Phys 521

Angular Momentum

Lectures 21-25

Angular Momentum

$\vec{L}$ — quantize classical orbital ang mom

$\vec{S}$ — intrinsic (spin) ang mom

$\vec{J}$ — total angular momentum (or general)

Orbital ang mom

Single particle: $\vec{\mathcal{L}} = \vec{r} \times \vec{p}$

$$\mathcal{L}_x = yp_z - zp_y$$
$$\mathcal{L}_y = zp_x - xp_z$$
$$\mathcal{L}_z = xp_y - yp_x$$
$$\mathcal{L}_j = \epsilon_{jkl} x_k p_l$$

Summation convention + Antisymmetric symbol

$$\epsilon_{jkl} L_l = \epsilon_{jkl}\epsilon_{lmn} X_m P_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) X_m P_n$$

Quantize:
$$L_x = YP_z - ZP_y$$
$$L_y = ZP_x - XP_z$$
$$L_z = XP_y - YP_x$$
$$L_j = \epsilon_{jkl} X_k P_l$$

Vector operator: $\vec{L} = L_x \vec{e}_x + L_y \vec{e}_y + L_z \vec{e}_z = \vec{R} \times \vec{P}$

No ordering problems

Commutators:
$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z]$$
$$= YP_x [P_z, Z] + [Z, P_z] XP_y$$
$$= i\hbar (XP_y - YP_x)$$
$$= i\hbar L_z$$

$$\epsilon_{jkl} L_l = \epsilon_{jkl}\epsilon_{lmn} X_m P_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) X_m P_n = X_j P_k - X_k P_j$$

Antisymmetric symbol

$$\epsilon_{jkl} = \begin{cases} +1, & \text{if } jkl \text{ is an even permutation of } 123 \\ -1, & \text{if } jkl \text{ is an odd permutation of } 123 \\ 0, & \text{otherwise} \end{cases}$$

Nonzero components: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321} = +1$

1. If M is a 3×3 matrix, $\det M = \epsilon_{jkl} M_{1j} M_{2k} M_{3l}$

2. $\vec{A} \times \vec{B} = \vec{e}_j \, \epsilon_{jkl} A_k B_l = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$

3. $\epsilon_{jkl}\,\epsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$

   $\Longrightarrow \epsilon_{jkl}\,\epsilon_{jkm} = 2\,\delta_{lm}$

4. $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$[L_j, L_k] = \epsilon_{jlm}\,\epsilon_{knq}\,\underbrace{[X_l P_m, X_n P_q]}$$

$$= X_l P_m X_n P_q - X_n P_q X_l P_m$$

$$= X_l (X_n P_m - i\hbar\,\delta_{nm}) P_q$$

$$- X_n (X_l P_q - i\hbar\,\delta_{ql}) P_m$$

$$= i\hbar\left(-\delta_{nm} X_l P_q + \delta_{ql} X_n P_m\right)$$

$$= i\hbar\left(-\epsilon_{jlm}\,\epsilon_{kmq}\,X_l P_q + \epsilon_{jlm}\,\epsilon_{knl}\,X_n P_m\right)$$

$$= i\hbar\,\underbrace{\epsilon_{ljm}\,\epsilon_{lkn}}_{\delta_{jk}\delta_{mn} - \delta_{jn}\delta_{km}}\,(X_m P_n - X_n P_m)$$

$$= i\hbar\,(X_j P_k - X_k P_j)$$

$$= i\hbar\,\epsilon_{jkl} L_l$$

$$\boxed{[L_j, L_k] = i\hbar\,\epsilon_{jkl} L_l}$$

Generalization to many particles

Generalization: $[J_j, J_k] = i\hbar\,\epsilon_{jkl} J_l$

Properties of rotations in 3-d space

Ex: S_x, S_y, S_z (have additional simple product property)

$$\vec{J}^2 = \vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2 = J_j J_j$$

$$[\vec{J}^2, J_k] = [J_j J_j, J_k] = J_j J_j J_k - J_k J_j J_j$$

Do this for a particular component

$$= J_j (J_k J_j + [J_j, J_k])$$

$$- (J_j J_k + [J_k, J_j]) J_j$$

$$= J_j [J_j, J_k] + [J_j, J_k] J_j$$

$$= \epsilon_{jkl} J_j J_l + \epsilon_{jkl} J_l J_j$$

$$\quad\quad\quad\quad\quad\quad\quad\quad l \; j \quad j \; l$$

$$= 0$$

Eigenvalue problem: Find simultaneous eigenstates of $\vec{J}^2$ and J_z

$$\vec{J}^2 |k, \lambda, m\rangle = \lambda \hbar^2 |k, \lambda, m\rangle \qquad \lambda \geq 0 \quad (\text{why})$$

$\hbar^2 \longrightarrow$ units

$$J_z |k, \lambda, m\rangle = m\hbar |k, \lambda, m\rangle$$

CSCO

Role of k? $A_1, A_2, \ldots, \vec{J}^2, J_z \ldots$ Example: Spin-1/2

$[A_\alpha, J_k] = 0 \longleftarrow$ $\boxed{\text{Commute with all Components of } \vec{J}}$

Raising and lowering operators:

$$J_\pm = J_x \pm i J_y \quad , \quad J_+^\dagger = J_-$$

Commutators: $[J_z, J_\pm] = [J_z, J_x \pm i J_y]$

$$= i\hbar(+J_y \mp i J_x)$$

$$= \pm\hbar(J_x \pm i J_y)$$

$$= \pm\hbar J_\pm$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = 2\hbar J_z$$

$$\boxed{\begin{aligned} [J_z, J_\pm] &= \pm\hbar J_\pm \\ [J_+, J_-] &= 2\hbar J_z \\ [\vec{J}^2, J_\pm] &= 0 \end{aligned}}$$

$$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i\underbrace{[J_x, J_y]}_{i\hbar J_z}$$

$$\boxed{\begin{aligned} J_+ J_- &= J_x^2 + J_y^2 + \hbar J_z = \vec{J}^2 - J_z^2 + \hbar J_z \\ J_- J_+ &= J_x^2 + J_y^2 - \hbar J_z = \vec{J}^2 - J_z^2 - \hbar J_z \\ \tfrac{1}{2}(J_+ J_- + J_- J_+) &= \vec{J}^2 - J_z^2 = J_x^2 + J_y^2 \end{aligned}}$$

<u>Theorem.</u> $|k, \lambda, m\rangle$, $\lambda \geq m^2$

<u>Proof:</u> $0 \leq \langle k, \lambda, m | (\vec{J}^2 - J_z^2) | k, \lambda, m\rangle = \hbar^2 (\lambda - m^2)$

$$A_\alpha J_\pm |k, \lambda, m\rangle = J_\pm A_\alpha |k, \lambda, m\rangle = (A_\alpha)_{k\lambda} J_\pm |k, \lambda, m\rangle$$

$$\vec{J}^2 J_\pm |k, \lambda, m\rangle = \lambda \hbar^2 J_\pm |k, \lambda, m\rangle$$

This argument shows this does not depend on m

$$J_z J_\pm |k, \lambda, m\rangle = (J_\pm J_z \pm \hbar J_\pm)|k, \lambda, m\rangle = (m \pm 1) J_\pm |k, \lambda, m\rangle$$

$$\Longrightarrow \quad J_\pm |k, \lambda, m\rangle = C_\pm(\lambda, m)\hbar |k, \lambda, m \pm 1\rangle$$

For a given k, λ, there must be a largest value of $m =: j \leq \sqrt{\lambda}$. $\therefore$

$$J_+ |k, \lambda, j\rangle = 0$$

$$0 = J_- J_+ |k, \lambda, j\rangle = (\vec{J}^2 - J_z^2 - \hbar J_z)|k, \lambda, j\rangle = \hbar^2 (\lambda - j^2 - j)$$

$$\Longrightarrow \quad \lambda = j(j+1) \qquad (\Rightarrow j \geq 0)$$

For a given k, λ, there must be a smallest value of $m = j' \geq -\sqrt{\lambda}$. $\therefore$

$$J_- |k, \lambda, j'\rangle = 0$$

$$0 = J_+ J_- |k, \lambda, j'\rangle = (\vec{J}^2 - J_z^2 + \hbar J_z) |k, \lambda, j'\rangle$$
$$= (\lambda^2 - j'^2 + j')$$

$$\Longrightarrow \quad \lambda = j'(j'-1) = j(j+1) \quad \Longrightarrow \quad j' = -j \text{ or } \cancel{j' = j+1}$$

Must be able to get from $|k, \lambda, j\rangle$ to $|k, \lambda, -j\rangle$ in a finite number of applications of J_- $\Longrightarrow$ j is integral or half-integral.

$$|k, \lambda, m\rangle = |k, j, m\rangle \qquad \lambda = j(j+1)$$

$j =$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\lambda =$	0	$\frac{3}{4}$	2	$\frac{15}{4}$	6
$m =$	0	$+\frac{1}{2}$, $-\frac{1}{2}$	1, 0, -1	$\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$	2, 1, 0, -1, -2

# of times, $g(j)$, that j is realized varies from system to system.

For given value of j, all possible values of m must be present

Structure of Hilbert space:

$$J_+ |k, j, m\rangle = C_+(j,m)\hbar\, |k, j, m+1\rangle$$

$$\hbar^2 |C_+(j,m)|^2 = \langle k,j,m| J_- J_+ |k,j,m\rangle$$

$$= \langle k,j,m| (\vec{J}^2 - J_z^2 - \hbar J_z) |k,j,m\rangle$$

$$= \hbar^2 (j(j+1) - m(m+1))$$

$$\Longrightarrow \quad |C_+(j,m)| = \sqrt{j(j+1) - m(m+1)} = \sqrt{(j-m)(j+m+1)}$$

Choose phases to be unity

$$\boxed{J_+ |k,j,m\rangle = \sqrt{\underbrace{(j-m)(j+m+1)}_{j(j+1)-m(m+1)}}\,\hbar\, |k,j,m+1\rangle} \qquad J_+|k,j,j\rangle = 0$$

$$J_- J_+ |k,j,m\rangle = \sqrt{(j-m)(j+m+1)}\,\hbar J_- |k,j,m+1\rangle$$

$$\parallel$$

$$\hbar^2 (j-m)(j+m+1)|k,j,m\rangle \qquad m = m'-1$$

$$\Longrightarrow \quad \boxed{J_- |k,j,m\rangle = \sqrt{\underbrace{(j+m)(j-m+1)}_{j(j+1)-m(m-1)}}\,\hbar\, |k,j,m-1\rangle} \qquad J_-|k,j,-j\rangle = 0$$

Determines matrix elements of J_+ and, hence, $J_- = J_+^\dagger$

(i) $j = 0$:

(ii) $j = \frac{1}{2}$:

$$J^2 \to \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$J_z \to \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_+ \to \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- \to \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{2}(J_+ + J_-) \longrightarrow \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{J} = \vec{S} = \frac{1}{2}\hbar\vec{\sigma}$$

$$J_y = -\frac{i}{2}(J_+ - J_-) \longrightarrow \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(iii) $j = 1$:

$$J^2 \longrightarrow 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_z \longrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J_+ \longrightarrow \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- \longrightarrow \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_x \longrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y \longrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(iv) General

(A)

$$\vec{L} \to \frac{\hbar}{i}\vec{r}\times\nabla = \frac{\hbar}{i}\vec{r}\times\left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= \frac{\hbar}{i}\left(\vec{e}_\varphi\frac{\partial}{\partial\theta} - \vec{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\vec{e}_r = \cos\theta\,\vec{e}_z + \sin\theta(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y)$$

$$\vec{e}_\theta = -\sin\theta\,\vec{e}_z + \cos\theta(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y)$$

$$\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)$$

$$L_x = i\hbar\left(-\underbrace{\vec{e}_x\cdot\vec{e}_\varphi}_{\sin\varphi}\frac{\partial}{\partial\theta} + \underbrace{\vec{e}_x\cdot\vec{e}_\theta}_{\cos\theta\cos\varphi}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$L_y = i\hbar\left(-\underbrace{\vec{e}_y\cdot\vec{e}_\varphi}_{-\cos\varphi}\frac{\partial}{\partial\theta} + \underbrace{\vec{e}_y\cdot\vec{e}_\theta}_{\cos\theta\sin\varphi}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$L_z = \frac{\hbar}{i}(-\vec{e}_z\cdot\vec{e}_\theta)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

$$\vec{L}^2 = \vec{L}\cdot\vec{L} = -\hbar^2(\vec{r}\times\nabla)\cdot(\vec{r}\times\nabla)$$

$$= -\hbar^2(\epsilon_{jkl}x_k\partial_l)(\epsilon_{jmn}x_m\partial_n)$$

$$= -\hbar^2\epsilon_{jkl}\epsilon_{jmn}x_k\partial_l(x_m\partial_n)$$

$$= -\hbar^2(\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm})x_k\partial_l(x_m\partial_n)$$

$$= -\hbar^2\left(x_k\partial_l(x_k\partial_l) - x_k\partial_l(x_l\partial_k)\right)$$

(B)

$$= -\hbar^2\left(x_k\partial_k + x_k x_k \partial_l \partial_l - 3x_k\partial_k - x_k x_l \partial_l \partial_k\right)$$

$$= -\hbar^2\left(\underbrace{-2x_k\partial_k - x_k x_l \partial_k \partial_l} + r^2\nabla^2\right)$$

$$= -x_k\partial_k - x_k\partial_k(x_l\partial_l)$$

$$= -\frac{1}{r}x_k\partial_k\left(r x_l \partial_l\right) \qquad \frac{1}{r}\vec{r}\cdot\nabla = \frac{\partial}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$\vec{L}^2 = -\hbar^2 r^2\left(\underbrace{\nabla^2 - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)}\right)$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$= -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Orbital angular momentum

Single spinless particle

$$L_x = YP_z - ZP_y \longrightarrow \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$L_y = ZP_x - XP_z \longrightarrow \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$L_z = XP_y - YP_x \longrightarrow \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

$\uparrow$

$|\vec{r}\rangle$ representation

Spherical polar coords: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$d^3x = r^2\sin\theta\, dr\, d\theta\, d\varphi = r^2\, dr\, d\Omega$

$$\vec{L}^2 = -\hbar^2\left(\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)}_{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$= -\hbar^2 r^2\left(\nabla^2 - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right)$$

$$L_\pm = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

Eigenvalue problem: $|k,l,m\rangle$, $\psi_{klm}(r,\theta,\varphi) = \langle\vec{r}|k,l,m\rangle$

$$\vec{L}^2\psi_{klm}(r,\theta,\varphi) = l(l+1)\hbar^2\psi_{klm}(r,\theta,\varphi)$$

$$L_z\psi_{klm}(r,\theta,\varphi) = m\hbar\,\psi_{klm}(r,\theta,\varphi)$$

① $\psi_{klm}(r,\theta,\varphi) = R_{klm}(r)\,Y_l^m(\theta,\varphi)$

$\longleftarrow$ undetermined by $\vec{L}^2$ and L_z

$$\vec{L}^2 Y_l^m = l(l+1)\hbar^2 Y_l^m$$

$$L_z Y_l^m = m\hbar Y_l^m$$

(2) $$\frac{\hbar}{i}\frac{\partial Y_\ell^m}{\partial\varphi} = m\hbar Y_\ell^m \Longrightarrow \frac{\partial Y_\ell^m}{\partial\varphi} = im Y_\ell^m$$

$$\Longrightarrow Y_\ell^m(\theta,\varphi) = F_\ell^m(\theta)\, e^{im\varphi}$$

Continuity $\Longrightarrow$ m integral $\Longrightarrow$ ℓ integral

(3) $$0 = L_+ Y_\ell^\ell = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) F_\ell^\ell(\theta)\, e^{i\ell\varphi}$$

$$\Longrightarrow 0 = \left(\frac{d}{d\theta} - \ell\cot\theta\right) F_\ell^\ell(\theta) = (\sin\theta)^\ell \frac{d}{d\theta}\left(\frac{F_\ell^\ell}{(\sin\theta)^\ell}\right)$$

$$\Longrightarrow F_\ell^\ell(\theta) = c_\ell (\sin\theta)^\ell$$

normalization constant

For each integral ℓ, $\exists$ a unique solution F_ℓ^ℓ.

Obtain other Y_ℓ^m by successive application of L_-

(4) $$L_\pm \Psi_{k\ell m}(r,\theta,\varphi) = \sqrt{\ell(\ell+1) - m(m\pm1)}\,\hbar\, \underbrace{\Psi_{k\ell,m\pm1}(r,\theta,\varphi)}_{R_{k\ell,m\pm1}(r)\, Y_\ell^{m\pm1}}$$

$$\overset{\shortparallel}{R_{k\ell m}(r)\, L_\pm Y_\ell^m}$$

$$\Longrightarrow R_{k\ell m}(r) = R_{k\ell,m\pm1}(r) = R_{k\ell}(r)$$

$$L_\pm Y_\ell^m = \sqrt{\ell(\ell+1) - m(m\pm1)}\,\hbar\, Y_\ell^{m\pm1}$$

Gather results:

$$\langle\vec{r}\,|\,k,\ell,m\rangle = \Psi_{k\ell m}(r,\theta,\varphi) = R_{k\ell}(r)\, Y_\ell^m(\theta,\varphi)$$

$$Y_\ell^m(\theta,\varphi) = F_\ell^m(\theta)\, e^{im\varphi}$$

$$\vec{L}^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m$$

$$L_z Y_\ell^m = m\hbar Y_\ell^m$$

$$L_\pm Y_\ell^m = \sqrt{\ell(\ell+1) - m(m\pm1)}\,\hbar\, Y_\ell^{m\pm1}$$

① Orthonormality

$$\delta_{kk'}\delta_{\ell\ell'}\delta_{mm'} = \langle k,\ell,m | k',\ell',m'\rangle$$

$$= \int d^3x \, \langle k,\ell,m|\vec{r}\rangle\langle\vec{r}|k',\ell',m'\rangle$$

$$= \int d^3x \, \psi^*_{k\ell m}(\vec{r})\,\psi_{k'\ell'm'}(\vec{r})$$

$$= \int r^2 dr \, R^*_{k\ell}(r)\, R_{k'\ell'}(r)$$

$$\times \int d\Omega \, Y_\ell^{m*}(\theta,\varphi)\, Y_{\ell'}^{m'}(\theta,\varphi)$$

$$\Longrightarrow \int r^2 dr\, R^*_{k\ell}(r)\, R_{k'\ell}(r) = \delta_{kk'}$$

$$\int d\Omega\, Y_\ell^{m*}(\theta,\varphi)\, Y_{\ell'}^{m'}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

choose normalization

② Completeness:

$$\mathbb{1} = \sum_{k,\ell,m} |k,\ell,m\rangle\langle k,\ell,m|$$

$$\Longrightarrow \delta(\vec{r}-\vec{r}\,') = \sum_{k,\ell,m} \langle\vec{r}|k,\ell,m\rangle\langle k,\ell,m|\vec{r}\,'\rangle$$

$$= \sum_{k,\ell,m} R_{k\ell}(r)\, Y_\ell^m(\theta,\varphi)\, Y_\ell^{m*}(\theta',\varphi')\, R^*_{k\ell}(r')$$

$$= \frac{1}{r^2\sin\theta}\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi') = 3\,\delta(r^3-r'^3)\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')$$

$$\frac{1}{r^2\sin\theta}\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

$$= \sum_{\ell,m} Y_\ell^m(\theta,\varphi)\,Y_\ell^{m*}(\theta',\varphi')\left(\sum_k R_{k\ell}(r)\,R_{k\ell}^*(r')\right)$$

$$\Rightarrow \quad \frac{1}{r^2}\delta(r-r') = \sum_{k,\ell} R_{k\ell}(r)\,R_{k\ell}^*(r') \longleftarrow$$ consistency with orthonormality

$$\frac{1}{\sin\theta}\delta(\theta-\theta')\,\delta(\varphi-\varphi') = \underbrace{\sum_{\ell m}}_{\text{range}} Y_\ell^m(\theta,\varphi)\,Y_\ell^{m*}(\theta',\varphi') \hookleftarrow$$

(3) Expansions:

$$f(\theta,\varphi) = \int d\Omega'\,\frac{1}{\sin\theta'}\delta(\theta-\theta')\,\delta(\varphi-\varphi')\,f(\theta',\varphi')$$

$$= \sum_{\ell m} Y_\ell^m(\theta,\varphi)\int d\Omega'\,Y_\ell^{m*}(\theta',\varphi')\,f(\theta',\varphi')$$

Calculation of Y_ℓ^m: Start with Y_ℓ^ℓ; lower using L_-.

$$Y_\ell^\ell(\theta,\varphi) = c_\ell(\sin\theta)^\ell e^{i\ell\varphi}$$

$$1 = \int d\Omega\,|Y_\ell^\ell(\theta,\varphi)|^2 = 2\pi|c_\ell|^2 \underbrace{\int_0^\pi d\theta\,(\sin\theta)^{2\ell+1}}_{I_\ell}$$

$$I_\ell = \int_0^\pi d\theta\,(\sin\theta)^{2\ell+1} = \int_0^\pi d\theta\,\underbrace{(\sin^2\theta)}_{1-\cos^2\theta}(\sin\theta)^{2\ell-1} \qquad g = (\sin\theta)^{2\ell}/2\ell$$

$$\ell \ge 1 \qquad = I_{\ell-1} - \underbrace{\int_0^\pi d\theta\,\underset{f}{\cos\theta}\,\underset{g'}{\cos\theta(\sin\theta)^{2\ell-1}}}$$

$$\cancel{\frac{1}{2\ell}\cos\theta\,(\sin\theta)^{2\ell}\Big|_0^\pi} + \frac{1}{2\ell}\int_0^\pi d\theta\,(\sin\theta)^{2\ell+1}$$

$$I_\ell = I_{\ell-1} - I_\ell / 2\ell$$

$$I_\ell = \frac{2\ell}{2\ell+1} I_{\ell-1} \quad \text{and} \quad I_0 = \int_0^\pi d\theta \sin\theta = 2$$

$$I_\ell = \frac{(2\ell)!!}{(2\ell+1)!!} I_0 = \frac{((2\ell)!!)^2}{(2\ell+1)!} I_0 = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!}$$

$$(2\ell)!! = (2\ell)(2\ell-2)\ldots 2 = 2^\ell \ell!$$

$$|c_\ell| = \frac{1}{\sqrt{2\pi I_\ell}} = \frac{1}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}$$

Choose $\quad c_\ell = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} \iff Y_\ell^0(\theta=0,\varphi) > 0$

$$Y_\ell^\ell(\theta,\varphi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} \underbrace{(\sin\theta)^\ell e^{i\ell\varphi}}_{(\sin\theta\, e^{i\varphi})^\ell = \left(\frac{x+iy}{r}\right)^\ell}$$

Apply L_- to get rest:

Examples:

① $Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$

② $Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$

$$L_- Y_1^1 = \hbar\sqrt{2}\, Y_1^0 = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right) Y_1^1$$

$$= -\hbar e^{-i\varphi}\sqrt{\frac{3}{8\pi}}\underbrace{\left(-\cos\theta + i\frac{\cos\theta}{\sin\theta}\sin\theta\, i\right)}_{-2\cos\theta} e^{i\varphi}$$

$$= \hbar\, 2\sqrt{\frac{3}{8\pi}} \cos\theta$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$L_- Y_1^0 = \hbar\sqrt{2}\, Y_1^{-1} = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right) Y_1^0$$

$$= \hbar e^{-i\varphi} \sqrt{\frac{3}{4\pi}} \sin\theta$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x - iy}{r}$$

Other properties:

① $Y_l^{-m} = (-1)^m Y_l^{m\,*}$

② $Y_l^m(0,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}$

③ $Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$

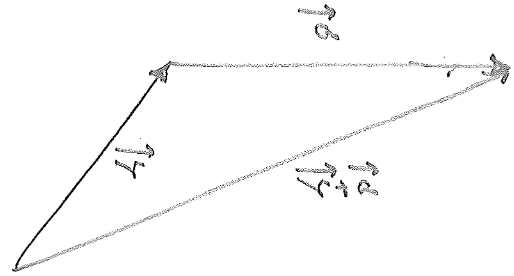
④ Parity: $(-1)^l$

Angular momentum and rotations

Example: translations

Single particle

Classical translation (active):



$$\mathcal{J}_{\vec{a}}\vec{r} = \vec{r}\,' = \vec{r} + \vec{a}$$

Commutative group

(1) States: $|\psi\rangle \xrightarrow[\mathcal{J}_{\vec{a}}]{} |\psi'\rangle = T_{\vec{a}}|\psi\rangle$

$\uparrow$ map

$$\langle \vec{r}\,'|\psi'\rangle = \langle \vec{r}|\psi\rangle = \langle \mathcal{J}_{\vec{a}}^{-1}\vec{r}\,'|\psi\rangle = \langle \vec{r}\,'-\vec{a}|\psi\rangle$$

$$\Longrightarrow \quad \langle \vec{r}|T_{\vec{a}}|\psi\rangle = \langle \vec{r}-\vec{a}|\psi\rangle$$

(i) $T_{\vec{a}}$ linear (matrix elements: $\langle \vec{r}|T_{\vec{a}}|\vec{r}\,'\rangle = \langle \vec{r}-\vec{a}|\vec{r}\,'\rangle = \delta(\vec{r}-\vec{r}\,'-\vec{a})$)

(ii) $T_{\vec{a}}T_{\vec{a}}^{\dagger} = \mathbb{1}$ $\quad \langle \vec{r}|T_{\vec{a}}^{\dagger}T_{\vec{a}}|\vec{r}\,'\rangle = \int d^3x''\, \underbrace{\langle \vec{r}|T_{\vec{a}}^{\dagger}|\vec{r}\,''\rangle}_{\delta(\vec{r}\,''-\vec{r}-\vec{a})}\underbrace{\langle \vec{r}\,''|T_{\vec{a}}|\vec{r}\,'\rangle}_{\delta(\vec{r}\,''-\vec{r}\,'-\vec{a})}$

$$= \delta(\vec{r}\,'-\vec{r})$$

$$\Longrightarrow T_{\vec{a}}^{\dagger}T_{\vec{a}} = \mathbb{1}$$

(iii) Rep of translation group: $T_{\vec{a}}T_{\vec{b}} = T_{\vec{a}+\vec{b}}$

$$\langle \vec{r}|T_{\vec{a}}|\vec{r}\,'\rangle = \langle \vec{r}-\vec{a}|\vec{r}\,'\rangle$$
$$= \langle \vec{r}|\vec{r}\,'+\vec{a}\rangle$$
$$= \langle \vec{r}|\mathcal{J}_{\vec{a}}\vec{r}\,'\rangle$$
$$\Rightarrow T_{\vec{a}}|\vec{r}\rangle = |\mathcal{J}_{\vec{a}}\vec{r}\rangle$$
$$= |\vec{r}+\vec{a}\rangle$$

Let $\vec{a} = a\vec{e}_x$:

$$\langle \vec{r}|T_{a\vec{e}_x}|\psi\rangle = \langle \vec{r}-\vec{a}|\psi\rangle$$

$$= \psi(x-a, y, z)$$

$$= \underbrace{\left(\sum_{k=0}^{\infty} \frac{1}{k!}(-a)^k \frac{\partial^k}{\partial x^k}\right)}_{e^{-a\partial/\partial x}} \psi(x,y,z)$$

$$\langle \vec{r} | T_{a\vec{e}_x} | \psi \rangle = \underbrace{e^{-a\,\partial/\partial x}}_{\exp\left(-\frac{i}{\hbar} a \frac{\hbar}{i}\frac{\partial}{\partial x}\right)} \psi(x,y,z)$$

$$= \langle \vec{r} | e^{-\frac{i}{\hbar} a P_x} | \psi \rangle$$

$$\Longrightarrow \quad T_{a\vec{e}_x} = e^{-\frac{i}{\hbar} a P_x}$$

Generalize: $\boxed{T_{\vec{a}} = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}}$ Check properties
Many particles – fields

Check: $T_{\vec{a}} |\vec{r}\rangle = |\vec{r} + \vec{a}\rangle$

② Operators

$\langle \psi' | A | \psi' \rangle$ is the expectation value of A with respect to the translated state $|\psi'\rangle = T_{\vec{a}} |\psi\rangle$. We can give an equivalent description of the translation in which the operators, instead of the states, are transformed.

$$\langle \psi' | A | \psi' \rangle = \langle \psi | \underbrace{T_{\vec{a}}^{\dagger} A T_{\vec{a}}}_{A'} | \psi \rangle = \langle \psi | A' | \psi \rangle$$

$A' \longleftarrow$ translated operator

C-T: $\tilde{A} = T_{\vec{a}} A T_{\vec{a}}^{\dagger} = T_{-\vec{a}}^{\dagger} A T_{-\vec{a}}$
$\langle \psi' | \tilde{A} | \psi' \rangle = \langle \psi | A | \psi \rangle$

Examples:

① $\vec{R}$: $$X'_k = e^{+\frac{i}{\hbar}\vec{a}\cdot\vec{P}} X_k e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}} = X_k + \frac{i}{\hbar}\overbrace{[\vec{a}\cdot\vec{P}, X_k]}^{a_j [P_j, X_k] = -i\hbar a_k} + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 [\vec{a}\cdot\vec{P}, \cancel{[\vec{a}\cdot\vec{P}, X_k]}] + \cdots \to 0$$

$$= X_k + a_k$$

$$\vec{r}\,' = \mathcal{J}_{\vec{a}}\vec{r} = \vec{r} + \vec{a} \iff \vec{R}\,' = T_{\vec{a}}^{\dagger}\vec{R}\,T_{\vec{a}} = \vec{R} + \vec{a}$$

$$\langle\psi|\vec{R}\,'|\psi\rangle = \langle\psi|\vec{R}|\psi\rangle + \vec{a}$$

$$\parallel$$

$$\langle\psi'|\vec{R}|\psi'\rangle$$

$$\boxed{\begin{aligned} &= \int d^3r\, \vec{r}\, |\psi'(\vec{r})|^2 \\ &= \int d^3r\, \vec{r}\, |\psi(\vec{r}-\vec{a})|^2 \\ &= \int d^3r\, (\vec{r}+\vec{a})\, |\psi(\vec{r})|^2 \\ &= \langle\psi|\vec{R}|\psi\rangle + \vec{a} \end{aligned}}$$

(2) $H = \vec{P}^2/2m + V(\vec{R})$

$H' = \vec{P}^2/2m + V(\vec{R}+\vec{a})$

(3) Symmetries and conservation laws

Conservation of momentum

$\iff [\vec{P}, H] = 0$

$\iff H' = H$, i.e., H is invariant under translations

$$\boxed{\begin{aligned} &\text{C-T:} \\ &\langle\psi|\vec{R}|\psi\rangle = \langle\psi'|\vec{R}\,'|\psi'\rangle \\ &\qquad = \langle\psi'|\vec{R}|\psi'\rangle - \vec{a} \\ &\qquad = \langle\psi|\vec{R}|\psi\rangle \end{aligned}}$$

Rotations

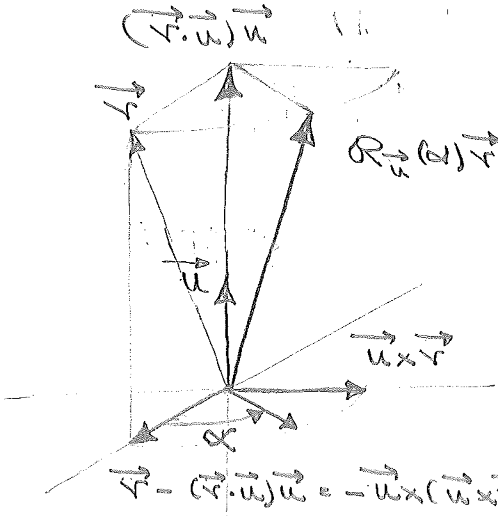
Single particle

Classical rotation (active)

$$R_{\vec u}(\alpha) = \begin{pmatrix}\text{rotation by angle}\\ \alpha \text{ about unit vector } \vec u\end{pmatrix} \longleftarrow$$

3 parameters
(Euler angles)

Noncommutative group



$$R_{\vec u}(\alpha)\vec r = \vec r\,' = (\vec r\cdot\vec u)\vec u - \vec u\times(\vec u\times\vec r)\cos\alpha + \vec u\times\vec r\sin\alpha$$

$$= \vec r\cos\alpha + (\vec r\cdot\vec u)\vec u(1-\cos\alpha) + \vec u\times\vec r\sin\alpha$$

Build up finite from infinitesimal

Infinitesimal rotation: $R_{\vec u}(d\alpha)\vec r = \vec r + d\alpha\,\vec u\times\vec r$

Orthogonal matrices:

$$R_{\vec u}(\alpha)\vec r = R_{\vec u}(\alpha)x_j\vec e_j = x_j \underbrace{R_{\vec u}(\alpha)\vec e_j}_{\vec e_k O_{kj}} = \vec e_k O_{kj} x_j$$

$\uparrow$ orthogonal matrix

$$x'_k = O_{kj}x_j = \vec r\,'\cdot\vec e_k = R\vec r\cdot\vec e_k = \vec r\cdot R^{-1}\vec e_k$$

Representations:

$$R_2R_1\vec r = R_2\left(\vec e_k O^{(1)}_{kj}x_j\right) = \vec e_\ell O^{(2)}_{\ell k}O^{(1)}_{kj}x_j = \vec e_\ell \left(O^{(2)}O^{(1)}\right)_{\ell j}x_j$$

$$R_{\vec u}(d\alpha)\vec e_j = \vec e_j + \vec u\times\vec e_j\,d\alpha$$

$$= \vec e_k(\delta_{kj}+\epsilon_{k\ell j}u_\ell\,d\alpha)$$

$$= \vec e_k\underbrace{(\delta_{kj} - d\alpha\,\epsilon_{kj\ell}u_\ell)}_{O_{kj}}$$

$$O = 1 + d\alpha M$$

$$O_{kj} = \delta_{kj} + d\alpha (M_{\vec{u}})_{kj} \; ; \; (M_{\vec{u}})_{kj} = -\epsilon_{kjl} u_l$$

$$R^{-1}_{\vec{u}_2}(d\alpha_2)\, R^{-1}_{\vec{u}_1}(d\alpha_1)\, R_{\vec{u}_2}(d\alpha_2)\, R_{\vec{u}_1}(d\alpha_1)$$

$$= \vec{e}_k \, O_{kj} \, x_j$$

$$O = e^{-d\alpha_2 M_2} \underbrace{e^{-d\alpha_1 M_1} e^{d\alpha_2 M_2} e^{d\alpha_1 M_1}} = 1 + d\alpha_1 d\alpha_2 [M_{\vec{u}_2}, M_{\vec{u}_1}]$$

$$e^{-d\alpha_1 M_1}\left(1 + d\alpha_2 M_2 + \tfrac{1}{2} d\alpha_2^2 M_2^2\right) e^{d\alpha_1 M_1}$$
$$= 1 + d\alpha_2 \left(M_2 - d\alpha_1 [M_1, M_2]\right) + \tfrac{1}{2} d\alpha_2^2 M_2^2$$
$$= 1 + d\alpha_2 M_2 + d\alpha_1 d\alpha_2 [M_2, M_1] + \tfrac{1}{2} d\alpha_2^2 M_2^2$$
$$= e^{d\alpha_2 M_2}\left(1 + d\alpha_1 d\alpha_2 [M_2, M_1]\right)$$

$$(M_{\vec{e}_l})_{kj} = -\epsilon_{kjl} = -\epsilon_{lkj}$$

$$M_{\vec{e}_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{\vec{e}_y} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$M_{\vec{e}_z} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[M_{\vec{e}_j}, M_{\vec{e}_k}] = M_{\vec{e}_j} M_{\vec{e}_k} - M_{\vec{e}_k} M_{\vec{e}_j}$$

$$([M_{\vec{e}_j}, M_{\vec{e}_k}])_{lm} = (M_{\vec{e}_j})_{ln} (M_{\vec{e}_k})_{nm} - (M_{\vec{e}_k})_{ln} (M_{\vec{e}_j})_{nm}$$

$$= +\epsilon_{jln}\epsilon_{knm} - \epsilon_{kln}\epsilon_{jnm}$$

$$= -\epsilon_{njl}\epsilon_{nkm} + \epsilon_{nkl}\epsilon_{njm}$$

$$= -(\cancel{\delta_{jk}\delta_{lm}} - \delta_{jm}\delta_{lk}) + (\cancel{\delta_{kj}\delta_{lm}} - \delta_{km}\delta_{jl})$$

$$= \delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km}$$

$$= -\epsilon_{jkn}\epsilon_{lmn}$$

$$= \epsilon_{jkn} (M_{\vec{e}_n})_{lm}$$

$$[M_{\vec e_j}, M_{\vec e_k}] = \epsilon_{jkl} M_{\vec e_l}$$

Notice that

$$\boxed{[iM_{\vec e_j}, iM_{\vec e_k}] = i\epsilon_{jkl}(iM_{\vec e_l})}$$

$$R^{-1}_{\vec e_y}(d\alpha_y)\, R^{-1}_{\vec e_x}(d\alpha_x)\, R_{\vec e_y}(d\alpha_y)\, R_{\vec e_x}(d\alpha_x) = R_{\vec e_z}(-d\alpha_x d\alpha_y)$$

$$\mathcal{O} = 1 + d\alpha_x d\alpha_y \underbrace{[M_{\vec e_y}, M_{\vec e_x}]}_{-M_{\vec e_z}} = 1 - d\alpha_x d\alpha_y M_{\vec e_z}$$

Demonstration

① States: $|\psi\rangle \xrightarrow[R_{\vec u}(\alpha)]{} |\psi'\rangle = R_{\vec u}(\alpha)|\psi\rangle$

$$\langle \vec r'|\psi'\rangle = \langle \vec r|\psi\rangle = \langle R^{-1}_{\vec u}(\alpha)\vec r'|\psi\rangle$$

$$\| $$

$$\langle \vec r'|R_{\vec u}(\alpha)|\psi\rangle$$

$$\Longrightarrow \boxed{\langle \vec r|R_{\vec u}(\alpha)|\psi\rangle = \langle R^{-1}_{\vec u}(\alpha)\vec r|\psi\rangle}$$

(i) R linear

matrix elements: $\langle \vec r|R|\vec r'\rangle = \langle R^{-1}\vec r|\vec r'\rangle$

$$= \delta(\vec r' - R^{-1}\vec r)$$

$$= \delta(R\vec r' - \vec r)$$

$$\langle \vec r'|R|\vec r\rangle = \delta(\vec r - R\vec r') = \langle \vec r'|R\,\vec r\rangle$$

$$\boxed{\langle R\,\vec r'|\cdot\rangle}$$

(ii) R unitary

$$\langle \vec r|R^\dagger R|\vec r'\rangle = \int d^3x'' \underbrace{\langle \vec r|R^\dagger|\vec r''\rangle}_{\langle \vec r''|R|\vec r\rangle^* = \delta(\vec r - R^{-1}\vec r'')} \underbrace{\langle \vec r''|R|\vec r'\rangle}_{\delta(\vec r'' - R\vec r')}$$

$$= \delta(\vec r - \vec r')$$

Or

$$\langle \vec r|R^\dagger R|\vec r'\rangle = \langle R\vec r|R\vec r'\rangle$$

$$= \delta(R\vec r - R\vec r')$$

$$= \delta(\vec r - \vec r')$$

$$\Longrightarrow R^\dagger R = 1$$

(iii) Rep of rotation group: $\mathcal{R}_2\mathcal{R}_1 = \mathcal{R}_3 \Rightarrow R_2R_1 = R_3$

$$\begin{aligned}
\langle \vec r | R_2 R_1 | \psi\rangle &= \langle \mathcal{R}_2^{-1}\vec r | R_1 | \psi\rangle \\
&= \langle \mathcal{R}_1^{-1}\mathcal{R}_2^{-1}\vec r | \psi\rangle \\
&= \langle (\mathcal{R}_2\mathcal{R}_1)^{-1}\vec r | \psi\rangle \\
&= \langle \mathcal{R}_3^{-1}\vec r | \psi\rangle \\
&= \langle \vec r | R_3 | \psi\rangle
\end{aligned}$$

What are they?

Let $\vec u = \vec e_z$:

$$\langle \vec r | R_{\vec e_z}(d\alpha) | \psi\rangle = \langle \underbrace{\mathcal{R}^{-1}_{\vec e_z}(d\alpha)\vec r}_{= \vec r - d\alpha\, \vec e_z \times \vec r \,=\, \vec r - d\alpha(x\vec e_y - y\vec e_x)} | \psi\rangle$$

$$= \langle x + d\alpha\, y,\ y - x\, d\alpha,\ z | \psi\rangle$$

$$= \langle x,y,z|\psi\rangle - d\alpha\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\langle x,y,z|\psi\rangle$$

$$= \langle \vec r|\psi\rangle - \frac{i}{\hbar}d\alpha\underbrace{\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\langle \vec r|\psi\rangle}_{=\langle \vec r|(XP_y - YP_x)|\psi\rangle \,=\, \langle \vec r|L_z|\psi\rangle}$$

$$= \langle \vec r|\left(1 - \frac{i}{\hbar}d\alpha\, L_z\right)|\psi\rangle$$

$$\boxed{\Longrightarrow \quad R_{\vec{e}_z}(d\alpha) = 1 - \frac{i}{\hbar}\, d\alpha\, L_z \quad \Longrightarrow \quad R_{\vec{e}_z}(\alpha) = e^{-\frac{i}{\hbar} d\alpha L_z}}$$

General: $\langle \vec{r} |\, R_{\vec{u}}(d\alpha) | \psi \rangle = \langle R^{-1}_{\vec{u}}(d\alpha)\vec{r} | \psi \rangle$

$$= \langle \vec{r} - \underbrace{d\alpha\, \vec{u} \times \vec{r}}_{d\alpha\, \vec{e}_j \epsilon_{jkl} u_k x_l} | \psi \rangle$$

$$= \langle \vec{r} | \psi \rangle - d\alpha\, \epsilon_{jkl} u_k x_l \frac{\partial}{\partial x_j} \langle \vec{r} | \psi \rangle$$

$$= \langle \vec{r} | \psi \rangle - \frac{i}{\hbar} d\alpha\, \underbrace{\epsilon_{jkl} u_k x_l \frac{\hbar}{i} \frac{\partial}{\partial x_j} \langle \vec{r} | \psi \rangle}_{\langle \vec{r} | X_l P_j | \psi \rangle}$$

$$= \langle \vec{r} | \psi \rangle - \frac{i}{\hbar} d\alpha\, u_k \langle \vec{r} | \underbrace{\epsilon_{klj} X_l P_j}_{L_k} | \psi \rangle$$

$$= \langle \vec{r} | \left(1 - \frac{i}{\hbar} d\alpha\, \vec{u} \cdot \vec{L}\right) | \psi \rangle$$

$$\boxed{R_{\vec{u}}(d\alpha) = 1 - \frac{i}{\hbar} d\alpha\, \vec{u} \cdot \vec{L} \quad \Longrightarrow \quad R_{\vec{u}}(\alpha) = e^{-(i/\hbar)\alpha\, \vec{u} \cdot \vec{L}}}$$

Relation to commutators:

$$R^{-1}_{\vec{e}_y}(d\alpha_y)\, R^{-1}_{\vec{e}_x}(d\alpha_x)\, R_{\vec{e}_y}(d\alpha_y)\, R_{\vec{e}_x}(d\alpha_x) = 1 + \left(-\frac{i}{\hbar}\right)^2 d\alpha_x d\alpha_y [L_y, L_x]$$

Rep of classical group $\longrightarrow$ $\|$

$$R_{\vec{e}_z}(-d\alpha_x d\alpha_y) = 1 + \frac{i}{\hbar} d\alpha_x d\alpha_y L_z$$

$$\Longrightarrow \quad L_z = \frac{i}{\hbar}[L_y, L_x] = -\frac{i}{\hbar}[L_x, L_y]$$

i.e., $[L_x, L_y] = i\hbar L_z$

$$e^{\frac{i}{\hbar} d\alpha_x L_x}\, e^{-\frac{i}{\hbar} d\alpha_y L_y}\, e^{-\frac{i}{\hbar} d\alpha_x L_x} = e^{\frac{i}{\hbar} d\alpha_x L_x}\left(1 - \frac{i}{\hbar} d\alpha_y L_y + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 d\alpha_y^2 L_y^2\right) e^{-\frac{i}{\hbar} d\alpha_x L_x}$$

$$= 1 - \frac{i}{\hbar} d\alpha_y \left(L_y + \frac{i}{\hbar} d\alpha_x [L_x, L_y]\right) + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 d\alpha_y^2 L_y^2$$

$$= 1 - \frac{i}{\hbar} d\alpha_y L_y + (\ $$

Generalize: orbital am of many particles
spin
fields

$$\boxed{R_{\vec{u}}(\alpha) = e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}}}$$

Angular momentum is generator of rotations

Spin-$\frac{1}{2}$ (half-angles)

$$R_{\vec{e}_z}(2\pi)|k,j,m\rangle = e^{-\frac{i}{\hbar}2\pi J_z}|k,j,m\rangle$$

$$= \underbrace{e^{-i2\pi m}}_{\begin{cases} +1, & j \text{ integral} \\ -1, & j \text{ half-integral}\end{cases}}|k,j,m\rangle$$

(2) Observables

C-T: $\tilde{A} = RAR^\dagger$ $\qquad \langle\psi'|\tilde{A}|\psi'\rangle = \langle\psi|A|\psi\rangle$

$A' = R^\dagger A R$ $\qquad \langle\psi|A'|\psi\rangle = \langle\psi'|A|\psi'\rangle$

$$\langle\vec{r}|\tilde{A}|\vec{r}'\rangle = \langle\vec{r}|RAR^\dagger|\vec{r}'\rangle = \langle\mathcal{R}^{-1}\vec{r}|A|\mathcal{R}^{-1}\vec{r}'\rangle$$

$$\langle\vec{r}|A'|\vec{r}'\rangle = \langle\vec{r}|R^\dagger A R|\vec{r}'\rangle = \langle\mathcal{R}\vec{r}|A|\mathcal{R}\vec{r}'\rangle$$

Example: $\vec{J} = \vec{L}$

(i) $\langle\vec{r}|\vec{R}'|\vec{r}'\rangle = \langle\mathcal{R}\vec{r}|\vec{R}|\mathcal{R}\vec{r}'\rangle$

$$= \mathcal{R}\vec{r}'\,\delta(\mathcal{R}\vec{r} - \mathcal{R}\vec{r}') \qquad \vec{r}' = x_j'\vec{e}_j$$

$$= \mathcal{R}\vec{r}'\,\delta(\vec{r} - \vec{r}') \qquad \mathcal{R}\vec{r}' = x_j'\mathcal{R}\vec{e}_j$$

$$= \langle\vec{r}|\mathcal{R}\vec{R}|\vec{r}'\rangle \qquad \vec{R} = X_j\vec{e}_j$$

$$\mathcal{R}\vec{R} = X_j\mathcal{R}\vec{e}_j$$

$$\Longrightarrow \vec{R}' = \mathcal{R}\vec{R}$$

(ii) $\langle \vec{r}|\vec{P}'|\vec{r}'\rangle = \langle R\vec{r}|\vec{P}|R\vec{r}'\rangle$

$$= \frac{\hbar}{i}\nabla_{R\vec{r}}\,\delta(R\vec{r}-R\vec{r}')$$

$$= \frac{\hbar}{i}\nabla_{R\vec{r}}\,\delta(\vec{r}-\vec{r}') \;, \qquad \vec{y} = R\vec{r}$$

$$\nabla_{R\vec{r}} = R\nabla_{\vec{r}}$$

$$= R\,\langle \vec{r}|\vec{P}|\vec{r}'\rangle$$

$$= \langle \vec{r}|R\vec{P}|\vec{r}'\rangle$$

$$\vec{P}' = R\vec{P}$$

(iii) $H = \vec{P}^2/2m + V(\vec{R})$

$$H' = \vec{P}'^2/2m + V(R\vec{R})$$

(3) Symmetries and conservation laws

Conservation of $\vec{J} \Leftrightarrow [\vec{J},H]=0 \Rightarrow H' = H$

Vector operators:

$$\vec{V}' = R^\dagger\vec{V}R = R\vec{V} = (\vec{u}\cdot\vec{V})\vec{u} - \vec{u}\times(\vec{u}\times\vec{V})\cos\alpha + \vec{u}\times\vec{V}\sin\alpha$$

$$\langle\psi|\vec{V}'|\psi\rangle = R\,\langle\psi|\vec{V}|\psi\rangle = \langle\psi|R^\dagger\vec{V}R|\psi\rangle = \langle\psi'|\vec{V}|\psi'\rangle$$

or

$$\vec{V} = R\,(R\vec{V}R^\dagger) = R\vec{V}$$

Notice that

$$\vec{V}'\cdot\vec{n} = R^\dagger(\vec{V}\cdot\vec{n})R$$

$$= R\vec{V}\cdot\vec{n}$$

$$= \vec{V}\cdot R^{-1}\vec{n}$$

$$V_j' = R^\dagger\vec{V}\cdot\vec{e}_j R = \vec{V}\cdot R^{-1}\vec{e}_j = \vec{V}\cdot\vec{e}_k g_{jk} = g_{jk}V_k$$

Apply to infinitesimal rotations

$$e^{\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}}\,\vec{V}_j\,e^{-\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}} = R\vec{V} = \vec{V} + d\alpha\,\vec{u}\times\vec{V}$$

$$e^{\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}}\,V_j\,e^{-\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}} = V_j + d\alpha\,\epsilon_{jkl}u_k V_l$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$V_j + \frac{i}{\hbar}d\alpha\,[\vec{u}\cdot\vec{J}, V_j] \qquad\qquad V_j + d\alpha\,u_k\epsilon_{jkl}V_l$$

$$= V_j + \frac{i}{\hbar}d\alpha\,u_k[J_k, V_j]$$

$$= V_j - \frac{i}{\hbar}d\alpha\,u_k[V_j, J_k]$$

$$\Longrightarrow \quad -\frac{i}{\hbar}[V_j, J_k] = \epsilon_{jkl}V_l$$

$$\boxed{[V_j, J_k] = i\hbar\,\epsilon_{jkl}V_l}$$

Direct check of finite rotations

$$R^\dagger V_j R = e^{\frac{i}{\hbar}\alpha\,\vec{u}\cdot\vec{J}}\,V_j\,e^{-\frac{i}{\hbar}\alpha\,\vec{u}\cdot\vec{J}}$$

$$= V_j + \frac{i}{\hbar}\alpha\,[\vec{u}\cdot\vec{J}, V_j] + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\alpha^2\,[\vec{u}\cdot\vec{J}, [\vec{u}\cdot\vec{J}, V_j]]$$

$$\frac{i}{\hbar}[\vec{u}\cdot\vec{J}, V_j] = \frac{i}{\hbar}u_k[J_k, V_j] = -\frac{i}{\hbar}u_k[V_j, J_k] = -\frac{i}{\hbar}u_k\,i\hbar\,\epsilon_{jkl}V_l = (\vec{u}\times\vec{V})_j$$

$$\frac{i}{\hbar}[\vec{u}\cdot\vec{J}, V_j] = (\vec{u}\times\vec{V})_j$$

Check

$$\left(\frac{i}{\hbar}\right)^2[\vec{u}\cdot\vec{J}, [\vec{u}\cdot\vec{J}, V_j]] = \frac{i}{\hbar}[\vec{u}\cdot\vec{J}, (\vec{u}\times\vec{V})_j] \overset{\downarrow}{=} (\vec{u}\times(\vec{u}\times\vec{V}))_j$$

$$\left(\frac{i}{\hbar}\right)^2 [\vec u\cdot\vec J, [\vec u\cdot\vec J, V_j]] = (\vec u\times(\vec u\times\vec V))_j = u_j(\vec u\cdot\vec V) - V_j$$

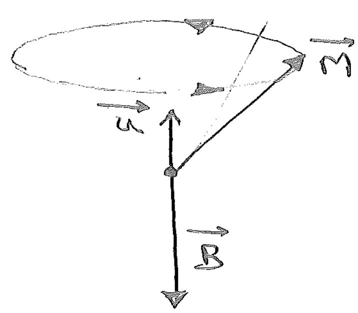
$$\left(\frac{i}{\hbar}\right)^3 [\vec u\cdot\vec J,[\vec u\cdot\vec J,[\vec u\cdot\vec J,V_j]]] = -(\vec u\times\vec V)_j$$

$$R^\dagger V_j R = V_j + \alpha(\vec u\times\vec V)_j + \frac{1}{2}\alpha^2(\vec u\times(\vec u\times\vec V))_j$$
$$- \frac{1}{3!}\alpha^3(\vec u\times\vec V)_j - \frac{1}{4!}\alpha^4(\vec u\times(\vec u\times\vec V))_j$$

$$= V_j + (\vec u\times(\vec u\times\vec V))_j - \cos\alpha\,(\vec u\times(\vec u\times\vec V))_j + \sin\alpha\,(\vec u\times\vec V)_j$$

$$R^\dagger\vec V R = \underbrace{\vec V + \vec u\times(\vec u\times\vec V)}_{\vec u(\vec u\cdot\vec V)} - \vec u\times(\vec u\times\vec V)\cos\alpha + \vec u\times\vec V\sin\alpha$$

Example:



$$H = -\vec M\cdot\vec B = \gamma B\,\vec J\cdot\vec u$$

$$\vec M = \gamma\vec J$$

$$\vec B = -B\vec u$$

$$U(t,0) = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}\overbrace{\gamma B t}^{\alpha=\omega_L t}\,\vec u\cdot\vec J} = R_{\vec u}(\alpha)$$

HP: $\vec V_H(t) = U^\dagger(t,0)\,\vec V_S\,U(t,0) = R^\dagger\vec V_S R$

$$i\hbar\frac{d\vec V_H}{dt} = [\vec V_H, H_H] = \vec e_j\,\gamma B\,u_k\underbrace{[(V_H)_j,(J_H)_k]}_{i\hbar\epsilon_{jkl}(V_H)_l}$$

(26)

$$\frac{d\vec{V}_H}{dt} = \gamma B \underbrace{\vec{e}_j \epsilon_{jkl} u_k (V_H)_l}_{\vec{u} \times \vec{V}_H} = \gamma B \vec{u} \times \vec{V}_H$$

$$\vec{V}_H(t) = R^\dagger_{\vec{u}}(\alpha) \vec{V}_S R_{\vec{u}}(\alpha) = R_{\vec{u}}(\alpha) \vec{V}_S$$

Ⓐ

Two-tensor operator:

$$\overleftrightarrow{\alpha} = \sum_{jk} \alpha_{jk} \vec{e}_j \otimes \vec{e}_k$$

$$R^\dagger \overleftrightarrow{\alpha} R = \sum_{jk} R^\dagger \alpha_{jk} R \vec{e}_j \otimes \vec{e}_k = \sum_{jk} \alpha_{jk} \underbrace{R\vec{e}_j}_{\vec{e}_\ell O_{\ell j}} \otimes \underbrace{R\vec{e}_k}_{\vec{e}_m O_{mk}}$$

$$= \sum_{\substack{\ell m \\ jk}} O_{\ell j} \alpha_{jk} O_{mk} \vec{e}_\ell \otimes \vec{e}_m$$

$$= \sum_{\substack{jk \\ \ell m}} O_{j\ell} \alpha_{\ell m} O_{km} \vec{e}_j \otimes \vec{e}_k$$

$$\Rightarrow R^\dagger \alpha_{jk} R = \sum_{\ell m} O_{j\ell} \alpha_{\ell m} O_{km}$$

Infinitesimal: $R = e^{-\frac{i}{\hbar}\theta \vec{n}\cdot\vec{J}} = 1 - \frac{i}{\hbar} d\theta \vec{n}\cdot\vec{J}$

$$O_{jk} = \delta_{jk} - d\theta\, \epsilon_{jk\ell} n_\ell$$

$$(1 + \frac{i}{\hbar} d\theta \vec{n}\cdot\vec{J})\, \alpha_{jk}\, (1 - \frac{i}{\hbar} d\theta \vec{n}\cdot\vec{J})$$

$$= (\delta_{j\ell} - d\theta\, \epsilon_{j\ell n} n_n)\, \alpha_{\ell m}\, (\delta_{km} - d\theta\, \epsilon_{kmp} n_p)$$

$$= \alpha_{jk} + \frac{i}{\hbar} d\theta [\vec{n}\cdot\vec{J}, \alpha_{jk}]$$

$$= \alpha_{jk} - d\theta\, \epsilon_{j\ell n} n_n \alpha_{\ell m} \delta_{km}$$

$$- d\theta\, \epsilon_{kmp} n_p \alpha_{\ell m} \delta_{j\ell}$$

B

$$= \alpha_{jk} - d\theta\,(\epsilon_{jln} n_n \alpha_{lk} + \epsilon_{kmp} n_p \alpha_{jm})$$

$$= \alpha_{jk} - d\theta\, n_n (\epsilon_{njl} \alpha_{lk} + \epsilon_{nkm} \alpha_{jm})$$

$$= \alpha_{jk} - d\theta\, n_n (\epsilon_{njl} \alpha_{lk} - \epsilon_{nmk} \alpha_{jm})$$

$$\frac{i}{\hbar} n_n [J_n, \alpha_{jk}] = -n_n (\epsilon_{njl} \alpha_{lk} - \epsilon_{nmk} \alpha_{jm})$$

$$\frac{i}{\hbar} [J_n, \alpha_{jk}] = -(\epsilon_{njl} \alpha_{lk} - \epsilon_{nmk} \alpha_{jm})$$

$$[\alpha_{jk}, J_n] = -i\hbar (\epsilon_{njl} \alpha_{lk} - \epsilon_{nmk} \alpha_{jm})$$

$$[\alpha_{jk}, J_l] = i\hbar (\epsilon_{lmk} \alpha_{jm} - \epsilon_{ljm} \alpha_{mk})$$

$$\boxed{[\alpha_{jk}, J_l] = i\hbar (\epsilon_{klm} \alpha_{jm} + \epsilon_{jlm} \alpha_{mk})}$$