Phys 521

HW #8

Solution Set

8.1. C-T $F_{VI}.8$

$[A, \vec{L}] = 0$

$A, \vec{L}^2, L_z$ are CSCO, with eigenkets $|n,l,m\rangle$

$$A|n,l,m\rangle = a_n|n,l,m\rangle$$

$$\vec{L}^2|n,l,m\rangle = l(l+1)\hbar^2|n,l,m\rangle$$

$$L_z|n,l,m\rangle = m\hbar|n,l,m\rangle$$

$$\mathcal{U}(\varphi) = e^{-i\varphi L_z/\hbar}$$

$$\tilde{K} = \mathcal{U}(\varphi) K \mathcal{U}^\dagger(\varphi)$$

(a) $L_\pm = L_x \pm i L_y$

$$\tilde{L}_\pm = \mathcal{U}(\varphi) L_\pm \mathcal{U}^\dagger(\varphi)$$

$$\tilde{L}_\pm|n,l,m\rangle = \mathcal{U}(\varphi) L_\pm \mathcal{U}^\dagger(\varphi)|n,l,m\rangle$$

$$= e^{-i\varphi L_z/\hbar} L_\pm \underbrace{e^{i\varphi L_z/\hbar}|n,l,m\rangle}_{e^{im\varphi}|n,l,m\rangle}$$

$$\underbrace{\hphantom{xx}}_{e^{im\varphi}\sqrt{l(l+1)-m(m\pm1)}\,|n,l,m\pm1\rangle}$$

$$\underbrace{\hphantom{xx}}_{e^{im\varphi}\sqrt{l(l+1)-m(m\pm1)}\, e^{-i(m\pm1)\varphi}|n,l,m\pm1\rangle}$$

$$\Longrightarrow \boxed{\begin{aligned} \tilde{L}_\pm|n,l,m\rangle &= e^{\mp i\varphi}\sqrt{l(l+1)-m(m\pm1)}\,|n,l,m\pm1\rangle \\ &= e^{\mp i\varphi} L_\pm|n,l,m\rangle \end{aligned}}$$

$$\Longrightarrow \boxed{\tilde{L}_\pm = e^{\mp i\varphi} L_\pm}$$

Because the ≈ operators have same action on a complete, orthonormal set

(b) $$\boxed{\tilde{L}_z = U(\varphi) L_z U^\dagger(\varphi) = L_z}$$

$$\tilde{L}_x \pm i\tilde{L}_y = \tilde{L}_\pm = e^{\mp i\varphi} L_\pm$$

$$= (\cos\varphi \mp i\sin\varphi)(L_x \pm iL_y)$$

$$= L_x\cos\varphi + L_y\sin\varphi$$
$$\pm i(-L_x\sin\varphi + L_y\cos\varphi)$$

$$\Longrightarrow \boxed{\begin{aligned}\tilde{L}_x &= L_x\cos\varphi + L_y\sin\varphi\\ \tilde{L}_y &= -L_x\sin\varphi + L_y\cos\varphi\end{aligned}}$$

This is a rotation by angle φ about the z-axis

(c) Use the results of problem 7.3.

$$[X_j, L_k] = i\hbar\,\epsilon_{jkl} X_l$$

$$[X \pm iY, L_z] = [X, L_z] \pm i[Y, L_z]$$

$$= i\hbar\,\epsilon_{xzy} Y \pm i\, i\hbar\,\epsilon_{yzx} X$$

$$= i\hbar(-Y \pm iX)$$

$$= \mp\hbar(X \pm iY)$$

$$\boxed{[X \pm iY, L_z] = \mp\hbar(X \pm iY)}$$

$$\boxed{[Z, L_z] = 0}$$

$$L_z(X \pm iY)|n,l,m\rangle = \Big((X \pm iY)L_z - \underbrace{[X \pm iY, L_z]}_{\mp\hbar(X \pm iY)}\Big)|n,l,m\rangle$$

$$= (m \pm 1)\hbar\,(X \pm iY)|n,l,m\rangle$$

$(X \pm iY)|n,l,m\rangle$ is an eigenvector of L_z with eigenvalue $(m \pm 1)\hbar$.

$$L_z Z|n,l,m\rangle = Z L_z|n,l,m\rangle = m\hbar\, Z|n,l,m\rangle$$

$Z|n,l,m\rangle$ is an eigenvector of L_z with eigenvalue $m\hbar$.

$\langle n',l',m'|(X \pm iY)|n,l,m\rangle$ is nonzero only if $m' = m \pm 1$. $\langle n',l',m'|Z|n,l,m\rangle$ is nonzero only if $m' = m$.

(d)

$$\langle n',\ell',m'|(\tilde{X}\pm i\tilde{Y})|n,\ell,m\rangle = \underbrace{\langle n',\ell',m'|U(\varphi)}_{e^{-im'\varphi}\langle n',\ell',m'|}(X\pm iY)\underbrace{U^\dagger(\varphi)|n,\ell,m\rangle}_{e^{im\varphi}|n,\ell,m\rangle}$$

$$= e^{-i(m'-m)\varphi}\langle n',\ell',m'|(X\pm iY)|n,\ell,m\rangle$$

Because the only nonzero matrix elements have $m'=m\pm 1$

$$= e^{\mp i\varphi}\langle n',\ell',m'|(X\pm iY)|n,\ell,m\rangle$$

$$\therefore\ (\tilde{X}\pm i\tilde{Y}) = e^{\mp i\varphi}(X\pm iY)$$

$$\Longrightarrow \boxed{\begin{aligned}\tilde{X} &= X\cos\varphi + Y\sin\varphi\\ \tilde{Z} &= -X\sin\varphi + Y\cos\varphi\end{aligned}}$$

$$\langle n',\ell',m'|\tilde{Z}|n,\ell,m\rangle = e^{-i(m'-m)\varphi}\langle n',\ell',m'|Z|n,\ell,m\rangle$$

$$= \langle n',\ell',m'|Z|n,\ell,m\rangle$$

$$\Longrightarrow \boxed{\tilde{Z} = Z}$$

18.2. C-T F$_{VI}$.10

(a) Suppose $\Delta J_x = \Delta J_y = \Delta J_z = 0$

$\Delta J_x = 0 \Rightarrow \langle\psi|(\Delta J_x)^2|\psi\rangle = 0$, where $\Delta J_x = J_x - \langle J_x\rangle$

$\Rightarrow \Delta J_x|\psi\rangle = 0$, i.e., $|\psi\rangle$ is an eigenstate of J_x (with eigenvalue $\langle J_x\rangle$).

$\therefore$ $|\psi\rangle$ must be a simultaneous eigenstate of J_x, J_y, and J_z

Eigenstate of J_z:

$$J_z|\psi\rangle = \underset{\uparrow}{m}\hbar|\psi\rangle \Rightarrow |\psi\rangle = \sum_{k,j} c_{kj}|k,j,m\rangle$$

for some value of m

Eigenstate of J_x and J_y

$$\left.\begin{array}{l} J_x|\psi\rangle = \langle J_x\rangle|\psi\rangle \\ J_y|\psi\rangle = \langle J_y\rangle|\psi\rangle \end{array}\right. \iff J_\pm|\psi\rangle = \langle J_\pm\rangle|\psi\rangle$$

$$\langle J_\pm\rangle|\psi\rangle = J_\pm|\psi\rangle = \sum_{k,j} c_{kj} J_\pm|k,j,m\rangle = \sum_{k,j} c_{kj}\sqrt{j(j+1)-m(m\pm1)}\,|k,j,m\pm1\rangle$$

The only way this can be true for J_+ is if $m = +j$, and the only way it can be true for J_- is if $m = -j$. We therefore must have

$m = j = -j \Rightarrow m = j = 0.$

Conclusion:

$$\boxed{\Delta J_x = \Delta J_y = \Delta J_z = 0 \quad \text{iff} \quad |\psi\rangle = \sum_k c_k |k,0,0\rangle}$$

(b)

(i) Let A and B be any two observables.

$$(\Delta A)^2 (\Delta B)^2 = \langle\psi|\underbrace{(\Delta A)^2}_{(A-\langle A\rangle)^2}|\psi\rangle\langle\psi|\underbrace{(\Delta B)^2}_{(B-\langle B\rangle)^2}|\psi\rangle$$

Schwarz inequality $|\langle\phi|\chi\rangle|^2 \le \langle\phi|\phi\rangle\langle\chi|\chi\rangle$, with $|\chi\rangle = \Delta B|\psi\rangle$ and $|\phi\rangle = \Delta A|\psi\rangle$. Equality iff $\Delta A|\psi\rangle = -i\lambda\Delta B|\psi\rangle$

$$\ge |\langle\psi|\Delta A\,\Delta B|\psi\rangle|^2$$

$$= \Big|\overbrace{\langle\psi|\tfrac{1}{2}(\Delta A\Delta B + \Delta B\Delta A)|\psi\rangle}^{\equiv C \text{ real}} + \tfrac{1}{2}\underbrace{\langle\psi|[\Delta A,\Delta B]|\psi\rangle}\Big|^2$$

$$= \langle\psi|[A,B]|\psi\rangle = \langle\psi|[A,B]^\dagger|\psi\rangle^* = -\langle\psi|[A,B]|\psi\rangle^*$$

this is pure imaginary

$$= C^2 + \frac{1}{4}|\langle[A,B]\rangle|^2$$

Equality iff $C=0$ $\longrightarrow$ $\ge \frac{1}{4}|\langle[A,B]\rangle|^2$

$$\boxed{\text{Robertson uncertainty principle:} \quad \Delta A\,\Delta B \ge \frac{1}{2}|\langle[A,B]\rangle|}$$

There is equality in the Robertson up iff

① $\mathcal{C} = \frac{1}{2}\langle \Delta A\, \Delta B + \Delta B\, \Delta A \rangle = 0$

and ② $(\Delta A + i\lambda\, \Delta B)|\psi\rangle = 0$

$$\mathcal{C} = \frac{1}{2}\underbrace{\langle\psi|\Delta A}_{= i\lambda^*\langle\psi|\Delta B}\,\Delta B|\psi\rangle + \frac{1}{2}\langle\psi|\Delta B\underbrace{\Delta A|\psi\rangle}_{-i\lambda\,\Delta B|\psi\rangle}$$

$$= -\frac{i}{2}(\lambda - \lambda^*)\langle\psi|(\Delta B)^2|\psi\rangle$$

$$= \mathrm{Im}(\lambda)\langle\psi|(\Delta B)^2|\psi\rangle$$

$\therefore\ \mathcal{C} = 0$ iff $\mathrm{Im}(\lambda) = 0$

We can replace the equality conditions ① and ② above by

> There is equality in the Robertson uncertainty principle iff
>
> $$(\Delta A + i\lambda\, \Delta B)|\psi\rangle = 0, \quad \text{where } \lambda = \lambda^*.$$

Notice that $\lambda^2 = \dfrac{\langle\psi|(\Delta A)^2|\psi\rangle}{\langle\psi|(\Delta B)^2|\psi\rangle}$ determines the ratios of the variances.

Applied to angular momentum, the Robertson up yields

$$\boxed{\begin{array}{l} \Delta J_x \, \Delta J_y \geq \frac{1}{2} \left| \langle \underbrace{[J_x, J_y]}_{i\hbar J_z} \rangle \right| = \frac{\hbar}{2} \left| \langle J_z \rangle \right| \\ \text{and cyclic permutations there of.} \\ \hline \text{Equality iff } \left(\Delta J_x + i \lambda \Delta J_y \right) |\psi\rangle = 0, \quad \lambda = \lambda^* \end{array}}$$

(ii) $\langle \vec{J} \rangle = \langle J_z \rangle \vec{e}_z \Rightarrow \underbrace{\langle J_x \rangle = \langle J_y \rangle = 0}$

$$(\Delta J_x)^2 + (\Delta J_y)^2 = \langle J_x^2 \rangle + \langle J_y^2 \rangle$$

$$= \langle \vec{J}^2 \rangle - \langle J_z^2 \rangle$$

$$= \underbrace{\langle J_\pm J_\mp \rangle}_{= \langle \psi | J_\pm J_\mp | \psi \rangle \geq 0} \mp \hbar \langle J_z \rangle$$

Equality iff $J_\mp |\psi\rangle = 0$

$$\boxed{\begin{array}{l} \langle J_z \rangle \geq 0: \quad (\Delta J_x)^2 + (\Delta J_y)^2 \geq \hbar \langle J_z \rangle = \hbar \left| \langle J_z \rangle \right| \\ \qquad\qquad \text{Equality iff } J_+ |\psi\rangle = 0 \\ \langle J_z \rangle \leq 0: \quad (\Delta J_x)^2 + (\Delta J_y)^2 \geq -\hbar \langle J_z \rangle = \hbar \left| \langle J_z \rangle \right| \\ \qquad\qquad \text{Equality iff } J_- |\psi\rangle = 0 \end{array}}$$

(c) Equality in (b)(i) and b(ii) $\Rightarrow$ $J_+ |\psi\rangle = 0$ or $J_- |\psi\rangle = 0$

Moreover, if $J_\pm |\psi\rangle = 0 \Rightarrow \Delta J_\pm |\psi\rangle = 0 \Rightarrow$

$\Rightarrow (\Delta J_x \pm i \Delta J_y) |\psi\rangle = 0 \Rightarrow$ equality in (b)(i) with $\lambda = \pm 1$

and equality in (b)(iii).

Direct check:

$$0 = J_\pm|\psi\rangle = (J_x \pm iJ_y)|\psi\rangle \implies 0 = \langle\psi|J_\pm|\psi\rangle = \underset{0}{\langle J_x\rangle} \pm i\underset{0}{\langle J_y\rangle}$$

$$\implies 0 = \langle\psi|(J_x \pm iJ_y)^2|\psi\rangle = \underbrace{\langle J_x^2\rangle - \langle J_y^2\rangle}_{0} \pm i\underbrace{\langle J_xJ_y + J_yJ_x\rangle}_{0}$$

<u>and</u> $0 = \langle\psi|(J_x \mp iJ_y)(J_x \pm iJ_y)|\psi\rangle$

$$= \langle J_x^2\rangle + \langle J_y^2\rangle \pm i\langle\underbrace{[J_x, J_y]}_{i\hbar J_z}\rangle$$

$$= \langle J_x^2\rangle + \langle J_y^2\rangle \mp \hbar\langle J_z\rangle$$

$$\implies \langle J_x^2\rangle = \langle J_y^2\rangle = \pm\tfrac{1}{2}\hbar\langle J_z\rangle$$

(d) $\vec{J} = \vec{L} = \vec{R}\times\vec{P}$

Suppose $\Delta L_x \Delta L_y = \frac{\hbar}{2}|\langle L_z\rangle|$ and $(\Delta L_x)^2 + (\Delta L_y)^2 = \hbar|\langle L_z\rangle|$.
Then $L_+|\psi\rangle = 0$ or $L_-|\psi\rangle = 0$

$$|\psi\rangle = \sum_{k,l,m} c_{klm}|k,l,m\rangle$$

$$0 = L_\pm|\psi\rangle = \sum_{k,l,m} c_{klm} L_\pm|k,l,m\rangle = \sum_{k,l,m} c_{klm}\sqrt{l(l+1) - m(m\pm1)}\,|k,l,m\pm1\rangle$$

$$\implies |\psi\rangle = \sum_{k,l} c_{kl,\pm l}|k,l,\pm l\rangle$$

$$\psi(\vec{r}) = \langle\vec{r}|\psi\rangle = \sum_{k,l} c_{kl,\pm l} R_{kl}(r)\, Y_l^{\pm l}(\theta,\varphi)$$

$$\propto (\sin\theta)^l e^{\pm il\varphi} = \left(\frac{x\pm iy}{r}\right)^l$$

$$\therefore \boxed{\psi(\vec{r}) = F(r, \sin\theta\, e^{\pm i\varphi})}$$

8.3.

$$H = \vec{P}^2/2\mu + \tfrac{1}{2}\mu\omega^2 \vec{R}^2$$

$$= \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \tfrac{3}{2}\right)$$

$$= \hbar\omega\left(a_R^\dagger a_R + a_L^\dagger a_L + a_z^\dagger a_z + \tfrac{3}{2}\right)$$

$$a_j = \sqrt{\frac{\mu\omega}{2\hbar}}\left(X_j + iP_j/\mu\omega\right), \quad j = 1,2,3$$

$$a_{\substack{R\\L}} = \frac{1}{\sqrt{2}}\left(a_x \mp i a_y\right)$$

$$L_z = \hbar\left(a_R^\dagger a_R - a_L^\dagger a_L\right)$$

$$L_+ = \sqrt{2}\hbar\left(a_z^\dagger a_L - a_R^\dagger a_z\right)$$

$$L_- = \sqrt{2}\hbar\left(a_L^\dagger a_z - a_z^\dagger a_R\right)$$

$$|\chi_{n_R n_L n_z}\rangle = \frac{(a_R^\dagger)^{n_R}(a_L^\dagger)^{n_L}(a_z^\dagger)^{n_z}}{\sqrt{n_R!\,n_L!\,n_z!}}|\chi_{000}\rangle$$

$$L_j = \epsilon_{jkl} X_k P_l$$

$$= \epsilon_{jkl}\sqrt{\frac{\hbar}{2\mu\omega}}\left(a_k + a_k^\dagger\right)(-i)\sqrt{\frac{\hbar\mu\omega}{2}}\left(a_l - a_l^\dagger\right)$$

$$= -i\frac{\hbar}{2}\epsilon_{jkl}\big(\underbrace{a_k a_l - a_k^\dagger a_l^\dagger}_{\text{symmetric in } k \text{ and } l,\ \text{so sum to zero}} - a_k a_l^\dagger + a_k^\dagger a_l\big)$$

$$= -i\frac{\hbar}{2}\epsilon_{jkl}\big(a_k^\dagger a_l - a_l^\dagger a_k - \underbrace{\delta_{lk}}_{\text{sums to zero}}\big)$$

$$= i\hbar\,\epsilon_{jkl}\, a_l^\dagger a_k$$

$$L_z = i\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right)$$

$$L_x = i\hbar\left(a_z^\dagger a_y - a_y^\dagger a_z\right)$$

$$L_y = i\hbar\left(a_x^\dagger a_z - a_z^\dagger a_x\right)$$

$$L_+ = L_x + iL_y$$

$$= i\hbar\big(a_z^\dagger\underbrace{(a_y - i a_x)}_{\substack{= -i(a_x + i a_y)\\ = -i\sqrt{2}\,a_L}} - \underbrace{(a_y^\dagger - i a_x^\dagger)}_{\substack{= -i(a_x^\dagger + i a_y^\dagger)\\ = -i\sqrt{2}\,a_R^\dagger}} a_z\big)$$

$$L_+ = \hbar\sqrt{2}\left(a_z^\dagger a_L - a_R^\dagger a_z\right)$$

$$L_- = L_+^\dagger = \hbar\sqrt{2}\left(a_L^\dagger a_z - a_z^\dagger a_R\right)$$

Work in the 6-d subspace with $n = n_R + n_L + n_z = 2$ ($E_n = \frac{7}{2}\hbar\omega$), which consists of the states

(a) $|\chi_{200}\rangle$

$|\chi_{101}\rangle$

$|\chi_{110}\rangle$, $|\chi_{002}\rangle$

$|\chi_{011}\rangle$

$|\chi_{020}\rangle$

$$\boxed{\begin{aligned} L_z|\chi_{200}\rangle &= 2\hbar|\chi_{200}\rangle \\ L_z|\chi_{101}\rangle &= \hbar|\chi_{101}\rangle \\ L_z|\chi_{110}\rangle &= 0,\quad L_z|\chi_{002}\rangle = 0 \\ L_z|\chi_{011}\rangle &= -\hbar|\chi_{011}\rangle \\ L_z|\chi_{020}\rangle &= -2\hbar|\chi_{020}\rangle \end{aligned}}$$

(b) The eigenvalues of L_z suggest that there is one $l=2$ subspace and one $l=0$ subspace. We know that $|\chi_{200}\rangle$, which has $m=2$, is the top state in an $l=2$ subspace because

$$L_+|\chi_{200}\rangle = \sqrt{2}\hbar(a_z^\dagger a_L - a_R^\dagger a_z)|\chi_{200}\rangle = 0$$

Since L_- cannot take us out of the 6-d subspace, the states in the $l=2$ subspace are

$|\chi_{200}\rangle$ $\quad m=2$

$|\chi_{101}\rangle$ $\quad m=1$

$\alpha|\chi_{110}\rangle + \beta|\chi_{002}\rangle$ $\quad m=0$

$|\chi_{011}\rangle$ $\quad m=-1$

$|\chi_{020}\rangle$ $\quad m=2$

where we must determine α and β. The orthogonal linear combination of $|\chi_{110}\rangle$ and $|\chi_{002}\rangle$ must make up a $l=0$ subspace. We can find this orthogonal linear combination because it must satisfy

$$0 = L_+(\gamma|\chi_{110}\rangle + \delta|\chi_{002}\rangle)$$

$$\Longrightarrow 0 = \gamma(a_z^\dagger a_L - a_R^\dagger a_z)|\chi_{110}\rangle + \delta(a_z^\dagger a_L - a_R^\dagger a_z)|\chi_{002}\rangle$$

$$= \gamma|\chi_{101}\rangle - \delta\sqrt{2}|\chi_{101}\rangle$$

$$\Longrightarrow \gamma = \delta\sqrt{2}$$

Normalization: $1 = |\gamma|^2 + |\delta|^2 = 3|\delta|^2 \Longrightarrow |\delta| = 1/\sqrt{3}$

Choose δ real: $\delta = 1/\sqrt{3}$, $\gamma = \sqrt{2/3}$

$$\boxed{l=0,\ m=0:\quad |0,0\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|\chi_{110}\rangle + |\chi_{002}\rangle\right)}$$

The orthogonal linear combination, which has $l=2$, $m=0$ is $\frac{1}{\sqrt{3}}\left(|\chi_{110}\rangle - \sqrt{2}\,|\chi_{002}\rangle\right)$

$$\boxed{\begin{array}{ll} l=2;\ m=2: & |2,2\rangle = |\chi_{200}\rangle \\ l=2,\ m=1: & |2,1\rangle = |\chi_{101}\rangle \\ l=2,\ m=0: & |2,0\rangle = \frac{1}{\sqrt{3}}\left(|\chi_{110}\rangle - \sqrt{2}\,|\chi_{002}\rangle\right) \\ l=2,\ m=-1: & |2,-1\rangle = |\chi_{011}\rangle \\ l=2,\ m=-2: & |2,-2\rangle = |\chi_{020}\rangle \end{array}}$$

Another way to proceed is to lower $|\chi_{200}\rangle$ successively to produce the other $l=2$ states

$$\boxed{|\chi_{200}\rangle = |2,2\rangle}$$

$$L_-|2,2\rangle = 2\hbar|2,1\rangle$$

$$\shortparallel$$

$$\sqrt{2}\hbar\left(a_L^\dagger a_z - a_z^\dagger a_R\right)|\chi_{200}\rangle = -\sqrt{2}\hbar\sqrt{2}\,|\chi_{101}\rangle = -2\hbar|\chi_{101}\rangle$$

$$\Longrightarrow \boxed{|2,1\rangle = -|\chi_{101}\rangle}$$

$$L_-|2,1\rangle = \sqrt{6}\,\hbar\,|2,0\rangle$$

$$\overset{\shortparallel}{-\sqrt{2}\,\hbar\,(a_L^\dagger a_z - a_z^\dagger a_R)|\chi_{101}\rangle} = -\sqrt{2}\,\hbar\left(|\chi_{110}\rangle - \sqrt{2}\,|\chi_{002}\rangle\right)$$

$$\Longrightarrow \boxed{|2,0\rangle = -\frac{1}{\sqrt{3}}\left(|\chi_{110}\rangle - \sqrt{2}\,|\chi_{002}\rangle\right)}$$

$$L_-|2,0\rangle = \sqrt{6}\,\hbar\,|2,-1\rangle$$

$$\overset{\shortparallel}{-\sqrt{2}\,\hbar\,(a_L^\dagger a_z - a_z^\dagger a_R)\frac{1}{\sqrt{3}}\left(|\chi_{110}\rangle - \sqrt{2}\,|\chi_{002}\rangle\right)}$$

$$= -\sqrt{\frac{2}{3}}\,\hbar\left(-|\chi_{011}\rangle - \sqrt{2}\,|\chi_{011}\rangle\right)$$

$$= \sqrt{6}\,\hbar\,|\chi_{011}\rangle$$

$$\Longrightarrow \boxed{|2,-1\rangle = |\chi_{011}\rangle}$$

When doing things this way, we find the conventional phases of the states $|l,m\rangle$

$$L_-|2,-1\rangle = 2\hbar\,|2,-2\rangle$$

$$\overset{\shortparallel}{\sqrt{2}\,\hbar\,(a_L^\dagger a_z - a_z^\dagger a_R)|\chi_{011}\rangle} = \sqrt{2}\,\hbar\,\sqrt{2}\,|\chi_{020}\rangle$$

$$\Longrightarrow \boxed{|2,-2\rangle = |\chi_{020}\rangle}$$

We find the other $m=0$ state as the orthogonal linear combination to $|2,0\rangle$ and then verify that it has $l=0$ by showing that it raises to 0.

8.4. Schwinger representation

(e) $J_z = \frac{1}{2}\hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$

$J_x = \frac{1}{2}\hbar\left(a_+^\dagger a_- + a_-^\dagger a_+\right)$

$J_y = -\frac{i}{2}\hbar\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$

$$\vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2 = \frac{1}{4}\hbar^2\left[\left(a_+^\dagger a_- + a_-^\dagger a_+\right)^2 - \left(a_+^\dagger a_- - a_-^\dagger a_+\right)^2 + \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)^2\right]$$

$$= \frac{1}{4}\hbar^2\left(2\underbrace{a_+^\dagger a_- a_-^\dagger a_+} + 2\underbrace{a_-^\dagger a_+ a_+^\dagger a_-} + a_+^\dagger a_+ a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_- - 2a_+^\dagger a_+ a_-^\dagger a_-\right)$$

$$a_+^\dagger a_+\left(a_-^\dagger a_- + 1\right) = a_+^\dagger a_+ a_-^\dagger a_- + a_+^\dagger a_+$$

$$a_-^\dagger a_-\left(a_+^\dagger a_+ + 1\right) \qquad a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_-$$

$$= \frac{1}{4}\hbar^2\left[\left(a_+^\dagger a_+\right)^2 + \left(a_-^\dagger a_-\right)^2 + 2a_+^\dagger a_+ a_-^\dagger a_- + 2a_+^\dagger a_+ + 2a_-^\dagger a_-\right]$$

$$= \frac{1}{4}\hbar^2\left(a_+^\dagger a_+ + a_-^\dagger a_- + 2\right)\left(a_+^\dagger a_+ + a_-^\dagger a_-\right)$$

$$\boxed{\vec{J}\cdot\vec{J} = \hbar^2\left(N/2 + 1\right)N/2}$$

$$[J_x, J_y] = -\frac{i}{4}\hbar^2\left[a_+^\dagger a_- + a_-^\dagger a_+,\; a_+^\dagger a_- - a_-^\dagger a_+\right]$$

$$= -\frac{i}{2}\hbar^2\underbrace{\left[a_-^\dagger a_+, a_+^\dagger a_-\right]}$$

$$= a_-^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_- a_-^\dagger a_+$$

$$= a_-^\dagger a_-\left(a_+^\dagger a_+ + 1\right) - a_+^\dagger a_+\left(a_-^\dagger a_- + 1\right)$$

$$= a_-^\dagger a_- - a_+^\dagger a_+$$

$$= i\hbar J_z$$

$$[J_x, J_z] = \frac{1}{4}\hbar^2 \left[a_+^\dagger a_- + a_-^\dagger a_+,\ a_+^\dagger a_+ - a_-^\dagger a_-\right]$$

$$= \frac{1}{4}\hbar^2 \Big(a_- \underbrace{[a_+^\dagger, a_+^\dagger a_+]} + a_-^\dagger \underbrace{[a_+, a_+^\dagger a_+]} - a_+^\dagger \overbrace{[a_-, a_-^\dagger a_-]}^{a_-} - a_+ \overbrace{[a_-^\dagger, a_-^\dagger a_-]}^{-a_-^\dagger} \Big)$$

$$[a_+^\dagger, a_+^\dagger a_+] = a_+^\dagger a_+^\dagger a_+ - a_+^\dagger a_+ a_+^\dagger = a_+^\dagger a_+^\dagger a_+ - a_+^\dagger (a_+^\dagger a_+ + 1) = -a_+^\dagger$$

$$[a_+, a_+^\dagger a_+] = [a_+^\dagger a_+, a_+^\dagger]^\dagger = a_+$$

$$= \frac{1}{2}\hbar^2 \left(-a_+^\dagger a_- + a_-^\dagger a_+ \right)$$

$$= -i\hbar J_y$$

Same commutators give $[J_y, J_z] = i\hbar J_x$

So $\boxed{[J_j, J_k] = i\hbar\, \epsilon_{jkl} J_l}$

(f) We just need to check that $|n_+, n_-\rangle$ is an eigenstate of $\vec{J}^2$ and J_z:

$$\vec{J}^2 |n_+, n_-\rangle = \hbar^2 (N/2+1)\overbrace{(N/2)}^{J} |n_+, n_-\rangle$$

$$= \hbar^2 \left(\frac{n_+ + n_-}{2} + 1 \right) \frac{n_+ + n_-}{2} |n_+, n_-\rangle$$

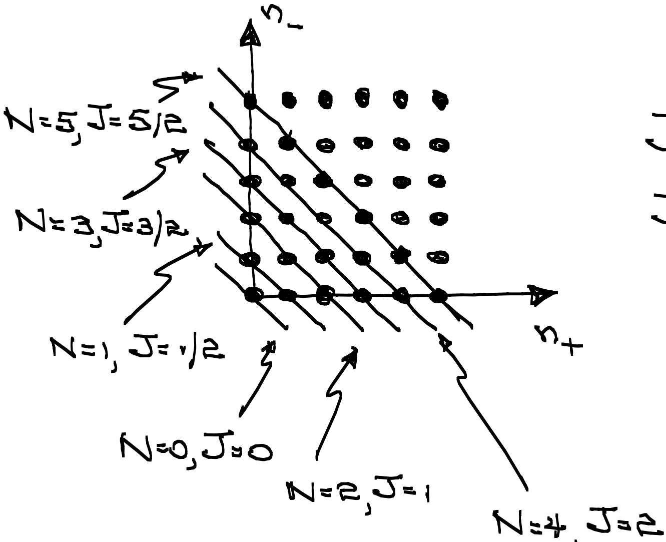
$$J_z |n_+, n_-\rangle = \frac{1}{2}\hbar \left(a_+^\dagger a_+ - a_-^\dagger a_- \right) |n_+, n_-\rangle$$

$$= \hbar \underbrace{\frac{n_+ - n_-}{2}}_{M} |n_+, n_-\rangle$$

$$\Longrightarrow \boxed{|n_+, n_-\rangle = |J = (n_+ + n_-)/2,\ M = (n_+ - n_-)/2\rangle}$$

The eigenvalue equations allow a phase in this relation. But, suppose we choose $|n_+ = N, n_- = 0\rangle = |J = N/2, M = N/2\rangle$ and then lower using $J_- = J_x - iJ_y = \hbar a_-^\dagger a_+$. Each lowering decreases n_+ by 1 and increases n_- by 1, i.e., decreases M by 1, without putting in any phase. So things are as stated.

<u>Digression:</u> The states $|n_+, n_-\rangle = |JM\rangle$ lie on a grid, with the angular-momentum subspaces corresponding to the antidiagonals.



$J_+ = \hbar a_+^\dagger a_-$ ⟵ moves down one unit and to the right one unit

$J_- = \hbar a_-^\dagger a_+$ ⟵ moves to the left one unit and up one unit