longleftrightarrow \hbar\gamma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

(a) $$H = H_0 + V \longleftrightarrow \boxed{\begin{pmatrix}\hbar\omega/2 & \hbar\gamma\\ \hbar\gamma & -\hbar\omega/2\end{pmatrix}}$$

(b) $$H = \frac{1}{2}\hbar\omega\,\sigma_z + \hbar\gamma\,\sigma_x = \hbar\left(\frac{1}{2}\omega\sigma_z + \gamma\sigma_x\right)$$

Define $\alpha \equiv \left(\gamma^2 + \omega^2/4\right)^{1/2}$

$$\vec{u} = \frac{\gamma\vec{e}_x + \frac{1}{2}\omega\vec{e}_z}{\alpha}, \qquad \vec{u}\cdot\vec{u} = 1$$

$$H = \hbar\alpha\,\vec{u}\cdot\vec{\sigma} = \hbar\alpha\left(|+\rangle_{\vec{u}\,\vec{u}}\langle+| - |-\rangle_{\vec{u}\,\vec{u}}\langle-|\right)$$

$\vec{u} \longleftrightarrow \varphi = 0 \qquad \cos\theta = u_z = \omega/2\alpha$

$\sin\theta = u_y = \gamma/\alpha$

$$\cos(\theta/2) = \left(\frac{1+\cos\theta}{2}\right)^{1/2} = \left(\frac{1+u_z}{2}\right)^{1/2} = \left(\frac{1+\omega/2\alpha}{2}\right)^{1/2}$$

$$\sin(\theta/2) = \left(\frac{1-\cos\theta}{2}\right)^{1/2} = \left(\frac{1-u_z}{2}\right)^{1/2} = \left(\frac{1-\omega/2\alpha}{2}\right)^{1/2}$$

Eigenvectors: $|+\rangle_{\vec{u}}$, $|-\rangle_{\vec{u}}$

$$|+\rangle_{\vec{u}} = \cos(\theta/2)|+\rangle + \sin(\theta/2)|-\rangle$$

$$\boxed{|+\rangle_{\vec{u}} = \left(\frac{1+\omega/2\alpha}{2}\right)^{1/2}|+\rangle + \left(\frac{1-\omega/2\alpha}{2}\right)^{1/2}|-\rangle \qquad \text{Eigenvalue } +\hbar\alpha}$$

$$|-\rangle_{\vec{u}} = |+\rangle_{-\vec{u}} = -i\sin(\theta/2)|+\rangle + i\cos(\theta/2)|-\rangle$$

$\hookrightarrow$ $\theta \to \pi - \theta$ $\quad \cos(\theta/2) \to \sin(\theta/2)$, $\sin(\theta/2) \to \cos(\theta/2)$

$\varphi = \pi$

$$\boxed{|-\rangle_{\vec{u}} = -i\left(\frac{1-\omega/2\alpha}{2}\right)^{1/2}|+\rangle + i\left(\frac{1+\omega/2\alpha}{2}\right)^{1/2}|-\rangle \qquad \text{Eigenvalue } -\hbar\alpha}$$

<u>Check</u>:

$$H \longrightarrow \hbar\begin{pmatrix} \omega/2 & \gamma \\ +\gamma & -\omega/2 \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} u_z & u_x \\ u_x & -u_z \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} \cos^2(\theta/2) - \sin^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ 2\sin(\theta/2)\cos(\theta/2) & -\cos^2(\theta/2) + \sin^2(\theta/2) \end{pmatrix}$$

$$H|+\rangle_{\vec{u}} \longrightarrow \hbar\alpha\begin{pmatrix} c^2 - s^2 & 2cs \\ 2cs & -c^2+s^2 \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} c^3 - s^2c + 2cs^2 \\ 2c^2s - sc^2 + s^3 \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} c(c^2+s^2) \\ s(s^2+c^2) \end{pmatrix}$$

$$= \hbar\alpha\begin{pmatrix} c \\ s \end{pmatrix} \longleftrightarrow \hbar\alpha|+\rangle_{\vec{u}}$$

$$H|-\rangle_{\vec{u}} \longrightarrow \hbar\alpha \begin{pmatrix} c^2-s^2 & 2cs \\ 2cs & -c^2+s^2 \end{pmatrix} \begin{pmatrix} -is \\ ic \end{pmatrix}$$

$$= i\hbar\alpha \begin{pmatrix} -c^2s + s^3 + 2c^2s \\ -2cs^2 - c^3 + s^2c \end{pmatrix}$$

$$= i\hbar\alpha \begin{pmatrix} s(s^2+c^2) \\ -c(c^2+s^2) \end{pmatrix}$$

$$= -\hbar\alpha \begin{pmatrix} -is \\ ic \end{pmatrix} \longrightarrow -\hbar\alpha|-\rangle_{\vec{u}}$$

(c) $|\psi(0)\rangle = |+\rangle$

$$|\psi(t)\rangle = U(t,0)|+\rangle = e^{-\frac{i}{\hbar}Ht}|+\rangle = e^{-i\alpha t\,\vec{u}\cdot\vec{\sigma}}|+\rangle$$

$$P(t) = |\langle + |\psi(t)\rangle|^2$$

Method 1:

$$|\psi(t)\rangle = e^{-i\alpha t\,\vec{u}\cdot\vec{\sigma}}|+\rangle$$

$$= (\mathbb{1}\cos\alpha t - i\,\vec{u}\cdot\vec{\sigma}\sin\alpha t)|+\rangle$$

$$\langle + |\psi(t)\rangle = \cos\alpha t - i\sin\alpha t\,\underbrace{\langle +|\vec{u}\cdot\vec{\sigma}|+\rangle}$$

$$= \langle +|(u_x\sigma_x + u_z\sigma_z)|+\rangle =$$

$$= u_x \underbrace{\langle +|\sigma_x|+\rangle}_{=0} + u_z \underbrace{\langle +|\sigma_z|+\rangle}_{=1}$$

$$= u_z = \cos\theta = \omega/2\alpha$$

$$\langle + | \psi(t)\rangle = \cos\alpha t - i\frac{\omega}{2\alpha}\sin\alpha t$$

$$\boxed{\begin{aligned} P(t) = |\langle + | \psi(t)\rangle|^2 &= \cos^2\alpha t + \underbrace{\left(\frac{\omega}{2\alpha}\right)^2}_{u_z^2 = 1 - u_x^2 = 1 - \gamma^2/\alpha^2}\sin^2\alpha t \\ &= 1 - \frac{\gamma^2}{\alpha^2}\sin^2\alpha t \\ &= \frac{1}{2}\left(1 + \left(\frac{\omega}{2\alpha}\right)^2\right) + \frac{1}{2}\underbrace{\left(1 - \left(\frac{\omega}{2\alpha}\right)^2\right)}_{u_x^2 = \gamma^2/\alpha^2}\cos 2\alpha t \end{aligned}}$$

Method 2:

$$|\psi(t)\rangle = \underbrace{e^{-\frac{i}{\hbar}E_+ t}}_{e^{-i\alpha t}} |+\rangle_{\vec{u}}\ {}_{\vec{u}}\langle + | + \rangle + \underbrace{e^{-\frac{i}{\hbar}E_- t}}_{e^{+i\alpha t}} |-\rangle_{\vec{u}}\ {}_{\vec{u}}\langle - | + \rangle$$

$$\begin{aligned} \langle + | \psi(t)\rangle &= e^{-i\alpha t}\langle + | + \rangle_{\vec{u}}\ {}_{\vec{u}}\langle + | + \rangle + e^{+i\alpha t}\langle + | - \rangle_{\vec{u}}\ {}_{\vec{u}}\langle - | + \rangle \\ &= e^{-i\alpha t}\underbrace{|\langle + | + \rangle_{\vec{u}}|^2}_{\cos^2(\theta/2)} + e^{i\alpha t}\underbrace{|\langle + | - \rangle_{\vec{u}}|^2}_{\sin^2(\theta/2)} \\ &= \cos\alpha t\left(\cos^2(\theta/2) + \sin^2(\theta/2)\right) \\ &\quad - i\sin\alpha t\left(\cos^2(\theta/2) - \sin^2(\theta/2)\right) \end{aligned}$$

$$\langle + | \psi(t)\rangle = \cos\alpha t - i\sin\alpha t\underbrace{\cos\theta}_{u_z = \omega/2\alpha}$$

G.R. C-T J_{IV}.1

Spin-$\frac{1}{2}$ particle

Magnetic moment $\vec{M} = \gamma \vec{S}$.

$$S_z|+\rangle = \frac{1}{2}\hbar|+\rangle_z \qquad S_z|-\rangle = -\frac{1}{2}\hbar|-\rangle_z$$

$$|\psi(0)\rangle = |+\rangle_z$$

(a) $S_x = \frac{1}{2}\hbar\,\sigma_x$

The eigenvalues of σ_x are ± 1; hence the possible results of a measurement of S_x are $\boxed{\pm \frac{1}{2}\hbar.}$ The eigenvectors $|\pm\rangle_x$ of σ_x lie along the $\pm x$-axes of the Bloch sphere

$$|+\rangle_x = |\theta = \tfrac{\pi}{2}, \varphi = 0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_z + |-\rangle_z\right)$$

$$|-\rangle_x = |\theta = \tfrac{\pi}{2}, \varphi = \pi\rangle = -\frac{i}{\sqrt{2}}\left(|+\rangle_z - |-\rangle_z\right)$$

Result	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$
Eigenvector	$\vert+\rangle_x$	$\vert-\rangle_x$
Probability	$\vert\underbrace{{}_x\langle+\vert+\rangle_z}_{1/\sqrt{2}}\vert^2 = \frac{1}{2}$	$\vert\underbrace{{}_x\langle-\vert+\rangle_z}_{i/\sqrt{2}}\vert^2 = \frac{1}{2}$

(b) $\vec{B} = B_0 \vec{e}_y$

$$H = -\vec{M}\cdot\vec{B} = -\gamma B_0 \vec{S}\cdot\vec{e}_y = -\gamma B_0 S_y = -\frac{\gamma B_0 \hbar}{2}\sigma_y$$

$$\underset{|\pm\rangle_z \text{ rep}}{\sigma_y \longleftrightarrow} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Schrödinger equation:

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

In the $|\pm\rangle_z$ representation

$$i\hbar \frac{d\, {}_z\langle +|\psi(t)\rangle}{dt} = {}_z\langle + | H | \psi(t)\rangle$$

$$= {}_z\langle + | H | +\rangle_z\, {}_z\langle + | \psi(t)\rangle$$

$$+ {}_z\langle + | H | -\rangle_z\, {}_z\langle - | \psi(t)\rangle$$

$$= -\frac{\gamma B_0 \hbar}{2}\Big(\overbrace{{}_z\langle + | \sigma_y | +\rangle_z}^{0}\, {}_z\langle + | \psi(t)\rangle$$

$$+ \underbrace{{}_z\langle + | \sigma_y | -\rangle_z}_{-i}\, {}_z\langle - | \psi(t)\rangle\Big)$$

$$\boxed{\frac{d\, {}_z\langle + | \psi(t)\rangle}{dt} = \frac{\gamma B_0}{2}\, {}_z\langle - | \psi(t)\rangle}$$

$$i\hbar \frac{d\, {}_z\langle - | \psi(t)\rangle}{dt} = {}_z\langle - | H | \psi(t)\rangle$$

$$= -\frac{\gamma B_0 \hbar}{2}\Big(\overbrace{{}_z\langle - | \sigma_y | +\rangle_z}^{i}\, {}_z\langle + | \psi(t)\rangle$$

$$+ \underbrace{{}_z\langle - | \sigma_y | -\rangle_z}_{0}\, {}_z\langle - | \psi(t)\rangle\Big)$$

$$\boxed{\frac{d\, {}_z\langle -|\psi(t)\rangle}{dt} = -\frac{\gamma B_0}{2}\, {}_z\langle +|\psi(t)\rangle}$$

General solution:

$${}_z\langle +|\psi(t)\rangle = \cos(\gamma B_0 t/2)\, {}_z\langle +|\psi(0)\rangle + \sin(\gamma B_0 t/2)\, {}_z\langle -|\psi(0)\rangle$$

$${}_z\langle -|\psi(t)\rangle = -\sin(\gamma B_0 t/2)\, {}_z\langle +|\psi(0)\rangle + \cos(\gamma B_0 t/2)\, {}_z\langle -|\psi(0)\rangle$$

In our case, $|\psi(0)\rangle = |+\rangle_z$, so

$${}_z\langle +|\psi(t)\rangle = \cos(\gamma B_0 t/2)$$

$${}_z\langle -|\psi(t)\rangle = -\sin(\gamma B_0 t/2)$$

$$\boxed{|\psi(t)\rangle = \cos(\gamma B_0 t/2)\,|+\rangle_z - \sin(\gamma B_0 t/2)\,|-\rangle_z}$$

Alternative derivation:

$$U(t,0) = \exp\left(-\frac{i}{\hbar} H t\right) = \exp\left(i \frac{\gamma B_0 t}{2} \sigma_y\right)$$

$$= \cos\left(\frac{\gamma B_0 t}{2}\right) 1 + i \sin\left(\frac{\gamma B_0 t}{2}\right) \sigma_y$$

$$|\psi(t)\rangle = U(t,0)\,|+\rangle_z = \cos\left(\frac{\gamma B_0 t}{2}\right)|+\rangle_z + i \sin\left(\frac{\gamma B_0 t}{2}\right) \sigma_y |+\rangle_z$$

$$\sigma_y |+\rangle_z = \left(-i|+\rangle_z\, {}_z\langle -| + i|-\rangle_z\, {}_z\langle +|\right)|+\rangle_z$$

$$= i|-\rangle_z$$

$$\Longrightarrow |\psi(t)\rangle = \cos(\gamma B_0 t/2)\,|+\rangle_z - \sin(\gamma B_0 t/2)\,|-\rangle_z$$

(c) $|\psi(t)\rangle = \cos(\gamma B_0 t/2)|+\rangle_z - \sin(\gamma B_0 t/2)|-\rangle_z$

S_z

Result	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$
Eigenvector	$\|+\rangle_z$	$\|-\rangle_z$
Probability	$\|{}_z\langle+\|\psi(t)\rangle\|^2 = \cos^2(\gamma B_0 t/2) = \frac{1}{2}(1+\underbrace{\cos(\gamma B_0 t)}_{\langle\sigma_z\rangle})$	$\|{}_z\langle-\|\psi(t)\rangle\|^2 = \sin^2(\gamma B_0 t/2) = \frac{1}{2}(1-\cos(\gamma B_0 t))$

S_x

Result	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$
Eigenvector	$\|+\rangle_x = \frac{1}{\sqrt{2}}(\|+\rangle_z + \|-\rangle_z)$	$\|-\rangle_x = -\frac{i}{\sqrt{2}}(\|+\rangle_z - \|-\rangle_z)$
Probability	$\|{}_x\langle+\|\psi(t)\rangle\|^2 = \sin^2(\gamma B_0 t/2 - \pi/4) = \frac{1}{2}[1-\cos(\gamma B_0 t - \pi/2)]$	$\|{}_x\langle-\|\psi(t)\rangle\|^2 = \cos^2(\gamma B_0 t/2 - \pi/4) = \frac{1}{2}[1+\cos(\gamma B_0 t - \pi/2)]$

$$\frac{1}{\sqrt{2}}\left(\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)\right) = -\sin(\gamma B_0 t/2 - \pi/4)$$

$\langle\sigma_x\rangle = -\sin(\gamma B_0 t)$

$$= \frac{i}{\sqrt{2}}\left(\cos(\gamma B_0 t/2) + \sin(\gamma B_0 t/2)\right) = i\cos(\gamma B_0 t/2 - \pi/4)$$

S_y

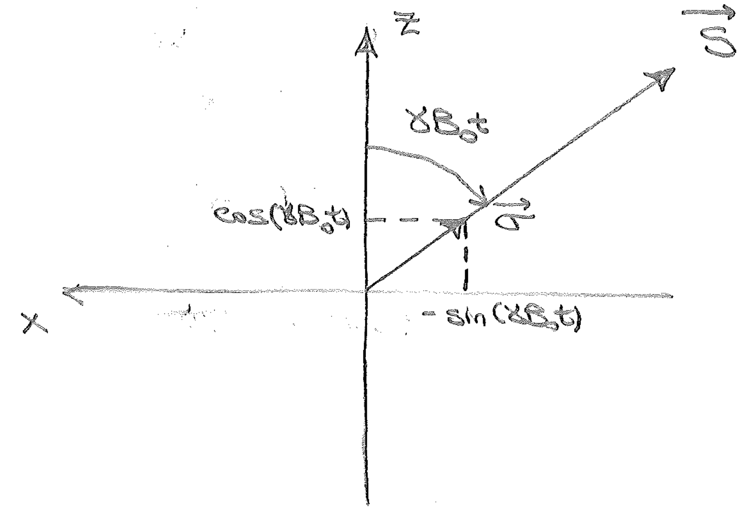
Result	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$
Eigenvector	$\|+\rangle_y = \frac{e^{-i\pi/4}}{\sqrt{2}}(\|+\rangle_z + i\|-\rangle_z)$	$\|-\rangle_y = \frac{e^{-i3\pi/4}}{\sqrt{2}}(\|+\rangle_z - i\|-\rangle_z)$
Probability	$\|{}_y\langle+\|\psi(t)\rangle\|^2 = \frac{1}{2}$	$\|{}_y\langle-\|\psi(t)\rangle\|^2 = \frac{1}{2}$

$$\frac{e^{i\pi/4}}{\sqrt{2}}\left(\cos(\gamma B_0 t/2) + i\sin(\gamma B_0 t/2)\right) = \frac{e^{i\pi/4}}{\sqrt{2}}\, e^{i\gamma B_0 t/2}$$

$$|+\rangle_y = |\theta=\pi/2, \varphi=\pi/2\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|+\rangle_z + e^{i\pi/4}|-\rangle_z\right) = \frac{e^{-i\pi/4}}{\sqrt{2}}\left(|+\rangle_z + i|-\rangle_z\right)$$

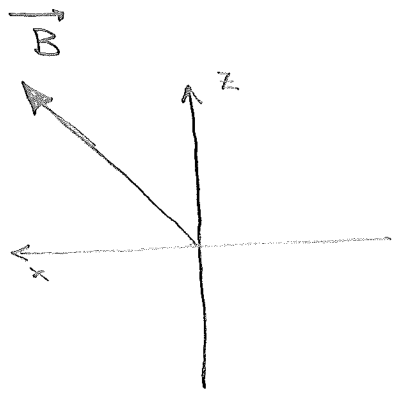
$$|-\rangle_y = |\theta=\pi/2, \varphi=3\pi/2\rangle = \frac{1}{\sqrt{2}}\left(e^{-i3\pi/4}|+\rangle_z + e^{i3\pi/4}|-\rangle_z\right) = \frac{e^{-i3\pi/4}}{\sqrt{2}}\left(|+\rangle_z - i|-\rangle_z\right)$$

The spin precesses about the $-y$-axis at the Larmor frequency γB_0



A measurement of S_y always yields both values with probability $1/2$. A measurement of S_z yields $+\frac{1}{2}\hbar$ with certainty when $\gamma B_0 t = 2n\pi$ and yields $-\frac{1}{2}\hbar$ with certainty when $\gamma B_0 t = (2n+1)\pi$. A measurement of S_x yields $-\frac{1}{2}\hbar$ with certainty when $\gamma B_0 t = (2n+\frac{1}{2})\pi$ and $+\frac{1}{2}\hbar$ with certainty when $\gamma B_0 t = (2n+\frac{3}{2})\pi$.

6.3. C-T $J_{IV}\cdot 3$



$$\vec{B} = \frac{1}{\sqrt{2}} B_0 (\vec{e}_x + \vec{e}_z)$$

$$= B_0 \vec{u}$$

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_z)$$

(a) $H = -\vec{M}\cdot\vec{B} = -\gamma \vec{S}\cdot\vec{B} = -\frac{\gamma B_0 \hbar}{2} \vec{\sigma}\cdot\vec{u}$

$$H = -\frac{\gamma B_0 \hbar}{2} \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \xrightarrow[|\pm\rangle_z \text{ rep}]{} -\frac{\gamma B_0 \hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) $H = -\frac{\gamma B_0 \hbar}{2} \vec{\sigma}\cdot u$

$$\vec{\sigma}\cdot\vec{u} = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|$$

state specified by $\vec{u}$ on Bloch sphere

State specified by $-\vec{u}$ on Bloch sphere

$$|+\rangle_{\vec{u}} = |\theta = \pi/4, \varphi = 0\rangle$$

$$= \underbrace{\cos(\pi/8)}_{\equiv \alpha_1}|+\rangle_z + \underbrace{\sin(\pi/8)}_{\beta_1}|-\rangle_z$$

$$|-\rangle_{\vec{u}} = |\theta = 3\pi/4, \varphi = \pi\rangle = -i\underbrace{\cos(3\pi/8)}_{\beta}|+\rangle_z + i\underbrace{\sin(3\pi/8)}_{\alpha}|-\rangle_z$$

Eigenvalue: $E_+ = -\gamma B_0 \hbar/2$ $\qquad$ $E_- = \gamma B_0 \hbar/2$

Eigenvector: $|+\rangle_{\vec{u}} = \alpha|+\rangle_z + \beta|-\rangle_z$ $\qquad$ $|-\rangle_{\vec{u}} = i(-\beta|+\rangle_z + \alpha|-\rangle_z)$

$$\alpha = \cos(\pi/8), \qquad \beta = \sin(\pi/8)$$

(c) $|\psi(0)\rangle = |-\rangle_z$

Energy is measured

Result	$-\gamma B_0\hbar/2$	$+\gamma B_0 \hbar/2$
Eigenvector	$\lvert+\rangle_{\vec{u}}$	$\lvert-\rangle_{\vec{u}}$
Probability	$\lvert\underbrace{{}_{\vec{u}}\langle+\lvert-\rangle_z}_{\beta}\rvert^2 = \beta^2$	$\lvert\underbrace{{}_{\vec{u}}\langle-\lvert-\rangle_z}_{-i\alpha}\rvert^2 = \alpha^2$

(d) $|\psi(t)\rangle = U(t,0)|-\rangle_z$

$$U(t,0) = \exp\left(-\frac{i}{\hbar}Ht\right) = e^{i(\gamma B_0 t/2)\vec{\sigma}\cdot\vec{u}}$$

1 $$U(t,0) = \underbrace{e^{i\gamma B_0 t/2}}_{e^{-\frac{i}{\hbar}E_+ t}} |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| + \underbrace{e^{-i\gamma B_0 t/2}}_{e^{-\frac{i}{\hbar}E_- t}} |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|$$

$$|\psi(t)\rangle = e^{i\gamma B_0 t/2}|+\rangle_{\vec{u}} \underbrace{{}_{\vec{u}}\langle +|-\rangle_z}_{\beta} + e^{-i\gamma B_0 t/2}|-\rangle_{\vec{u}} \underbrace{{}_{\vec{u}}\langle -|-\rangle_z}_{-i\alpha}$$

$$= \beta e^{i\gamma B_0 t/2}(\alpha|+\rangle_z + \beta|-\rangle_z) + i\alpha e^{-i\gamma B_0 t/2}\, i(-\beta|+\rangle_z + \alpha|-\rangle_z)$$

$$|\psi(t)\rangle = \alpha\beta\left(e^{i\gamma B_0 t/2} - e^{-i\gamma B_0 t/2}\right)|+\rangle_z$$
$$+ \left(\beta^2 e^{i\gamma B_0 t/2} + \alpha^2 e^{-i\gamma B_0 t/2}\right)|-\rangle_z$$

$$|\psi(t)\rangle = i\underbrace{2\alpha\beta}_{2\cos(\pi/8)\sin(\pi/8) = \sin(\pi/4) = \frac{1}{\sqrt{2}}} \sin(\gamma B_0 t/2)|+\rangle_z + \left[\cos(\gamma B_0 t/2) + i\underbrace{(\beta^2 - \alpha^2)}_{\sin^2(\pi/8) - \cos^2(\pi/8) = -\cos(\pi/4) = -\frac{1}{\sqrt{2}}} \sin(\gamma B_0 t/2)\right]|-\rangle_z$$

$$\boxed{|\psi(t)\rangle = \frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)|+\rangle_z + \left[\cos(\gamma B_0 t/2) - \frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)\right]|-\rangle_z}$$

(2) $$U(t,0) = e^{i(\gamma B_0 t/2)\vec{\sigma}\cdot\vec{u}} = \cos(\gamma B_0 t/2) + i\,\vec{\sigma}\cdot\vec{u}\,\sin(\gamma B_0 t/2)$$

$$|\psi(t)\rangle = U(t,0)\,|-\rangle_z$$
$$= \cos(\gamma B_0 t/2)|-\rangle_z + i\sin(\gamma B_0 t/2)\,\underbrace{\vec{\sigma}\cdot\vec{u}\,|-\rangle_z}$$

$$|\pm\rangle_z \text{ rep: } \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\left(|+\rangle_z - |-\rangle_z\right)$$

$$\boxed{|\psi(t)\rangle = \frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)|+\rangle_z + \left[\cos(\gamma B_0 t/2) - \frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)\right]|-\rangle_z}$$

Method 2 is much easier because the operator algebra does all the work.

$$\langle S_x \rangle_t = \langle \psi(t) | S_x | \psi(t) \rangle$$

$$= \tfrac{1}{2}\hbar \langle \psi(t) | \sigma_x | \psi(t) \rangle$$

$$|+\rangle_z \, {}_z\langle -| + |-\rangle_z \, {}_z\langle +|$$

$$= \tfrac{1}{2}\hbar \left(\langle \psi(t) | + \rangle_z \, {}_z\langle - | \psi(t) \rangle + \langle \psi(t) | - \rangle_z \, {}_z\langle + | \psi(t) \rangle \right)$$

$$= \hbar \, \mathrm{Re} \left(\underbrace{\langle \psi(t) | + \rangle_z}_{-\frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)} \; \underbrace{{}_z\langle - | \psi(t) \rangle}_{\cos(\gamma B_0 t/2) - \frac{i}{\sqrt{2}} \sin(\gamma B_0 t/2)} \right)$$

$$= -\tfrac{1}{2}\hbar \sin^2(\gamma B_0 t/2)$$

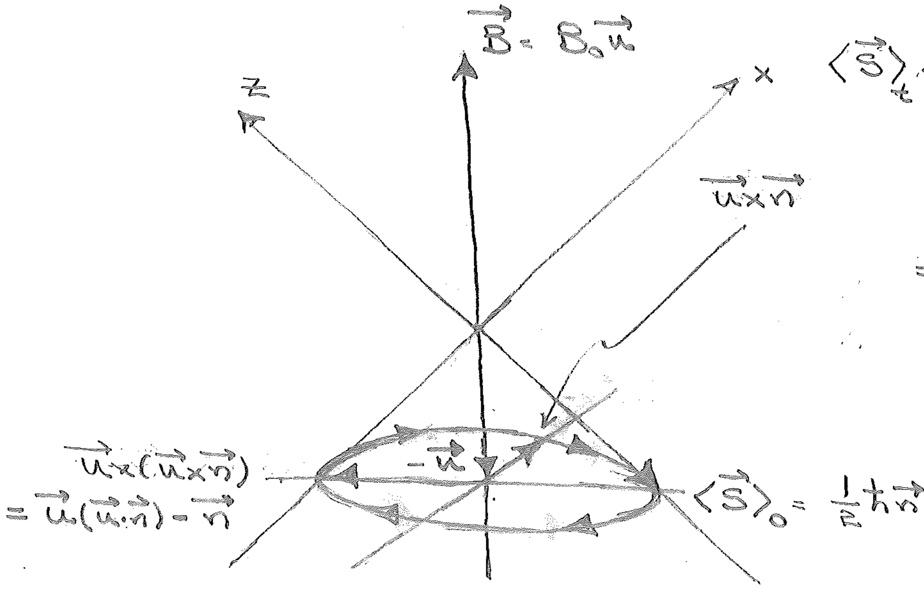
$$\boxed{\langle S_x \rangle_t = -\tfrac{1}{2}\hbar \, \tfrac{1}{2} \left(1 - \cos(\gamma B_0 t) \right)}$$

Geometric interpretation: The expectation value $\langle \vec{S} \rangle$ of the spin precesses about $-\vec{B}$ at the Larmor frequency γB_0.

$$\langle \vec{S} \rangle_0 = \tfrac{1}{2}\hbar \left((\vec{n}\cdot\vec{u})\vec{u} - \vec{u}\times(\vec{u}\times\vec{n}) \right)$$

$$\langle \vec{S} \rangle_t = \tfrac{1}{2}\hbar \Big((\vec{n}\cdot\vec{u})\vec{u} - \vec{u}\times(\vec{u}\times\vec{n}) \cos(\gamma B_0 t) - \vec{u}\times\vec{n} \sin(\gamma B_0 t) \Big)$$

$$= \tfrac{1}{2}\hbar \Big(\vec{n} \cos(\gamma B_0 t) + (\vec{n}\cdot\vec{u})\vec{u} \left(1 - \cos(\gamma B_0 t)\right) - \vec{u}\times\vec{n} \sin(\gamma B_0 t) \Big)$$



In our case, $\vec{n} = -\vec{e}_z$ and $\vec{u} = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_z)$, so

$\vec{n} \cdot \vec{u} = -\frac{1}{\sqrt{2}}$ and $\vec{u} \times \vec{n} = -\frac{1}{\sqrt{2}}\vec{e}_x \times \vec{e}_z = +\frac{1}{\sqrt{2}}\vec{e}_y$

$$\therefore \langle \vec{S} \rangle_t = \frac{1}{2}\hbar\left(-\vec{e}_z\cos(\gamma B_0 t) - \frac{1}{2}(\vec{e}_x + \vec{e}_z)(1 - \cos(\gamma B_0 t)) - \frac{1}{\sqrt{2}}\vec{e}_y \sin(\gamma B_0 t)\right)$$

$$= \frac{1}{2}\hbar\left(-\frac{1}{2}(1 - \cos(\gamma B_0 t))\vec{e}_x - \frac{1}{\sqrt{2}}\sin(\gamma B_0 t)\vec{e}_y - \frac{1}{2}(1 + \cos(\gamma B_0 t))\vec{e}_z\right)$$

$$\Longrightarrow \boxed{\begin{aligned} \langle S_x \rangle &= -\frac{1}{2}\hbar\,\frac{1}{2}(1 - \cos(\gamma B_0 t)) \\ \langle S_y \rangle &= -\frac{1}{2}\hbar\,\frac{1}{\sqrt{2}}\sin(\gamma B_0 t) \\ \langle S_z \rangle &= -\frac{1}{2}\hbar\,\frac{1}{2}(1 + \cos(\gamma B_0 t)) \end{aligned}}$$

6.4. C-T J_{IV}. 5

$$\vec{M} = \gamma \vec{S}$$

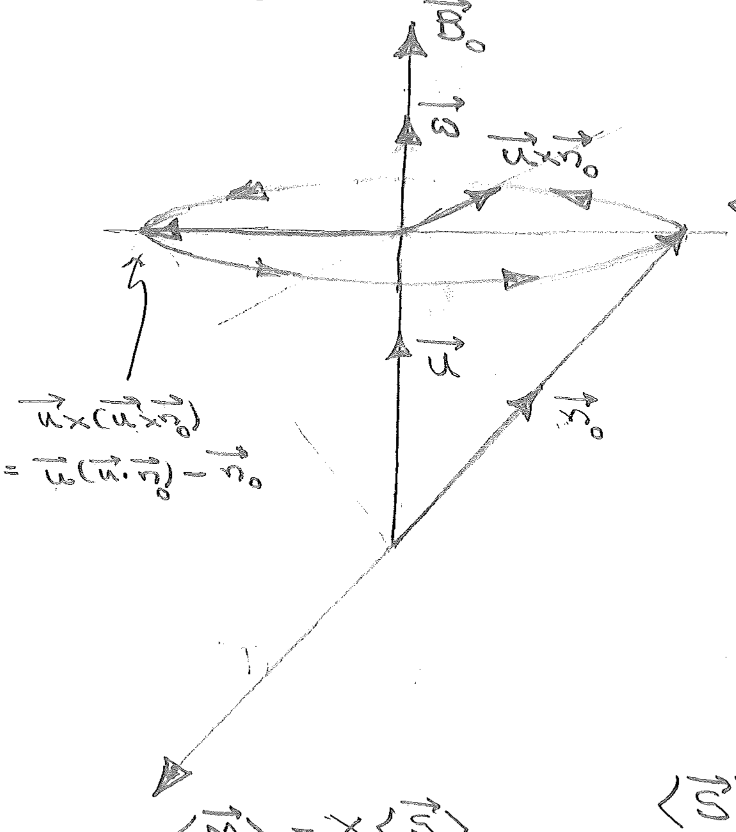
$$\vec{B}_0 = -\vec{\omega}/\gamma = |\vec{B}_0| \vec{u}$$

$$\omega_0 \equiv -\gamma |\vec{B}_0| \longleftarrow$$

Notice that ω_0 is positive if γ is negative; we shall act as if γ is negative

$$\Rightarrow \vec{\omega} = \omega_0 \vec{u}$$

Geometric interpretation



$$\langle \vec{S} \rangle_0 = \tfrac{1}{2}\hbar \vec{n}_0 = \tfrac{1}{2}\hbar \left((\vec{n}_0 \cdot \vec{u})\vec{u} - \vec{u} \times (\vec{u} \times \vec{n}_0) \right)$$

The expectation value of spin, $\langle \vec{S} \rangle_t$, precesses about $\vec{B}$ at the Larmor frequency $-\gamma |\vec{B}_0| = \omega_0$

$$\langle \vec{S} \rangle_t = \tfrac{1}{2}\hbar \Big((\vec{n}_0 \cdot \vec{u})\vec{u} - \vec{u} \times (\vec{u} \times \vec{n}_0) \cos \omega_0 t + (\vec{u} \times \vec{n}_0) \sin \omega_0 t \Big)$$

$$= \tfrac{1}{2}\hbar \Big(\vec{n}_0 \cos \omega_0 t + (\vec{n}_0 \cdot \vec{u})\vec{u}(1 - \cos \omega_0 t) + \vec{u} \times \vec{n}_0 \sin \omega_0 t \Big)$$

$$= \tfrac{1}{2}\hbar \vec{n}_t$$

The expectation value of the spin component along $\vec{n}_t$ is $\frac{1}{2}\hbar$, i.e., $\langle \vec{S}\cdot\vec{n}_t \rangle_t = \frac{1}{2}\hbar$, which means that the state at time t is a eigenstate of $\vec{S}\cdot\vec{n}_t$ with eigenvalue $\frac{1}{2}\hbar$, i.e.,

$$|\psi(t)\rangle = e^{i\delta(t)}|+\rangle_{\vec{n}_t},$$

$$\vec{n}_t = \vec{n}_0 \cos\omega_0 t + (\vec{n}_0\cdot\vec{u})\vec{u}(1-\cos\omega_0 t) + \vec{u}\times\vec{n}_0 \cos\omega_0 t.$$

(a) $H = -\vec{M}\cdot\vec{B}_0 = -(\gamma\vec{S})\cdot(-\vec{\omega}/\gamma) = \vec{S}\cdot\vec{\omega} = \frac{1}{2}\hbar\,\vec{\sigma}\cdot\vec{\omega}$

$$U(t,0) = \exp\left(-\frac{i}{\hbar}Ht\right) = \exp\left(-i\frac{\vec{S}\cdot\vec{\omega}}{\hbar}t\right)$$

$$\therefore \boxed{U(t,0) = e^{-iMt}, \quad M \equiv \frac{\vec{S}\cdot\vec{\omega}}{\hbar} = \frac{1}{2}\vec{\sigma}\cdot\vec{\omega}}$$

(b) $M = \frac{1}{2}\vec{\sigma}\cdot\vec{\omega} = \frac{1}{2}\omega_0\,\vec{\sigma}\cdot\vec{u} = \frac{1}{2}\omega_0\,\sigma_{\vec{u}}$

$$\sigma_{\vec{u}}^2 = \hat{1} \implies \sigma_{\vec{u}}^n = \begin{cases} \hat{1}, & n \text{ even} \\ \sigma_{\vec{u}}, & n \text{ odd} \end{cases}$$

$$e^{-iMt} = e^{-i(\omega_0 t/2)\sigma_{\vec{u}}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n(\omega_0 t/2)^n \sigma_{\vec{u}}^n$$

$$= \left(\sum_{n\text{ even}} \frac{1}{n!}(-i)^n(\omega_0 t/2)^n\right)\hat{1} + \left(\sum_{n\text{ odd}} \frac{1}{n!}(-i)^n(\omega_0 t/2)^n\right)\sigma_{\vec{u}}$$

$$\therefore \boxed{\mathcal{U}(t,0) = e^{-iMt} = \hat{1}\cos(\omega_0 t/2) - i\underbrace{\vec{\sigma}\cdot\vec{u}}_{\frac{1}{\omega_0}\vec{\sigma}\cdot\vec{\omega} = \frac{2}{\omega_0}M}\sin(\omega_0 t/2)}$$

(c) $|\psi(0)\rangle = |+\rangle_z$

$$\vec{n}_0 = \vec{e}_z$$

$$\vec{n}_t = \vec{e}_z\cos\omega_0 t + u_z\vec{u}(1-\cos\omega_0 t) + \vec{u}\times\vec{e}_z\cos\omega_0 t$$

$$P_{++}(t) = \left|{}_z\langle +|\psi(t)\rangle\right|^2$$

$$= \left|{}_z\langle +|\mathcal{U}(t,0)|+\rangle_z\right|^2$$

$${}_z\langle +|\mathcal{U}(t,0)|+\rangle_z = \cos(\omega_0 t/2) - i\,\underbrace{{}_z\langle +|\vec{\sigma}\cdot\vec{u}|+\rangle_z}\sin(\omega_0 t/2)$$

$$= u_x\underbrace{{}_z\langle +|\sigma_x|+\rangle_z}_{0} + u_y\underbrace{{}_z\langle +|\sigma_y|+\rangle_z}_{0} + u_z\underbrace{{}_z\langle +|\sigma_z|+\rangle_z}_{1}$$

$${}_z\langle +|\mathcal{U}(t,0)|+\rangle_z = \cos(\omega_0 t/2) - i\,u_z\sin(\omega_0 t/2)$$

$$P_{++}(t) = \cos^2(\omega_0 t/2) + u_z^2\sin^2(\omega_0 t/2)$$

$$= 1 + \underbrace{(u_z^2 - 1)}_{-(u_x^2+u_y^2)}\sin^2(\omega_0 t/2)$$

$$\boxed{P_{++}(t) = 1 + \underbrace{(u_x^2 + u_y^2)}_{\frac{\omega_x^2+\omega_y^2}{\omega_0^2}} \sin^2(\omega_0 t/2)}$$

Geometric interpretation:

$$\langle S_z \rangle_t = \frac{1}{2}\hbar \underbrace{\vec{n}_t \cdot \vec{e}_z}$$

$$= \cos\omega_0 t + u_z^2 (1 - \cos\omega_0 t)$$

$$\langle S_z \rangle_t = \frac{1}{2}\hbar\, P_{++}(t) - \frac{1}{2}\hbar \underbrace{P_{-+}(t)}_{1 - P_{++}(t)}$$

$$= \frac{1}{2}\hbar \left(-1 + 2P_{++}(t) \right)$$

$$\therefore\ -1 + 2P_{++}(t) = \cos\omega_0 t + u_z^2 (1 - \cos\omega_0 t)$$

$$P_{++}(t) = \underbrace{\frac{1}{2}(1 + \cos\omega_0 t)}_{\cos^2(\omega_0 t/2)} + u_z^2 \underbrace{\frac{1}{2}(1 - \cos\omega_0 t)}_{\sin^2(\omega_0 t/2)}$$

$$P_{++}(t) = \cos^2(\omega_0 t/2) + u_z^2 \sin^2(\omega_0 t/2)$$