Phys 521

Lectures 15-17

Harmonic Oscillator

Harmonic oscillator

Examples:

1. Mass m on spring k

   Classical description x, p.

$$\mathcal{H}(x,p) = p^2/2m + \frac{1}{2}\underbrace{m\omega^2}_{k}x^2$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = p/m \qquad \Longrightarrow \quad \ddot{x} + \omega^2 x = 0$$

2. LC circuit $\quad Q, \Pi$

3. Microwave cavity: modes $\quad x, p$?

4. Free em field: modes $\quad x, p$?

5. Phonons

6. Molecular vibrations

Quantum description

$$x \longrightarrow \hat{x}, \quad p \longrightarrow \hat{p} \qquad [\hat{x}, \hat{p}] = i\hbar$$

$$\mathcal{H} \longrightarrow \hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$$

$\longrightarrow$ HP eqns. of motion and solution

$$\hat{H}_H = \hat{U}^\dagger(t,0)\hat{H}_S\hat{U}(t,0) = \hat{H}_S = \frac{\hat{p}_S^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_S^2$$

$$= \frac{\hat{p}_H^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2$$

$\hat{p}_S = \hat{p}(0)$, $\hat{x}_S = \hat{x}(0)$

$\hat{p}_H = \hat{p}(t)$, $\hat{x}_H = \hat{x}(t)$

$$i\hbar\frac{d\hat{x}}{dt} = [\hat{x},\hat{H}] = (\hat{p}/m)[\hat{x},\hat{p}] = i\hbar\,\hat{p}/m$$

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m}$$

$$i\hbar\frac{d\hat{p}}{dt} = [\hat{p},\hat{H}] = m\omega^2\hat{x}[\hat{p},\hat{x}] = -i\hbar\, m\omega^2\hat{x}$$

$$\frac{d\hat{p}}{dt} = -m\omega^2\hat{x}$$

$\rightarrow$ classical eqns of motion

Soln: $\hat{x}(t) = \hat{x}(0)\cos\omega t + \dfrac{\hat{p}(0)}{m\omega}\sin\omega t$

$\hat{p}(t) = -m\omega\hat{x}(0)\sin\omega t + \hat{p}(0)\cos\omega t$

Creation and annihilation operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) = \hat{a}^\dagger(0)e^{i\omega t}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

gs half widths

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \hbar\omega(\underbrace{\hat{a}^\dagger\hat{a}}_{\hat{N}} + \tfrac{1}{2})$$

Canonical commutator: $[\hat{a},\hat{a}^\dagger] = 1$

Dimensionless position and momentum

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{P} = \sqrt{\frac{1}{\hbar m\omega}}\,\hat{p} = -\frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P})$$

$$[\hat{X},\hat{P}] = i$$

Energy eigenstates:

$$\hat{N}|\varphi_{\lambda,k}\rangle = \lambda|\varphi_{\lambda,k}\rangle \implies \hat{H}|\varphi_{\lambda,k}\rangle = \hbar\omega(\lambda+\tfrac{1}{2})|\varphi_{\lambda,k}\rangle$$

1. $\lambda \geq 0$

2. $\hat{N}(\hat{a}|\varphi_{\lambda,k}\rangle) = \hat{a}(\hat{N}-1)|\varphi_{\lambda,k}\rangle = (\lambda-1)(\hat{a}|\varphi_{\lambda,k}\rangle)$

   $\hat{N}(\hat{a}^\dagger|\varphi_{\lambda,k}\rangle) = \hat{a}^\dagger(\hat{N}+1)|\varphi_{\lambda,k}\rangle = (\lambda+1)(\hat{a}^\dagger|\varphi_{\lambda,k}\rangle)$

   a) $\hat{a}|\varphi_{\lambda,k}\rangle$ is an eigenvector with eigenvalue $\lambda-1$

   $\hat{a}^\dagger|\varphi_{\lambda,k}\rangle$ is an eigenvector with eigenvalue $\lambda+1$

   b) Lowest eigenvalue satisfies $0 \leq \lambda < 1$ and then $\lambda = 0$.

   c) Ladders of states starting from lowest eigenvalue; same degeneracy at each rung

3. Ground state – differential equation $\hat{a}|0\rangle = 0$ – nd

Ladder of states: $\lambda = n = 0, 1, 2, \ldots$

$$|\varphi_{n+1}\rangle = C_n \hat{a}^\dagger |\varphi_n\rangle$$

C_n ← normalization constant

$$1 = \langle\varphi_{n+1}|\varphi_{n+1}\rangle = |C_n|^2 \langle\varphi_n|\underbrace{\hat{a}\hat{a}^\dagger}_{\hat{a}^\dagger\hat{a}+1}|\varphi_n\rangle = |C_n|^2(n+1)$$

$$|C_n| = \frac{1}{\sqrt{n+1}} \quad \text{(choose real)}$$

$$C_n = \frac{1}{\sqrt{n+1}}$$

$$\hat{a}^\dagger|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$$

$$\hat{a}|\varphi_{n+1}\rangle = \frac{1}{\sqrt{n+1}}\underbrace{\hat{a}\hat{a}^\dagger}_{\hat{a}^\dagger\hat{a}+1}|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_n\rangle$$

Summarize

$$\hat{a}|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle$$

$$\hat{a}^\dagger|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$$

Creation & annihilation (destruction) operators

How remember

$$|\varphi_n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|\varphi_0\rangle$$

$\leftarrow$ gs

Expectation values: $|\varphi_n\rangle$

$\langle\hat{a}\rangle$, $\langle\hat{a}^\dagger\rangle = \langle\hat{a}\rangle^*$

$\langle\hat{a}^\dagger\hat{a}\rangle$, $\langle\hat{a}^2\rangle$, $\langle\hat{a}^{\dagger 2}\rangle = \langle\hat{a}^2\rangle^*$

Convert to $\hat{x}, \hat{p}$

Time dependence

Position rep:

Problem 1.2 (4 points) Number-state wave functions. Consider a simple harmonic oscillator that has mass m, angular frequency ω, position operator $\hat{x}$, momentum operator $\hat{p}$, and annihilation operator

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right).$$

(a) Use the fact that the annihilation operator annihilates the ground state, i.e., $\hat{a}|0\rangle = 0$, to derive a differential equation for the ground-state wave function. Show that the solution of this differential equation is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

where you must use normalization and a choice of phase to determine the constant in front of the Gaussian.

(b) Use the raising condition $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|\,n+1\rangle$ to derive a (differential) recursion relation for $\langle x|\,n+1\rangle$ in terms of $\langle x\,|n\rangle$, and use your result to derive explicitly the wave function $\langle x|1\rangle$ for the one-quantum state from the wave function in part (a).

(c) Use the definition of the n-quantum state,

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle,$$

to write the wave function of the n-quantum state in the form

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\frac{1}{2}\xi^2}\Big|_{\xi=\sqrt{\frac{m\omega}{\hbar}}x}.$$

Use this result to write the wave function of the n-quantum state in the conventional form

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\xi^2} H_n(\xi), \qquad \xi \equiv \sqrt{\frac{m\omega}{\hbar}}x,$$

where $H_n(\xi)$ is the nth Hermite polynomial. You may use the following definition of the Hermite polynomials:

$$H_n(\xi) \equiv (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}.$$

Number-state wave functions

HO with $m, \omega, \hat{x}, \hat{p}, \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + i\hat{p}/m\omega)$

(a) Ground-state wave function

Ground state: $|0\rangle$, where $\hat{a}|0\rangle = 0$

Wave function: $\langle x|0\rangle$

$$0 = \langle x|\hat{a}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|0\rangle$$

$$\frac{d\langle x|0\rangle}{dx} = -\frac{m\omega}{\hbar}x\langle x|0\rangle$$

$$\Longrightarrow \quad \langle x|0\rangle = C\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

normalization constant - chosen real

$$|\langle x|0\rangle|^2 = C^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)$$

Normalization $\Longrightarrow \quad C^2 = \frac{1}{\sqrt{\pi\hbar/m\omega}} = \sqrt{\frac{m\omega}{\pi\hbar}}$

$$\Longrightarrow \boxed{\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)}$$

(b) Number-state wave functions:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\langle x|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,\langle x|n+1\rangle$$

$$\sqrt{n+1}\,\langle x|n+1\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,\underbrace{\langle x|(\hat{x} - i\hat{p}/m\omega)|n\rangle}_{\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|n\rangle}$$

$$\boxed{\langle x|n+1\rangle = \sqrt{\frac{m\omega}{2\hbar(n+1)}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|n\rangle}$$

$$\langle x|1\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,\underbrace{\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)}_{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(x + x)\,\exp\left(-\frac{m\omega}{2\hbar}x^2\right)}$$

$$\boxed{\langle x|1\rangle = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\Big(\underbrace{\sqrt{\frac{m\omega}{\hbar}}\,2x}_{H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right)}\Big)}$$

(c) Number-state wave functions

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\underbrace{\langle x|(\hat{x} - i\hat{p}/m\omega)^n|0\rangle}_{\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\langle x|0\rangle}$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\underbrace{\left(\sqrt{\frac{m\omega}{\hbar}}\,x - \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)^n\exp\left(-\frac{m\omega}{2\hbar}x^2\right)}_{\left.\left(\xi - \frac{d}{d\xi}\right)^n e^{-\frac{1}{2}\xi^2}\right|_{\xi = \sqrt{\frac{m\omega}{\hbar}}x}}$$

Notice that

$$\left(\xi - \frac{d}{d\xi}\right) e^{\frac{1}{2}\xi^2} f(\xi) = -e^{\frac{1}{2}\xi^2} \frac{df(\xi)}{d\xi}$$

$$\left(\xi - \frac{d}{d\xi}\right)^n \left(e^{\frac{1}{2}\xi^2} f(\xi)\right) = (-1)^n e^{\frac{1}{2}\xi^2} \frac{d^n f(\xi)}{d\xi^n}$$

$$\therefore \langle x|n\rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (-1)^n e^{\frac{1}{2}\xi^2} \underbrace{\left.\frac{d^n}{d\xi^n} e^{-\xi^2}\right|_{\xi = \sqrt{\frac{m\omega}{\hbar}}\,x}}_{(-1)^n e^{-\xi^2} H_n(\xi)}$$

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\xi^2} H_n(\xi), \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

Quasiclassical coherent states

$\Delta x \Delta p \geq \hbar/2$

$[\hat{x}, \hat{p}] = i\hbar$

$[\hat{X}, \hat{P}] = i \quad \Delta X \Delta P \geq \frac{1}{2}$

$[\hat{a}, \hat{a}^\dagger] = 1$

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \frac{1}{2}(\hat{P}^2 + \hat{X}^2) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Motivation: The c.s. are mus for $\hat{X}$ and $\hat{P}$, i.e., $\Delta X \Delta P = 1/2$ ($\Delta x \Delta p = \hbar/2$), with the additional requirement that $\Delta X = \Delta P = 1/\sqrt{2}$ ($\Delta x = \sqrt{\hbar/2m\omega}$, $\Delta p = \sqrt{\hbar m\omega/2}$).

Review:

$$(\Delta X)^2(\Delta P)^2 = \langle\psi|(\overbrace{\hat{X} - \langle\hat{X}\rangle}^{\Delta\hat{X}})^2|\psi\rangle\langle\psi|(\overbrace{\hat{P} - \langle\hat{P}\rangle}^{\Delta\hat{P}})^2|\psi\rangle$$

Schwarz inequality

$$\geq |\langle\psi|\Delta\hat{X}\,\Delta\hat{P}|\psi\rangle|$$

$$\Delta\hat{X}\,\Delta\hat{P} = \frac{1}{2}(\Delta\hat{X}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{X}) + \frac{1}{2}\underbrace{[\Delta\hat{X}, \Delta\hat{P}]}_{[\hat{X},\hat{P}] = i}$$

$$= \Big|\underbrace{\frac{1}{2}\langle\psi|(\Delta\hat{X}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{X})|\psi\rangle}_{C} + \frac{i}{2}\Big|^2$$

$$= \frac{1}{4} + C^2$$

$$\geq \frac{1}{4}$$

MUS: ① $0 = C = \frac{1}{2}\langle\psi|(\Delta\hat{x}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{x})|\psi\rangle$

② $\Delta\hat{x}|\psi\rangle = -i\lambda\,\Delta\hat{p}|\psi\rangle \Leftrightarrow (\Delta\hat{x} + i\lambda\,\Delta\hat{p})|\psi\rangle = 0$

① $\Rightarrow$ $0 = C = \frac{1}{2}\underbrace{\langle\psi|\Delta\hat{x}\Delta\hat{p}|\psi\rangle}_{i\lambda^*(\Delta p)^2} + \frac{1}{2}\underbrace{\langle\psi|\Delta\hat{p}\Delta\hat{x}|\psi\rangle}_{-i\lambda(\Delta p)^2}$

$= \frac{i}{2}(\lambda^* - \lambda)(\Delta p)^2$

$= \mathrm{Im}(\lambda)(\Delta p)^2$

$\Rightarrow \lambda = \lambda^*$

② $\Rightarrow$ $(\Delta x)^2 = \langle\psi|(\Delta\hat{x})^2|\psi\rangle = \lambda^2\langle\psi|(\Delta\hat{p})^2|\psi\rangle = \lambda^2(\Delta p)^2$

MUS: $(\Delta\hat{x} + i\underset{\substack{\uparrow\\ \text{real}}}{\lambda}\Delta\hat{p})|\psi\rangle$ $\qquad \frac{(\Delta x)^2}{(\Delta p)^2} = \lambda^2$

CS: $\lambda = 1$ $\quad (\Delta\hat{x} + i\Delta\hat{p})|\psi\rangle = 0 \Leftrightarrow \Delta\hat{a}|\psi\rangle = 0$

$$\boxed{\hat{a}|\varphi_\alpha\rangle = \alpha|\varphi_\alpha\rangle}$$

There is an eigenstate for each complex number α

Ex: gs

Moments:

Position rep:

$\langle x|\hat{a}|\varphi_\alpha\rangle = \alpha\langle x|\varphi_\alpha\rangle$

$$\sqrt{\frac{m\omega}{2\hbar}}\, \underbrace{\langle x| \hat{x} + i \frac{\hat{p}}{m\omega} |\varphi_\alpha\rangle}_{\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|\varphi_\alpha\rangle} = \alpha \langle x|\varphi_\alpha\rangle$$

$$\frac{d}{dx}\langle x|\varphi_\alpha\rangle = -\frac{m\omega}{\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)\langle x|\varphi_\alpha\rangle$$

$$\ln\langle x|\varphi_\alpha\rangle = (\text{const}) - \frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)^2$$

$$= (\text{const}) - \frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_R - i\sqrt{\frac{2\hbar}{m\omega}}\,\alpha_I\right)^2$$

$$= (\text{const}) - \frac{m\omega}{2\hbar}\left[\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_R\right)^2 - 2i\sqrt{\frac{2\hbar}{m\omega}}\, x\alpha_I\right]$$

$$= (\text{const}) - \frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_R\right)^2 + \frac{i}{\hbar}\sqrt{2\hbar m\omega}\,\alpha_I x$$

$$\langle x|\varphi_\alpha\rangle = C\, e^{\frac{i}{\hbar}\sqrt{2\hbar m\omega}\,\alpha_I x} \exp\left[-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_R\right)^2\right]$$

$$|C| = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \qquad \text{choose } C \text{ real}$$

Notice that $\langle x|\varphi_{\alpha=0}\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$ is the wave function of the ground state $|\varphi_{n=0}\rangle$, so $|\varphi_{\alpha=0}\rangle = |\varphi_{n=0}\rangle$. Or we know this from

$$\hat{a}|\varphi_{\alpha=0}\rangle = 0 \Rightarrow \hat{a}^\dagger\hat{a}|\varphi_{\alpha=0}\rangle = 0 \Rightarrow |\varphi_{\alpha=0}\rangle = |\varphi_{n=0}\rangle \text{ within a phase}$$

Displacement operator: $\boxed{D(\hat{a},\alpha) = \hat{D}(\alpha) \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}$

conventional notation

unitarity

$$[D(\hat{a},\alpha)]^{-1} = [D(\hat{a},\alpha)]^\dagger = D(\hat{a},-\alpha) = D(-\hat{a},\alpha)$$

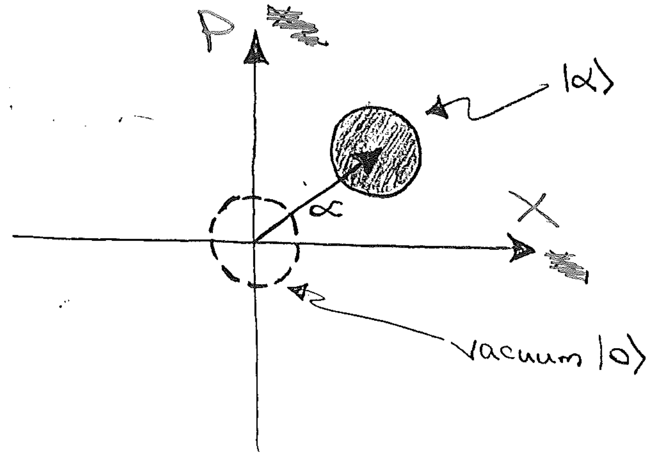
Crucial property $\quad [D(\hat{a},\alpha)]^\dagger \hat{a}\, D(\hat{a},\alpha) = \hat{a} + \alpha \quad \longleftarrow$ displaces $\hat{a}$

$$\Longrightarrow \quad \hat{a}\, D(\hat{a},\alpha)|0\rangle = \alpha D(\hat{a},\alpha)|0\rangle$$

$$\boxed{|\alpha\rangle = D(\hat{a},\alpha)|0\rangle}$$

Alternative definition

Fixes phase



$$[D(\hat{a},\beta)]^\dagger D(\hat{a},\alpha)\, D(\hat{a},\beta) = D(\hat{a}+\beta,\alpha) = \underbrace{D(\beta,\alpha)}\, D(\hat{a},\alpha)$$

phase factor $\longrightarrow e^{\alpha\beta^* - \alpha^*\beta}$

$$D(\hat{a},\beta)\, D(\hat{a},\alpha) = \underbrace{D(\tfrac{1}{2}\beta,-\alpha)}\, D(\hat{a},\alpha+\beta) \quad \longleftarrow \text{use BCH identity}$$

$e^{\frac{1}{2}(\alpha^*\beta - \alpha\beta^*)} \longleftarrow$ phase factor

$$D(\hat{a},\alpha) = \underbrace{e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}}}_{\text{normally ordered form}} = \underbrace{e^{|\alpha|^2/2}\, e^{-\alpha^*\hat{a}}\, e^{\alpha\hat{a}^\dagger}}_{\text{antinormally ordered form}}$$

Further properties of coherent states:

① $$\langle \beta | \alpha \rangle = \langle 0 | \underbrace{[D(\hat{a},\beta)]^{\dagger} D(\hat{a},\alpha)}_{D(\frac{1}{2}\beta,\alpha)\, D(\hat{a},\alpha-\beta)} | 0 \rangle = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}\, e^{-\frac{1}{2}|\alpha-\beta|^2}$$

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha-\beta|^2} \longleftarrow$$ Coherent states are not orthogonal

② $$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^{\dagger}} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle n | \alpha \rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$$

$$P_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \longleftarrow$$ Poisson distribution w/ mean $\langle \hat{n} \rangle = |\alpha|^2$

Time dependence