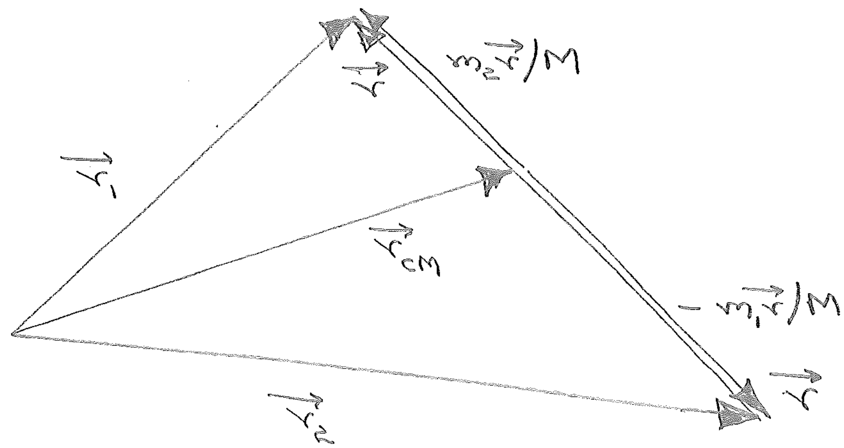
Phys 521

Central Forces

Lectures 26-28

Two interacting particles

Classical: $\mathcal{H} = \vec{p}_1^{\,2}/2m_1 + \vec{p}_2^{\,2}/2m_2 + V(\vec{r}_1 - \vec{r}_2)$



$\vec{r}_{cm} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$

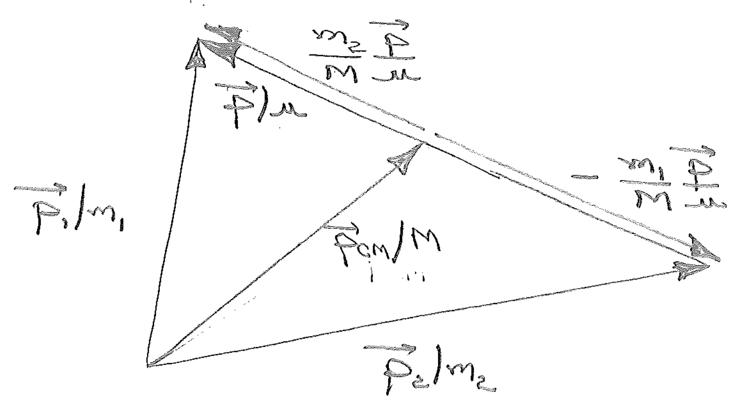
$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{r}_1 = \vec{r}_{cm} + m_2\vec{r}/M$

$\vec{r}_2 = \vec{r}_{cm} - m_1\vec{r}/M$

$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$\mu \leq m_1, m_2$



$\vec{p}_{cm} = \vec{p}_1 + \vec{p}_2$

$\vec{p}/\mu = \vec{p}_1/m_1 - \vec{p}_2/m_2$

$\vec{p}_1 = \vec{p} + m_1\vec{p}_{cm}/M$

$\vec{p}_2 = -\vec{p} + m_2\vec{p}_{cm}/M$

Canonical transformation: show w/ x-component

$$\mathcal{H} = \frac{\vec{P}_{CM}^{\,2}}{2M} + \frac{\vec{P}^{\,2}}{2\mu} + V(\vec{r})$$

Eqns. of motion $\longleftarrow$ CM and relative motion decouple

CM frame

$\vec{p}/\mu = \vec{p}_1/m_1 - \vec{p}_2/m_2 = \dot{\vec{r}}_1 - \dot{\vec{r}}_2 =$ (relative velocity)

QM: Same transformation $\begin{matrix} \vec{R}_1, \vec{R}_2 \longleftrightarrow \vec{R}_{CM}, \vec{R} \\ \vec{P}_1, \vec{P}_2 \longleftrightarrow \vec{P}_{CM}, \vec{P} \end{matrix}$

Commutators: Use x-components as example

$[X_{CM}, X] = 0$ (all X's commute)

$[X_{CM}, P_{CM}] = \left[\frac{m_1}{M}X_1 + \frac{m_2}{M}X_2, P_1 + P_2\right] = i\hbar$

$[X_{CM}, P] = \left[\frac{m_1}{M}X_1 + \frac{m_2}{M}X_2, \frac{\mu}{m_1}P_1 - \frac{\mu}{m_2}P_2\right] = 0$

$[X, P_{CM}] = [X_1 - X_2, P_1 + P_2] = 0$

$[X, P] = \left[X_1 - X_2, \frac{\mu}{m_1}P_1 - \frac{\mu}{m_2}P_2\right] = i\hbar$

$[P_{CM}, P] = 0$ (all P's commute)

$$\boxed{\begin{aligned} &[(X_{cm})_j, (X_{cm})_k] = [(X_{cm})_j, X_k] = [(X_{cm})_j, P_k] = [X_j, X_k] = \\ &[X_j, (P_{cm})_k] = [(P_{cm})_j, (P_{cm})_k] = [(P_{cm})_j, P_k] = 0 \\ &[(X_{cm})_j, (P_{cm})_k] = [X_j, P_k] = i\hbar\,\delta_{jk} \end{aligned}}$$

$$H = \vec{P}_{cm}^2/2M + \vec{P}^2/2\mu + V(\vec{R})$$

Consequences:

(1) Indep of Hamiltonian: $\mathcal{E}_{\vec{r}_1\vec{r}_2} = \mathcal{E}_{\vec{r}_{cm}} \otimes \mathcal{E}_{\vec{r}}$

direct product

Basis $|\vec{R}_{cm}\rangle \otimes |\vec{R}\rangle = |\vec{R}_{cm}, \vec{R}\rangle$

$$|\psi\rangle = \int d^3x_{cm}\, d^3x \underbrace{\left(|\vec{R}_{cm}\rangle\langle\vec{R}_{cm}| \otimes |\vec{R}\rangle\langle\vec{R}|\right)|\psi\rangle}_{|\vec{R}_{cm},\vec{R}\rangle\langle\vec{R}_{cm},\vec{R}|\psi\rangle}$$

$$= \int d^3x_{cm}\, d^3x\, |\vec{R}_{cm},\vec{R}\rangle \cdot \psi(\vec{R}_{cm},\vec{R})$$

Doesn't mean wave function factors or that state vector factors as $|\psi\rangle = |\psi_{cm}\rangle \otimes |\psi_{rel}\rangle$

↓

Consequences for CM frame

Entanglement if

$$|\psi\rangle = |\psi_{cm}\rangle \otimes \sum_m |k,l,m\rangle \longrightarrow \sum_m |\psi_{cm,k}\rangle \otimes |k,l,m\rangle$$

(2) Hamiltonian as a sum:

Gravity: $-(m_1\vec{r}_1 + m_2\vec{r}_2)\cdot\vec{g} = -M\vec{r}_{cm}\cdot\vec{g}$

$$H = H_{cm} + H_{rel}$$

$$\left.\begin{array}{l} H_{cm}|\chi_{cm}\rangle = E_{cm}|\chi_{cm}\rangle \\ H_{rel}|\chi_{rel}\rangle = E_{rel}|\chi_{rel}\rangle \end{array}\right] \longrightarrow \text{position rep?}$$

$$H\,|\chi_{cm}\rangle \otimes |\chi_{rel}\rangle = (E_{cm} + E_{rel})|\chi_{cm}\rangle \otimes |\chi_{rel}\rangle$$

$$\vec{L} = \vec{L}_1 + \vec{L}_2$$
$$= \vec{R}_1 \times \vec{P}_1 + \vec{R}_2 \times \vec{P}_2$$
$$= \left(\vec{R}_{cm} + \frac{m_2}{M}\vec{R}\right) \times \left(\frac{m_1}{M}\vec{P}_{cm} + \vec{P}\right) + \left(\vec{R}_{cm} - \frac{m_1}{M}\vec{R}\right) \times \left(\frac{m_2}{M}\vec{P}_{cm} - \vec{P}\right)$$
$$= \vec{R}_{cm} \times \vec{P}_{cm} + \vec{R} \times \vec{P}$$

Central force – relative motion

$$H_{rel} = \vec{P}^2/2\mu + V(|\vec{R}|)$$

Important cases
$1/r$, const, r^2

Classical: $E = \mathcal{H}$ and $\vec{L}$ conserved

Orbit confined to a plane (z component of $\vec{L}$)

$$\mathcal{H} = \frac{p_r^2}{2\mu} + \frac{p_\varphi^2}{2\mu r^2} + V(r)$$

$$\Longrightarrow \quad p_\varphi = \text{constant} = \mu r^2 \dot{\varphi} = \mathcal{L}$$

$$\mathcal{H} = \frac{p_r^2}{2\mu} + \underbrace{\frac{\mathcal{L}^2}{2\mu r^2} + V(r)}_{V_{eff}(r)}$$

$$\dot{r} = + \frac{\partial \mathcal{H}}{\partial p_r} = p_r/\mu$$
$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\frac{\partial V_{eff}}{\partial r} = -\frac{\partial V}{\partial r} + \frac{\mathcal{L}^2}{\mu r^3}$$

$\mu r \dot{\varphi}^2 \leftarrow$ centrifugal force

Quantum: Time-indep SE in $|\vec{R}\rangle$-rep

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right)\varphi(\vec{r}) = E\,\varphi(\vec{r})$$

relative energy

$$\nabla^2 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)}_{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r} - \frac{\vec{L}^2}{\hbar^2 r^2}$$

$$H = -\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2}r + \frac{\vec{L}^2}{2\mu r^2} + V(r)$$

Since $\vec{L}$ commutes with H, can find simultaneous eigenstates of H, $\vec{L}^2$, and L_z.

$$H\varphi(\vec{r}) = E_{k\ell}\varphi(\vec{r})$$
$$\vec{L}^2\varphi(\vec{r}) = \ell(\ell+1)\hbar^2\varphi(\vec{r}) \quad \Longrightarrow \quad \varphi_{k\ell m}(\vec{r}) = R_{k\ell}(r)\,Y_\ell^m(\theta,\phi)$$
$$L_z\varphi(\vec{r}) = m\hbar\,\varphi(\vec{r}) \qquad E = E_{k\ell}$$

Radial equation

$$\left[-\frac{\hbar^2}{2\mu r}\frac{d^2}{dr^2}r + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r)\right]R_{k\ell}(r) = E_{k\ell}R_{k\ell}(r)$$

$$R_{k\ell}(r) = \frac{u_{k\ell}(r)}{r}$$

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \underbrace{\frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r)}_{V_{eff}(r)}\right]u_{k\ell}(r) = E_{k\ell}u_{k\ell}(r)}$$

$$\boxed{\frac{d^2u_{k\ell}}{dr^2} = -\frac{2\mu}{\hbar^2}\left(E_{k\ell} - V_{eff}(r)\right)u_{k\ell}(r)}$$

Boundary condition at $r=0$:

(1) Facile approach: $V(r)=\infty,\ r<0 \implies u_{n\ell}(0)=0$

(2) Assume centrifugal potential dominates near $r=0$; this restricts us to $\ell \geq 1$.

near $r=0$: $$\frac{d^2 u_{n\ell}}{dr^2} + \frac{\ell(\ell+1)}{r^2} u_{n\ell} = 0$$

$u_{n\ell} = Cr^s$: $s(s-1) + \ell(\ell+1) = 0$

$\implies s = \ell+1$ or $s = -\ell$ two l.i. solutions

not a solution to $H\psi = E\psi$ at $r=0$
not normalizable

$u_{n\ell} = Cr^{\ell+1}$

(3) Hermiticity of H (completeness of solutions)

$$\langle \varphi_{n'\ell'm'} | H | \varphi_{n\ell m} \rangle = \int r^2 dr\, d\Omega\, \varphi^*_{n'\ell'm'}(\vec{r}) \underbrace{\left(-\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2} r + \frac{\vec{L}^2}{2\mu r^2} + V(r) \right) \varphi_{n\ell m}(\vec{r})}_{\left(-\frac{\hbar^2}{2\mu r}\frac{d^2}{dr^2} r + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r) \right) R_{n\ell}(r) Y_\ell^m(\Omega)}$$

$R^*_{n'\ell'}(r) Y_{\ell'}^{m'*}(\Omega)$

$$= \delta_{\ell\ell'}\delta_{mm'} \int_0^\infty dr\, u^*_{n'\ell}(r) \left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_{eff}(r) \right) u_{n\ell}(r)$$

$$\int_0^\infty dr\, u^*_{n'\ell}(r) \frac{d^2 u_{n\ell}}{dr^2} = u^*_{n'\ell}(r) \frac{du_{n\ell}}{dr}\Big|_0^\infty - \int_0^\infty dr\, \frac{du^*_{n'\ell}}{dr}\frac{du_{n\ell}}{dr}$$

$$= u^*_{n'\ell}(r) \frac{du_{n\ell}}{dr}\Big|_0^\infty - \frac{du^*_{n'\ell}}{dr} u_{n\ell}(r)\Big|_0^\infty + \int_0^\infty dr\, \frac{d^2 u^*_{n'\ell}}{dr^2} u_{n\ell}(r)$$

$$\langle \varphi_{n'l'm'} | H | \varphi_{nlm} \rangle = \langle \varphi_{nlm} | H | \varphi_{n'l'm'} \rangle^*$$
$$+ \underbrace{\left(u^*_{n'l}(r) \frac{du_{nl}}{dr} - \frac{du^*_{n'l}}{dr} u_{nl}(r) \right) \Big|_0^\infty}_{=0}$$

BC at ∞: automatic for bound states

B.C. at $r=0$:

$$\lim_{r\to 0} \left(u^*_{n'l}(r) \frac{du_{nl}}{dr} - \frac{du^*_{n'l}}{dr} u_{nl}(r) \right) = 0$$

Back to (12): certainly true for $l \geq 1$.

BC at $r=\infty$: Assume $V \longrightarrow 0$ at $r=\infty$.

$$\frac{d^2 u_{nl}}{dr^2} + \frac{2\mu E_{nl}}{\hbar^2} u_{nl} = 0 , \qquad E_{nl} < 0 \ \left(\begin{matrix}\text{bound}\\ \text{states}\end{matrix}\right)$$

$$u_{nl}(r) \sim \exp\left(- \sqrt{-\frac{2\mu E_{nl}}{\hbar^2}}\, r \right)$$

Define $\rho = \kappa r = \sqrt{-\frac{2\mu E_{nl}}{\hbar^2}}\, r$

$$u(r) = \rho^{l+1} e^{-\rho} w(\rho)$$

$$\Longrightarrow \frac{d^2 w}{d\rho^2} + 2\left(\frac{l+1}{\rho} - 1 \right) \frac{dw}{d\rho} + \left[\frac{V}{E} - \frac{2(l+1)}{\rho} \right] w = 0$$

Coulomb potential: $V(r) = -\frac{e^2}{r}$

$l=0$: $u = A(r - \frac{k}{2}r^2 + \dots) + B(1 - kr\log r + \dots)$

↑ throw away

Introduce $\rho_0 = \frac{e^2 K}{|E|} = \sqrt{\frac{2\mu}{|E|}}\frac{e^2}{\hbar}$

$$\frac{E}{V} = -\frac{Er}{e^2} = \frac{|E|}{e^2 K}\rho = \frac{\rho}{\rho_0}$$

$$\rho\frac{d^2w}{d\rho^2} + 2(l+1-\rho)\frac{dw}{d\rho} + [\rho_0 - 2(l+1)]w = 0$$

Assume $w(\rho) = a_0 + a_1\rho + a_2\rho^2 + \dots$

$$0 = (k+1)k\,a_{k+1} + 2(l+1)(k+1)\,a_{k+1} - 2k\,a_k + [\rho_0 - 2(l+1)]a_k$$

$$\Longrightarrow \frac{a_{k+1}}{a_k} = \frac{2(k+l+1) - \rho_0}{(k+1)(k+2l+2)}$$

If not terminate, $a_{k+1}/a_k \longrightarrow 2/k \Longrightarrow w \sim e^{2\rho}$

$\therefore\ \rho_0 = 2\underbrace{(N+l+1)}_{n}$, $\quad l = 0,1,2,\dots$ $\quad N = 0,1,2,\dots$

$$E = -\frac{2\mu e^4}{\hbar^2\rho_0^2} = -\frac{\mu e^4}{2\hbar^2 n^2} = -\frac{e^2}{2a_0 n^2}$$

$a_0 \equiv \frac{\hbar^2}{\mu e^2} = 0.529\times10^{-10}\,m$

Wave functions:

$$W(\rho) \propto L_N^{2\ell+1}(\rho) = L_{n-\ell-1}^{2\ell+1}(\rho)$$

$\uparrow$

Associated Laguerre polynomial

$$\frac{\exp\left(-\frac{sz}{1-s}\right)}{(1-s)^{p+1}} = \sum_{n=0}^{\infty} \frac{L_n^p(z)}{(n+p)!} s^n$$

$\uparrow$

generating fcn.

Normalization: $\int_0^\infty dx\, e^{-x} x^{2(\ell+1)} \left[L_{n-\ell-1}^{2\ell+1}(x)\right]^2 = \frac{2n[(n+\ell)!]^3}{(n-\ell-1)!}$

$$\psi_{n\ell m}(r,\theta,\varphi) = \left\{(2K)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2}$$

$$\times\ e^{-Kr}(2Kr)^\ell L_{n-\ell-1}^{2\ell+1}(2Kr)\, Y_\ell^m(\theta,\varphi)$$