Phys 521

Lectures 18-20

Two-level systems

Two-level systems

Examples:

1. Two atomic energy levels
2. Spin-1/2
3. Polarization of a single photon
4. Double-well potential

Kinematics:

Example 1: Polarization of a single photon

Plane wave propagating in +z direction

Linear polarization along $\vec{f}_x$

$$\vec{E} = \mathrm{Re}\left(E_0 e^{i(kz-\omega t)}\vec{f}_x\right)$$

Linear polarization along $\vec{f}_y$

$$\vec{E} = \mathrm{Re}\left(E_0 e^{i(kz-\omega t)}\vec{f}_y\right)$$

Inside phasor: real linear combinations $\Rightarrow$ other lin pol

complex linear combination $\Longrightarrow$ phase delay elliptical pol

QM in 1-particle sector: $\hat{a}_x^\dagger|0\rangle = |x\rangle$ $\qquad$ $\hat{a}_y^\dagger|0\rangle = |y\rangle$

$$\vec{f}_x \leftrightarrow |x\rangle \longleftrightarrow \hat{a}_x^\dagger \qquad \vec{f}_y \longleftrightarrow |y\rangle \longleftrightarrow \hat{a}_y^\dagger$$

Right circular polarization (positive helicity)

$$\vec{E} = \mathrm{Re}\left(E_0 e^{i(kz-\omega t)} \underbrace{\frac{1}{\sqrt{2}}(\vec{f}_x + i\vec{f}_y)}_{\equiv \vec{f}_R}\right)$$

$z=0$
$$= \frac{1}{\sqrt{2}} E_0 (\cos\omega t\, \vec{f}_x + \sin\omega t\, \vec{f}_y)$$

$1 \leftrightarrow \cos\omega t$

$i \leftrightarrow \sin\omega t$

$$\hat{a}_R^\dagger |0\rangle = |R\rangle$$

$$\vec{f}_R \leftrightarrow |R\rangle \leftrightarrow \hat{a}_R^\dagger$$

$$\vec{f}_R = \frac{1}{\sqrt{2}}(\vec{f}_x + i\vec{f}_y)$$

$\hat{a}_R^\dagger$

Left circular polarization (negative helicity)

$$\vec{E} = \mathrm{Re}\left(E_0 e^{i(kz-\omega t)} \underbrace{\frac{1}{\sqrt{2}}(\vec{f}_x - i\vec{f}_y)}_{\equiv \vec{f}_L}\right)$$

$$= \frac{1}{\sqrt{2}} E_0 (\cos\omega t\, \vec{f}_x - \sin\omega t\, \vec{f}_y)$$

$$\hat{a}_L^\dagger |0\rangle = |L\rangle$$

$$\vec{f}_L \leftrightarrow |L\rangle \leftrightarrow \hat{a}_L^\dagger$$

$$\vec{e}_L = \frac{1}{\sqrt{2}}(\vec{f}_x - i\vec{f}_y)$$

$\hat{a}_L^\dagger$

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle) \qquad |L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$$

$$|x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$$

$$|y\rangle = -\frac{i}{\sqrt{2}}(|R\rangle - |L\rangle)$$

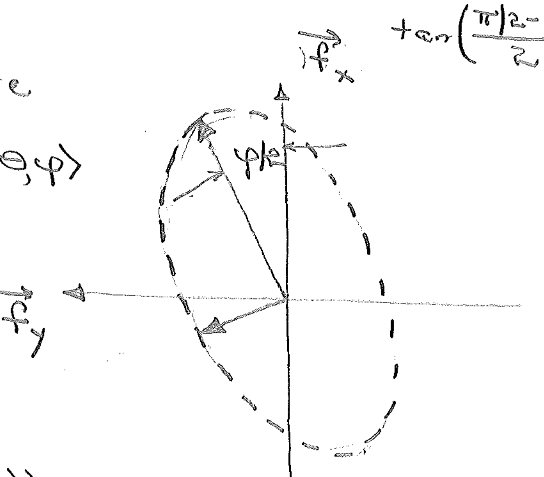
General polarization state:

$u_x = \sin\theta\cos\varphi$
$u_y = \sin\theta\sin\varphi$
$u_z = \cos\theta$

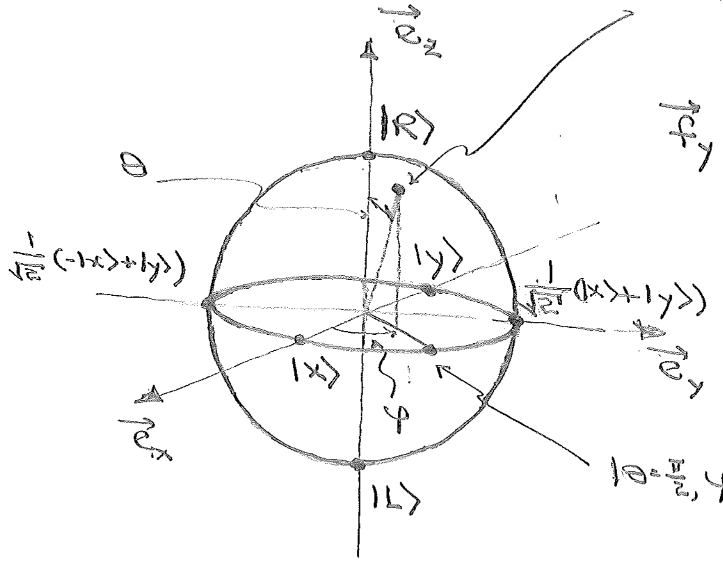
$$\cos(\theta/2)\, e^{-i\varphi/2} |R\rangle + \sin(\theta/2)\, e^{i\varphi/2} |L\rangle \equiv |\theta,\varphi\rangle = |+\rangle_{\vec{u}}$$

$0 \le \theta \le \pi$
$0 \le \varphi \le 2\pi$
→ all physical states, overall phase freedom removed

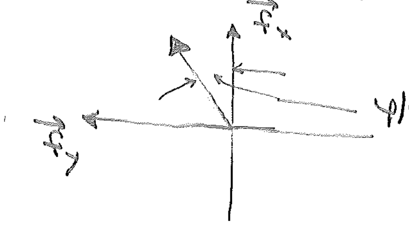
Minor-to-major axis ratio: $\tan\left(\frac{\pi/2-\theta}{2}\right)$



Poincare or Bloch sphere



$|\theta=\frac{\pi}{2},\varphi\rangle = |x\rangle\cos(\varphi/2) + |y\rangle\sin(\varphi/2)$



$|\theta=0,\varphi\rangle = |R\rangle = |+\rangle_{\vec{e}_z}$

$|\theta=\pi,\varphi\rangle = |L\rangle = |+\rangle_{-\vec{e}_z}$

$|\theta=\frac{\pi}{2},\varphi=0\rangle = |x\rangle = |+\rangle_{\vec{e}_x}$

$|\theta=\frac{\pi}{2},\varphi=\pi\rangle = |y\rangle = |+\rangle_{-\vec{e}_x}$

$|\theta=\frac{\pi}{2},\varphi=\frac{\pi}{2}\rangle = \frac{1}{\sqrt{2}}(|x\rangle+|y\rangle) = |+\rangle_{\vec{e}_y}$

$|\theta=\frac{\pi}{2},\varphi=\frac{3\pi}{2}\rangle = \frac{1}{\sqrt{2}}(-|x\rangle+|y\rangle) = |+\rangle_{-\vec{e}_y}$

$$|\theta,\varphi\rangle \doteq \cos\left(\frac{\pi/2-\theta}{2}\right)\left(|x\rangle\cos(\varphi/2)+|y\rangle\sin(\varphi/2)\right) + i\sin\left(\frac{\pi/2-\theta}{2}\right)\left(-|x\rangle\sin(\varphi/2)+|y\rangle\cos(\varphi/2)\right)$$

Why the half angles?
Why the change of sign when $\varphi \to \varphi + 2\pi$?

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$$|\theta,\varphi\rangle = \cos(\theta/2)\, e^{-i\varphi/2} \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$$

$$+ \sin(\theta/2)\, e^{+i\varphi/2} \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$$

$$= \frac{1}{\sqrt{2}}|x\rangle\left(c\, e^{-i\varphi/2} + s\, e^{+i\varphi/2}\right)$$

$$+ \frac{i}{\sqrt{2}}|y\rangle\left(c\, e^{-i\varphi/2} - s\, e^{+i\varphi/2}\right)$$

$$= \frac{1}{\sqrt{2}}|x\rangle\left((c+s)\cos(\varphi/2) - i(c-s)\sin(\varphi/2)\right)$$

$$+ \frac{1}{\sqrt{2}}|y\rangle\left(i(c-s)\cos(\varphi/2) + (c+s)\sin(\varphi/2)\right)$$

$$\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}\left(\cos(\theta/2) + \sin(\theta/2)\right)$$

$$\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}\left(\cos(\theta/2) - \sin(\theta/2)\right)$$

$$|\theta,\varphi\rangle = |x\rangle\left(\cos(\varphi/2)\cos\left(\frac{\pi/2-\theta}{2}\right) - i\sin(\varphi/2)\sin\left(\frac{\pi/2-\theta}{2}\right)\right)$$

$$+ |y\rangle\left(i\cos(\varphi/2)\sin\left(\frac{\pi/2-\theta}{2}\right) + \sin(\varphi/2)\cos\left(\frac{\pi/2-\theta}{2}\right)\right)$$

$$= \cos\left(\frac{\pi/2-\theta}{2}\right)\left(|x\rangle\cos(\varphi/2) + |y\rangle\sin(\varphi/2)\right)$$

$$+ i\sin\left(\frac{\pi/2-\theta}{2}\right)\left(-|x\rangle\sin(\varphi/2) + |y\rangle\cos(\varphi/2)\right)$$

Orthogonal polarizations lie at antipodes of the Poincare sphere ($\vec{u}$ and $-\vec{u}$)

$$|+\rangle_{\vec{u}} \qquad\qquad |+\rangle_{-\vec{u}} = |-\rangle_{\vec{u}}$$

Note: C-T uses $|-\rangle_{\vec{u}} = -i|+\rangle_{-\vec{u}}$

This is the ordinary dot product in the plane of polarization

$$_{\vec{u}'}\langle + | + \rangle_{\vec{u}} = \cos(\theta/2)\cos(\theta'/2)\, e^{-i(\varphi-\varphi')/2} + \sin(\theta/2)\sin(\theta'/2)\, e^{i(\varphi-\varphi')/2}$$

$$= \cos[(\theta-\theta')/2]\cos[(\varphi-\varphi')/2] - i\cos[(\theta+\theta')/2]\sin[(\varphi-\varphi')/2]$$

$$\longrightarrow 0 \quad \text{if } \vec{u}' = -\vec{u}$$

Observables:

Ex: Suppose light in polarization state $|+\rangle_{\vec{u}}$ is incident on a linear polarizer that transmits $\vec{e}_x$ polarization.

(fraction of intensity transmitted) = (probability that single photon is transmitted) = (probability that single photon has $\vec{e}_x$-pol)

$$= \Big|\underbrace{\langle x|}_{{}_{\vec{e}_x}\langle +|}+\rangle_{\vec{u}}\Big|^2 = \left|\cos\left(\frac{\pi/2-\theta}{2}\right)\cos(\varphi/2) - i\cos\left(\frac{\pi/2-\theta}{2}\right)\sin(\varphi/2)\right|^2$$

$\theta' = \pi/2$
$\varphi' = 0$

More generally, if light with polarization $\vec{u}$ falls on a polarizer that distinguishes $\vec{u}'$ and $-\vec{u}'$,

$$(\qquad)\cdot(\qquad)\cdot(\qquad)$$

$$= \left| {}_{\vec{u}'}\langle + | + \rangle_{\vec{u}} \right|^2$$

$$= \tfrac{1}{2}\left(1+\cos(\theta-\theta')\right)\tfrac{1}{2}\left(1+\cos(\varphi-\varphi')\right) + \tfrac{1}{2}\left(1+\cos(\theta+\theta')\right)\tfrac{1}{2}\left(1-\cos(\varphi-\varphi')\right)$$

$$= \tfrac{1}{2}\left(1+\cos\theta\cos\theta' + \sin\theta\sin\theta'\right)\tfrac{1}{2}\left(1+\cos(\varphi-\varphi')\right) + \tfrac{1}{2}\left(1+\cos\theta\cos\theta' - \sin\theta\sin\theta'\right)\tfrac{1}{2}\left(1-\cos(\varphi-\varphi')\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\underbrace{\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right)}_{\vec{u}'\cdot\vec{u}}$$

$$= \tfrac{1}{2}\left(1+\vec{u}'\cdot\vec{u}\right)$$

$$\boxed{\left| {}_{\vec{u}'}\langle + | + \rangle_{\vec{u}} \right|^2 = \tfrac{1}{2}\left(1+\vec{u}'\cdot\vec{u}\right)}$$

Geometry of Poincaré sphere means something

Let $\hat{\sigma}_{\vec{u}}$ be the observable that has value $+1$ for $\vec{u}$-polarization and -1 for $-\vec{u}$-polarization

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$$\hat{\sigma}_{\vec{u}} = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| - |+\rangle_{-\vec{u}}\,{}_{-\vec{u}}\langle +| = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|$$

$$|+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| = \cos^2(\theta/2)\,|R\rangle\langle R| + \sin^2(\theta/2)\,|L\rangle\langle L|$$
$$+ \cos(\theta/2)\sin(\theta/2)\left(e^{-i\varphi}|R\rangle\langle L| + e^{i\varphi}|L\rangle\langle R|\right)$$

matrix rep

$$|-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -| = \sin^2(\theta/2)\,|R\rangle\langle R| + \cos^2(\theta/2)\,|L\rangle\langle L|$$
$$- \cos(\theta/2)\sin(\theta/2)\left(+e^{-i\varphi}|R\rangle\langle L| + e^{i\varphi}|L\rangle\langle R|\right)$$

$\theta \to \pi - \theta$

$\varphi \to \varphi + \pi$

$\cos(\theta/2) \to \sin(\theta/2)$

$\sin(\theta/2) \to \cos(\theta/2)$

$e^{i\varphi} \to -e^{i\varphi}$

matrix rep

$$|+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| + |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -| = \hat{1}$$

$$\hat{\sigma}_{\vec{u}} = \cos\theta\left(|R\rangle\langle R| - |L\rangle\langle L|\right)$$
$$+ \sin\theta\left(e^{-i\varphi}|R\rangle\langle L| + e^{i\varphi}|L\rangle\langle R|\right)$$
$$= \overbrace{\sin\theta\cos\varphi}^{u_x}\,\overbrace{\left(|R\rangle\langle L| + |L\rangle\langle R|\right)}^{\equiv\hat{\sigma}_x}$$
$$+ \overbrace{\sin\theta\sin\varphi}^{u_y}\,\overbrace{\left(-i|R\rangle\langle L| + i|L\rangle\langle R|\right)}^{\equiv\hat{\sigma}_y}$$
$$+ \underbrace{\cos\theta}_{u_z}\,\underbrace{\left(|R\rangle\langle R| - |L\rangle\langle L|\right)}_{\hat{\sigma}_z}$$

$$\boxed{\hat{\sigma}_{\vec{u}} = \vec{u} \cdot \hat{\vec{\sigma}} = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|}$$

$\hat{\sigma}_{\vec{u}}|+\rangle_{\vec{u}} = |+\rangle_{\vec{u}}$

$\hat{\sigma}_{\vec{u}}|-\rangle_{\vec{u}} = -|-\rangle_{\vec{u}}$

$$|x\rangle\langle x| - |y\rangle\langle y| = \hat{\sigma}_{x_1} = |R\rangle\langle L| + |L\rangle\langle R| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

45 + 135

$$\hat{\sigma}_{y_2} = -i|R\rangle\langle L| + i|L\rangle\langle R| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\sigma}_{z_3} = |R\rangle\langle R| - |L\rangle\langle L| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

Properties of Pauli matrices:

① Hermitian: $\hat{\sigma}_j^\dagger = \hat{\sigma}_j$ $\qquad (\hat{\sigma}_{\vec{n}}^\dagger = \hat{\sigma}_{\vec{n}})$

② $\hat{\sigma}_j^2 = \hat{1}$ $\qquad (\hat{\sigma}_{\vec{n}}^2 = \hat{\sigma}_n)$

③ $\hat{\sigma}_1\hat{\sigma}_2 = -\hat{\sigma}_2\hat{\sigma}_1 = i\hat{\sigma}_3, \quad \hat{\sigma}_2\hat{\sigma}_3 = -\hat{\sigma}_3\hat{\sigma}_2 = i\hat{\sigma}_1, \quad \hat{\sigma}_3\hat{\sigma}_1 = -\hat{\sigma}_1\hat{\sigma}_3 = i\hat{\sigma}_2$

$$\boxed{\hat{\sigma}_j\hat{\sigma}_k = \hat{1}\,\delta_{jk} + i\epsilon_{jkl}\hat{\sigma}_l}$$

sum on repeated indices

antisymmetric symbol

All products can be reduced to a single matrix.

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\epsilon_{jkl}\hat{\sigma}_l \qquad \begin{cases} [\hat{\sigma}_1, \hat{\sigma}_2] = 2i\hat{\sigma}_3 \\ [\hat{\sigma}_2, \hat{\sigma}_3] = 2i\hat{\sigma}_1 \\ [\hat{\sigma}_3, \hat{\sigma}_1] = 2i\hat{\sigma}_2 \end{cases}$$

$$[\hat{\sigma}_j, \hat{\sigma}_k]_+ = \hat{\sigma}_j\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_j = 2\delta_{jk}\hat{1}$$

anticommutator

④ $\mathrm{tr}(\hat{\sigma}_j) = 0$ $\qquad [\mathrm{tr}(\hat{\sigma}_{\vec{n}}) = 0]$

⑤ Any (linear) operator (2×2-matrix) can be written as a complex linear combination of $\hat{1}$ and the $\hat{\sigma}_j$'s (the operators are a complex 4-d vector space; they are also a group called GL(2, C)):

$$\hat{A} = A_0\hat{1} + A_j\hat{\sigma}_j = A_0\hat{1} + \vec{A}\cdot\hat{\vec{\sigma}} = A_\alpha\hat{\sigma}_\alpha$$

$(\hat{\sigma}_0 \equiv \hat{1})$

$\alpha = 0, 1, 2, 3$

If $\hat{A}^\dagger = \hat{A}$, the coefficients are real.

Antisymmetric symbol

$$\epsilon_{jkl} = \begin{cases} +1, & \text{if } jkl \text{ is an even permutation of } 123 \\ -1, & \text{if } jkl \text{ is an odd permutation of } 123 \\ 0, & \text{otherwise} \end{cases}$$

Nonzero components: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321} = +1$

1. If M is a 3×3 matrix, $\det M = \epsilon_{jkl} M_{1j} M_{2k} M_{3l}$

2. $\vec{A}\times\vec{B} = \vec{e}_j\,\epsilon_{jkl} A_k B_l = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$

3. $\epsilon_{jkl}\,\epsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$

   $\Rightarrow \epsilon_{jkl}\,\epsilon_{jkm} = 2\delta_{lm}$

4. $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

(a) $\hat{A} = A_\alpha \hat{\sigma}_\alpha \iff A_\alpha = \frac{1}{2}\mathrm{tr}(\hat{\sigma}_\alpha \hat{A}) = \frac{1}{2}\mathrm{tr}(\hat{A}\hat{\sigma}_\alpha)$

Use the "orthogonality" property $\frac{1}{2}\mathrm{tr}(\hat{\sigma}_\alpha \hat{\sigma}_\beta) = \delta_{\alpha\beta}$

(b) If $\hat{A} = A_\alpha \hat{\sigma}_\alpha = A_0 \hat{1} + \vec{A}\cdot\hat{\vec{\sigma}}$ and $\hat{B} = B_0\hat{1} + \vec{B}\cdot\hat{\vec{\sigma}}$

then

$$\hat{A}\hat{B} = (A_0 B_0 + \vec{A}\cdot\vec{B})\hat{1} + A_0(\vec{B}\cdot\hat{\vec{\sigma}}) + B_0(\vec{A}\cdot\hat{\vec{\sigma}}) + i\,\vec{A}\times\vec{B}\cdot\hat{\vec{\sigma}}$$

$$\Longrightarrow \mathrm{tr}(\hat{A}\hat{B}) = 2(A_0 B_0 + \vec{A}\cdot\vec{B}$$

(c) If $\hat{A} = A_0\hat{1} + \vec{A}\cdot\hat{\vec{\sigma}}$ is Hermitian, then

$$\hat{A} = (A_0 + |\vec{A}|)\,|\vec{n}\rangle\langle\vec{n}| + (A_0 - |\vec{A}|)\,|-\vec{n}\rangle\langle-\vec{n}|$$

eigenvalues

eigenvectors

where $\vec{n} = \vec{A}/|\vec{A}|$.

(6) Raising and lowering operators

$$\hat{\sigma}_\pm \equiv \frac{1}{2}(\hat{\sigma}_1 \pm i\hat{\sigma}_2) \iff \begin{aligned}\hat{\sigma}_1 &= \hat{\sigma}_+ + \hat{\sigma}_- \\ \hat{\sigma}_2 &= -i(\hat{\sigma}_+ - \hat{\sigma}_-)\end{aligned}$$

$$\hat{\sigma}_+^\dagger = \hat{\sigma}_-$$

$$\hat{\sigma}_+ = |R\rangle\langle L| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ "raising" operator

$$\hat{\sigma}_- = |L\rangle\langle R| \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ "lowering" operator

$$\hat{\sigma}_{\vec{n}} = \vec{n}\cdot\hat{\vec{\sigma}} = \sin\theta\left(e^{-i\varphi}\hat{\sigma}_+ + e^{i\varphi}\hat{\sigma}_-\right) + \cos\theta\,\hat{\sigma}_3$$

$$\hat{\sigma}_\pm^2 = \pm\frac{i}{4}[\hat{\sigma}_1,\hat{\sigma}_2]_+ = 0$$

$$\hat{\sigma}_\pm\hat{\sigma}_\mp = \frac{1}{2} \mp \frac{i}{4}\underbrace{[\hat{\sigma}_1,\hat{\sigma}_2]}_{2i\hat{\sigma}_3} = \frac{1}{2}(\hat{1} \pm \hat{\sigma}_3)$$

$$\hat{\sigma}_\pm\hat{\sigma}_3 = \mp\hat{\sigma}_\pm$$

$$\hat{\sigma}_3\hat{\sigma}_\pm = \pm\hat{\sigma}_\pm$$

$$[\hat{\sigma}_\pm,\hat{\sigma}_\mp] = \pm\hat{\sigma}_3$$

$$[\hat{\sigma}_\pm,\hat{\sigma}_\mp]_+ = \hat{1}$$

$$[\hat{\sigma}_\pm,\hat{\sigma}_3] = \mp 2\hat{\sigma}_\pm$$

$$[\hat{\sigma}_\pm,\hat{\sigma}_3]_+ = 0$$

Note that

$$|e\rangle\langle e| = \frac{1}{2}(\hat{1}+\hat{\sigma}_3) \leftrightarrow \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$|g\rangle\langle g| = \frac{1}{2}(\hat{1}-\hat{\sigma}_3) \leftrightarrow \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

(7)
$$\hat{P}_{\vec{n}} = |\vec{n}\rangle\langle\vec{n}| = \cos^2(\theta/2)\,\frac{1}{2}(\hat{1}+\hat{\sigma}_3) + \sin^2(\theta/2)\,\frac{1}{2}(\hat{1}-\hat{\sigma}_3)$$

$$+ \cos(\theta/2)\sin(\theta/2)\left(\cos\varphi\,\hat{\sigma}_1 + \sin\varphi\,\hat{\sigma}_2\right)$$

$$= \frac{1}{2}\left(\hat{1} + \vec{n}\cdot\hat{\vec{\sigma}}\right)$$

$$|\langle\vec{n}'|\vec{n}\rangle|^2 = \mathrm{tr}\left(\hat{P}_{\vec{n}}\hat{P}_{\vec{n}'}\right) = \frac{1}{2}\left(1 + \vec{n}'\cdot\vec{n}\right)$$

Unitary operators: The most general unitary operator $\hat{U}$ can be written as $e^{i(\text{Hermitian operator})}$, so we can write $\hat{U}$ as

$$\hat{U} = \exp\Big[i\big(\mu\hat{1} - \tfrac{1}{2}\gamma\,\vec{m}\cdot\hat{\vec{\sigma}}\big)\Big] = e^{i\mu}\underbrace{\exp\left(-\tfrac{i}{2}\gamma\,\vec{m}\cdot\hat{\vec{\sigma}}\right)}_{\text{rotation by angle }\gamma\text{ about }\vec{m}}$$

unit vector

these operators make up a group $U(2)$

(8) $e^{-i\vec{m}\cdot\vec{\sigma}\alpha} = \cos\alpha - i\vec{m}\cdot\vec{\sigma}\sin\alpha$

Example 2: Spin-$\frac{1}{2}$

State space: $|R\rangle = |+\rangle_{\vec{e}_z} = |+\rangle_z$; $|L\rangle = |-\rangle_{\vec{e}_z} = |-\rangle_z$

Observables: $\hat{\vec{S}} = \frac{1}{2}\hbar\hat{\vec{\sigma}}$ $\quad [S_j, S_k] = i\hbar\epsilon_{jkl}S_l$

$$\hat{S}_x = \frac{1}{2}\hbar\hat{\sigma}_x = \frac{1}{2}\hbar\left(|+\rangle_x\,{}_x\langle +| - |-\rangle_x\,{}_x\langle -|\right) \quad \text{reps.}$$

$$= \frac{1}{2}\hbar\left(|+\rangle_z\,{}_z\langle -| + |-\rangle_z\,{}_z\langle -|\right)$$

$\hat{S}_y$, $\hat{S}_z$, $\hat{\vec{S}}\cdot\vec{n}$ — reps.

The Poincare sphere is called the Bloch sphere, and it <u>is</u> the 3-d space where the spin resides.

Magnetic moment $\vec{M} = \gamma\vec{S}$ (Sign of γ?)

Magnetic field $\vec{B} = B_0\vec{m}$

Hamiltonian $H = -\vec{M}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\gamma B_0\vec{S}\cdot\vec{m}$

Heisenberg equation for $\vec{S}$:

$$\frac{dS_j}{dt} = -\frac{i}{\hbar}[S_j, H] = +\frac{i}{\hbar}\gamma\underbrace{[S_j, S_k]}_{i\hbar\epsilon_{jkl}S_l}B_k = -\gamma\epsilon_{jkl}B_kS_l$$

rotation about $\vec{m}$ at angular velocity $-\gamma B_0$

$$\frac{d\vec{S}}{dt} = -\gamma\vec{B}\times\vec{S} = -\gamma B_0\vec{m}\times\vec{S}$$

$$U = e^{-iHt/\hbar} = e^{i\gamma B_0 t\,\vec{m}\cdot\vec{S}/\hbar} = e^{i\vec{m}\cdot\vec{\sigma}(\gamma B_0 t/2)}$$

Density operator: Suppose a light source emits various pure polarizations. Let $P(\vec{u})\,d\Omega$ be the probability that it emits polarization $\vec{u}$ within solid angle $d\Omega$ on the Poincaré sphere. Let this light be incident on a $\vec{u}'$ polarizer.

$$\begin{pmatrix}\text{fraction of intensity}\\ \text{transmitted}\end{pmatrix} = \begin{pmatrix}\text{probability that a single}\\ \text{photon is transmitted}\end{pmatrix}$$

$$= \int d\Omega\, P(\vec{u}) \underbrace{\left|{}_{\vec{u}'}\langle + | + \rangle_{\vec{u}}\right|^2}_{\frac{1}{2}(1+\vec{u}'\cdot\vec{u})}$$

$$= \frac{1}{2}\left(1 + \vec{u}'\cdot\vec{P}\right)$$

$$\vec{P} \equiv \int d\Omega\, P(\vec{u})\,\vec{u} = \begin{pmatrix}\text{polarization}\\ \text{vector}\end{pmatrix}, \quad |\vec{P}| \le 1$$

The three components of $\vec{P}$ determine the statistics of any polarization measurement

$$= {}_{\vec{u}'}\langle + | \underbrace{\left(\int d\Omega\, P(\vec{u})\, |+\rangle_{\vec{u}}\, {}_{\vec{u}}\langle + |\right)}_{\hat{\rho} = \text{(density operator}} | + \rangle_{\vec{u}'}$$

$$= {}_{\vec{u}'}\langle + | \hat{\rho} | + \rangle_{\vec{u}'}$$

$$= \mathrm{tr}\left(\hat{\rho}\, \hat{P}_{\vec{u}'}\right)$$

$$\hat{\rho} = \int d\Omega\, P(\vec{u})\, \hat{P}_{\vec{u}} = \frac{1}{2}(\hat{1} + \vec{P}\cdot\hat{\vec{\sigma}})$$ Interior of Poincaré Sphere

The density operator is determined by $\vec{P}$. Letting $\vec{P} = |\vec{P}|\vec{u}$,

① $\hat{\rho} = \frac{1}{2}(\hat{1} + |\vec{P}|\vec{u}\cdot\hat{\vec{\sigma}})$

$$= \frac{1}{2}(1 + |\vec{P}|)\,|+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle +| + \frac{1}{2}(1 - |\vec{P}|)\,|-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle -|$$

② $\hat{\rho}$ is a pure state iff $|\vec{P}| = 1$

③ $\langle \hat{\sigma}_j \rangle = \mathrm{tr}(\hat{\rho}\,\hat{\sigma}_j) = P_j$

Matrix form: $\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{P}\cdot\hat{\vec{\sigma}})$

$$\longrightarrow \begin{pmatrix} \frac{1}{2}(1+P_3) & \frac{1}{2}(P_1 - iP_2) \\ \frac{1}{2}(P_1 + iP_2) & \frac{1}{2}(1-P_3) \end{pmatrix}$$