Lectures 3-5

Phys 521

1d square potentials

1-d, time-indep potential $V(x)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Stationary states: $\psi(x,t) = \varphi(x)\, e^{-\frac{i}{\hbar}Et}$

Bound, discrete states
Continuum states
Wave packets

$$-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi$$

① If $\varphi(x)$ is a stationary state, so is $\varphi^*(x)$.

Time-reversal invariance: $\varphi^*(x)\, e^{-\frac{i}{\hbar}Et} = \psi^*(x,-t)$

Two possibilities:

① φ and φ^* are the same solution

$\varphi^* = e^{i\alpha}\varphi \Rightarrow (e^{i\alpha/2}\varphi)^* = e^{i\alpha/2}\varphi$

Choose φ to be real

② φ and φ^* are l.i. solutions

$\varphi+\varphi^*$ and $-i(\varphi-\varphi^*)$ are a pair of degenerate real solutions

Look at free particle: real solns. are standing waves

② $V(x) = V(-x)$: If $\varphi(x)$ is a stationary state, so is $\varphi(-x)$

Two possibilities:

① $\varphi(x)$ and $\varphi(-x)$ are the same solution

$\varphi(-x) = e^{i\alpha}\varphi(x)$ $\qquad$ $\varphi(0) = e^{i\alpha}\varphi(0)$

$\Rightarrow -\varphi'(-x) = e^{i\alpha}\varphi'(x)$ $\quad \Longrightarrow \quad$ $-\varphi'(0) = e^{i\alpha}\varphi'(0)$

$e^{i\alpha} = 1$, $\varphi'(0) = 0$, and $\varphi(x) = \varphi(-x)$ (even)

or $e^{i\alpha} = -1$, $\varphi(0) = 0$, and $\varphi(-x) = -\varphi(x)$ (odd)

② $\varphi(x)$ and $\varphi(-x)$ are l.i. solutions

$\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are a pair of degenerate even and odd solutions

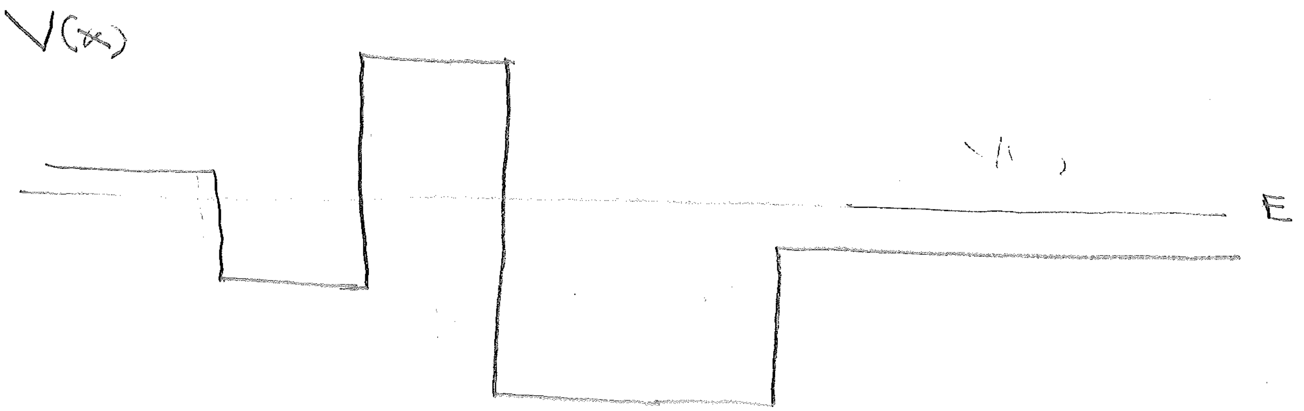
Look at free particle

$$0 = \varphi'' + \underbrace{\frac{2m}{\hbar^2}(E - V(x))}_{k^2(x) = \frac{p^2(x)}{\hbar^2}}\varphi = \varphi'' + k^2(x)\varphi$$

$$k(x) = \frac{\sqrt{2m}}{\hbar}(E - V(x))^{1/2}$$

Approaches: (1) WKB

(2) Square potentials (piecewise-constant)
(approximation to sudden changes in V)



$$\varphi'' + k^2\varphi = 0 \Rightarrow \varphi(x) = Ae^{ikx} + A'e^{-ikx}$$

or

$$\varphi(x) = A\cos kx + A'\sin kx$$

$E > V$, k real

$$p^2 = -k^2 = \frac{2m}{\hbar^2}(V - E)$$

$$p = ik = \frac{\sqrt{2m}}{\hbar}(V - E)^{1/2}$$

$$\varphi(x) = Be^{px} + B'e^{-px}$$

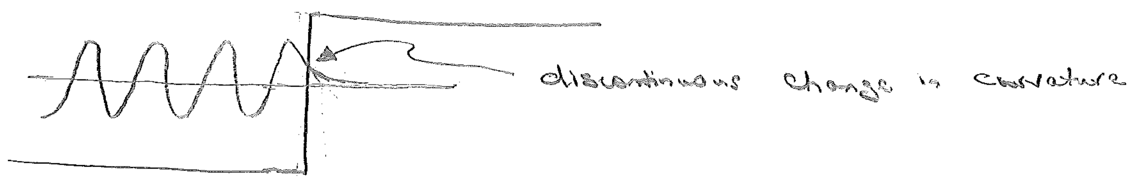
or

$$\varphi(x) = B\cosh px + B'\sinh px$$

$E < V$, p real

Boundary conditions at a finite step

1. Replace step by continuous, differentiable sudden change
2. Derive BC's from Schrödinger equation

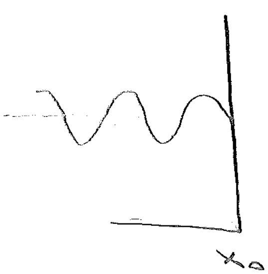


$$\varphi'' + k^2(x)\,\varphi = 0$$

1. φ is continuous at x_0 (discontinuity in φ gives a $\delta'(x-a)$ potential)
2. φ' is continuous at x_0 (discontinuity in φ' gives a $\delta(x-a)$ potential)

$$\varphi'(x_0+\epsilon) - \varphi'(x_0-\epsilon) = -\int_{x_0-\epsilon}^{x_0+\epsilon} dx\, k^2(x)\,\varphi(x) \longrightarrow 0$$

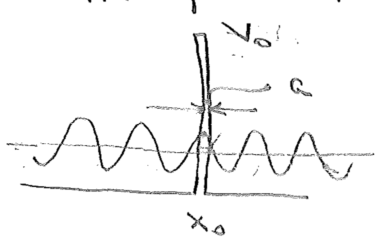
Infinite step



1. $\varphi(x_0) = 0$

φ' discontinuous at x_0

Delta-potential:



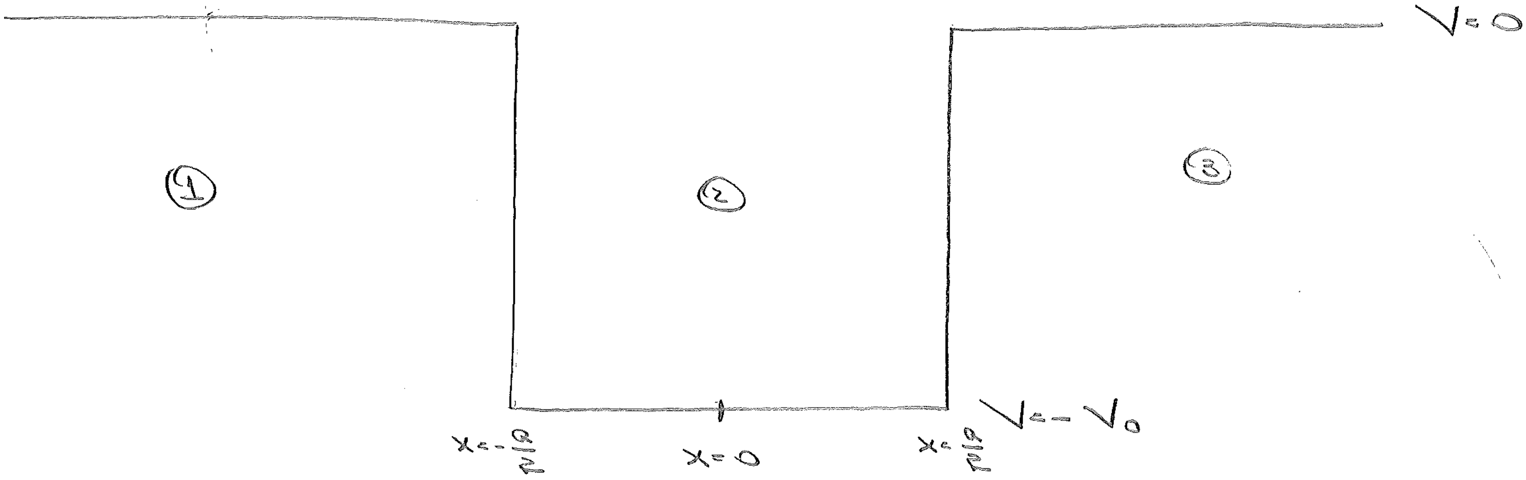
$V(x) = V_0\, a\, \delta(x-x_0)$

1. φ is continuous at x_0
2. $\varphi'(x_0+\epsilon) - \varphi'(x_0-\epsilon) = -\int_{x_0-\epsilon}^{x_0+\epsilon} dx\, k^2(x)\,\varphi(x)$

$$+\frac{2m}{\hbar^2}\int_{x_0-\epsilon}^{x_0+\epsilon} dx\, V(x)\,\varphi(x)$$

$$= +\frac{2m}{\hbar^2} V_0\, a\, \varphi(x_0)$$

Example: Square well



Bound states: $E < 0$

$$\varphi(x) = B_1 e^{px} \qquad \varphi = A_2 e^{ikx} + A_2' e^{-ikx} \qquad \varphi = B_3' e^{-px}$$

$$p = \sqrt{\frac{-2mE}{\hbar^2}} \qquad k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \qquad p = \sqrt{\frac{-2mE}{\hbar^2}}$$

3 constants and 4 BC's

$\Longrightarrow$ only certain values of E allowed

$$\varphi = \pm B e^{+px} \qquad \varphi = A\begin{cases}\cos kx \\ \sin kx\end{cases} \qquad \varphi = B e^{-px}$$

Even: $A\cos(ka/2) = Be^{-pa/2}$

$Ak\sin(ka/2) = +Bpe^{-pa/2}$

Odd: $A\sin(ka/2) = Be^{-pa/2}$

$Ak\cos(ka/2) = -Bpe^{-pa/2}$

Even: $Ak\cos(ka/2) = Bk e^{-pa/2}$

$Ak\sin(ka/2) = +Bp e^{-pa/2}$

$$\Rightarrow \quad A^2k^2 = B^2\underbrace{(k^2+p^2)}_{\frac{2mV_0}{\hbar^2} = k_0^2} e^{-pa} \quad \Rightarrow \left(\frac{B}{A}\right)^2 = \left(\frac{k}{k_0}\right)^2 e^{pa}$$

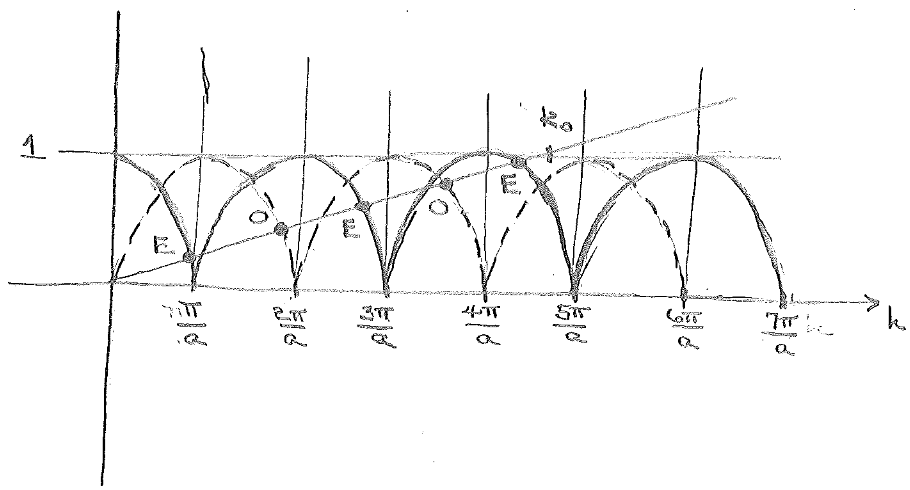
$k_0^2 = k^2 + p^2 = \frac{2mV_0}{\hbar^2}$

$E = \frac{\hbar^2}{2m}(k^2 - k_0^2)$

$$\longrightarrow \tan(ka/2) = +\frac{p}{k} > 0$$

$$\frac{1}{\cos^2(ka/2)} = 1 + \tan^2(ka/2) = \frac{k_0^2}{k^2}$$

$$\longrightarrow |\cos(ka/2)| = \frac{k}{k_0}$$



Odd: $Ak\sin(ka/2) = Bk e^{-pa/2}$

$Ak\cos(ka/2) = -Bp e^{-pa/2}$

$\left(\frac{B}{A}\right)^2 = \left(\frac{k}{k_0}\right)^2 e^{pa}$

$$\longrightarrow \tan(ka/2) = -\frac{k}{p} < 0$$

$$\frac{1}{\sin^2(ka/2)} = 1 + \frac{1}{\tan^2(ka/2)} = \frac{k_0^2}{k^2}$$

$p_{max} = \sqrt{2mV_0}$

$\longrightarrow \quad |\sin(ka/2)| = \dfrac{k}{k_0}$

Comments: ① $(\#\text{ of bound states}) = 1 + \mathrm{Int}\left(\dfrac{k_0 a}{\pi}\right) = 1 + \mathrm{Int}\left(\underbrace{\dfrac{\sqrt{2mV_0}\,a}{\pi\hbar}}_{\frac{(2p_{max})a}{h}}\right)$

1 bound state if $\dfrac{k_0 a}{\pi} = \dfrac{\sqrt{2mV_0}\,a}{\pi\hbar} \le 1$

δ-potential: $\begin{array}{l} a \to 0 \\ V_0 \to \infty \\ aV_0 = \text{constant} \end{array}$ $\quad \dfrac{k_0 a}{\pi} = \dfrac{\sqrt{2mV_0}\,a}{\pi\hbar} \longrightarrow 0 \Rightarrow$ 1 bound state

$\sin^2(ka/2) = 1 - (k/k_0)^2$

$(k/k_0)^2 = 1 - \sin^2(k_0 a/2) = 1 - (k_0 a/2)^2 = 1 - (k_0 a/2)^2$

$E = \dfrac{\hbar^2}{2m}k_0^2\left(\dfrac{k^2}{k_0^2} - 1\right) = -\dfrac{\hbar^2}{2m}\dfrac{k_0^4 a^2}{4} = -\dfrac{m(V_0 a)^2}{2\hbar^2}$

② $\dfrac{k_0 a}{\pi} = \dfrac{\sqrt{2mV_0}\,a}{\pi\hbar} \gg 1$

lowest levels: $k_n = \dfrac{n\pi}{a} \Rightarrow E_n = \dfrac{\hbar^2 k_n^2}{2m} - V_0$

$= \dfrac{n^2\pi^2\hbar^2}{2ma^2} - V_0$

Do the accounting of constants vs. BC's for 3 situations that can arise, using potential on ⑥ as an example.

Continuum states: $E > 0$

$\varphi(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \quad \varphi(x) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x} \quad \varphi(x) = A_3 e^{ik_1 x} + A_3' e^{-ik_1 x}$

$k_1 = \sqrt{\dfrac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\dfrac{2m(E+V_0)}{\hbar^2}}$

5 constants and 4 BC's

$\Longrightarrow$ there are 2 linearly independent solutions for each E

$A_3' = 0$

wave from left

Consequences of parity and time-reversal

Easier to put boundaries at $x=0$ and $x=a$ and to work with $A_3' \neq 0$

BC's at $x=a$

$$A_2 e^{ik_2 a} + A_2' e^{-ik_2 a} = A_3 e^{ik_3 a} + A_3' e^{-ik_3 a}$$

$$k_2 A_2 e^{ik_2 a} - k_2 A_2' e^{-ik_2 a} = k_3 A_3 e^{ik_3 a} - k_3 A_3' e^{-ik_3 a}$$

$$\begin{pmatrix} e^{ik_2 a} & e^{-ik_2 a} \\ k_2 e^{ik_2 a} & -k_2 e^{-ik_2 a} \end{pmatrix} \begin{pmatrix} A_2 \\ A_2' \end{pmatrix} = \begin{pmatrix} e^{ik_3 a} & e^{-ik_3 a} \\ k_3 e^{ik_3 a} & -k_3 e^{-ik_3 a} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$$\begin{pmatrix} A_3 \\ A_3' \end{pmatrix} = \frac{1}{2k_3} \begin{pmatrix} k_3 e^{-ik_3 a} & e^{-ik_3 a} \\ k_3 e^{ik_3 a} & -e^{ik_3 a} \end{pmatrix} \begin{pmatrix} e^{ik_2 a} & e^{-ik_2 a} \\ k_2 e^{ik_2 a} & -k_2 e^{-ik_2 a} \end{pmatrix} \begin{pmatrix} A_2 \\ A_2' \end{pmatrix}$$

$$= M^{32} \begin{pmatrix} A_2 \\ A_2' \end{pmatrix}$$

$$M^{32} = \frac{1}{2k_3} \begin{pmatrix} (k_3 + k_2) e^{-i(k_3 - k_2)a} & (k_3 - k_2) e^{-i(k_3 + k_2) a} \\ (k_3 + k_2) e^{i(k_3 + k_2) a} & (k_3 + k_2) e^{i(k_3 - k_2) a} \end{pmatrix}$$

Contrast S-matrix which relates inputs $\begin{pmatrix} A_2 \\ A_3' \end{pmatrix}$ to outputs $\begin{pmatrix} A_3' \\ A_2' \end{pmatrix}$:

$$\begin{pmatrix} \sqrt{k_3} A_3 \\ \sqrt{k_2} A_2' \end{pmatrix} = S^{32} \begin{pmatrix} \sqrt{k_2} A_2 \\ \sqrt{k_3} A_3' \end{pmatrix}.$$

↑ why?

S-matrix is what we want, but M-matrices can be concatenated (why can't S's).

In our problem $k_3 = k_1$ and

$$M^{21} = \frac{1}{2k_2} \begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix}.$$

$$\begin{pmatrix} A_3 \\ A_3' \end{pmatrix} = \underbrace{M^{32} M^{21}}_{M} \begin{pmatrix} A_1 \\ A_1' \end{pmatrix}$$

$$M = \frac{1}{4k_1k_2} \begin{pmatrix} (k_1+k_2)\, e^{-i(k_1-k_2)a} & (k_1-k_2)\, e^{-i(k_1+k_2)a} \\ (k_1-k_2)\, e^{i(k_1+k_2)a} & (k_1+k_2)\, e^{i(k_1-k_2)a} \end{pmatrix}$$

$$\times \begin{pmatrix} k_2+k_1 & k_2-k_1 \\ k_2-k_1 & k_2+k_1 \end{pmatrix}$$

Meaning of $2k_2a$ phase differences

$$= \frac{1}{4k_1k_2} \begin{pmatrix} e^{-ik_1a}\left[(k_1+k_2)^2 e^{ik_2a} - (k_1-k_2)^2 e^{-ik_2a}\right] & -2ie^{-ik_1a}(k_1^2-k_2^2)\sin k_2a \\ 2ie^{ik_1a}(k_1^2-k_2^2)\sin k_2a & e^{ik_1a}\left[(k_1+k_2)^2 e^{-ik_2a} - (k_1-k_2)^2 e^{ik_2a}\right] \end{pmatrix}$$

General theory: $M = \frac{1}{\sqrt{T}} \begin{pmatrix} e^{i\alpha} & \sqrt{R}\, e^{i\mu} \\ \sqrt{R}\, e^{-i\mu} & e^{-i\alpha} \end{pmatrix} = \tilde{M}$ (in this case)

$$\frac{1}{\sqrt{T}} e^{i\alpha} = M_{11} = e^{-ik_1a} \frac{(k_1+k_2)^2 e^{ik_2a} - (k_1-k_2)^2 e^{-ik_2a}}{4k_1k_2}$$

$$= e^{-ik_1a} \frac{2k_1k_2\cos k_2a + i(k_1^2+k_2^2)\sin k_2a}{2k_1k_2}$$

$$\frac{\sqrt{R}}{\sqrt{T}} e^{i\mu} = M_{12} = \frac{-ie^{-ik_1a}(k_1^2-k_2^2)\sin k_2a}{2k_1k_2}$$

Without general theory: $A_3' = 0$

Use $M_{22} = M_{11}^*$, $M_{21} = M_{12}^*$, $\det M = |M_{11}|^2 - |M_{12}|^2 = 1$

$$A_3 = M_{11}A_1 + M_{12}A_1'$$

$$0 = M_{12}^* A_1 + M_{11}^* A_1'$$

$$A_1' = -\frac{M_{12}^*}{M_{11}^*} A_1$$

$$A_3 = \left(M_{11} + \frac{|M_{12}|^2}{M_{11}^*} \right) A_1 = \frac{1}{M_{11}^*} A_1$$

$$T = \left| \frac{A_3}{A_1} \right|^2 = \frac{1}{|M_{11}|^2} = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 \cos^2 k_2 a + (k_1^2 + k_2^2)^2 \sin^2 k_2 a}$$

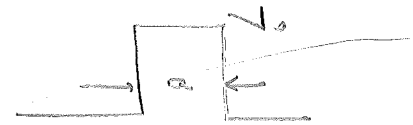
$$= \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2 k_2 a}$$

$$R = \left| \frac{A_1'}{A_1} \right|^2 = T\,|M_{12}|^2 = \frac{(k_1^2 - k_2^2)^2 \sin^2 k_2 a}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2 k_2 a}$$

Comments:

① $1 = R + T$

② Potential barrier: $V_0 \longrightarrow -V_0$



③ Resonances: $\sin^2 k_2 a = 0 \implies R = 0,\ T = 1$

$$k_2 a = n\pi \iff n\lambda_2 = 2a$$

④ Tunneling: $V_0 \mapsto -V_0$, $k_2 = -i p_2$, $k_2^2 = -p_2^2 = \frac{2m(E - V_0)}{\hbar^2}$

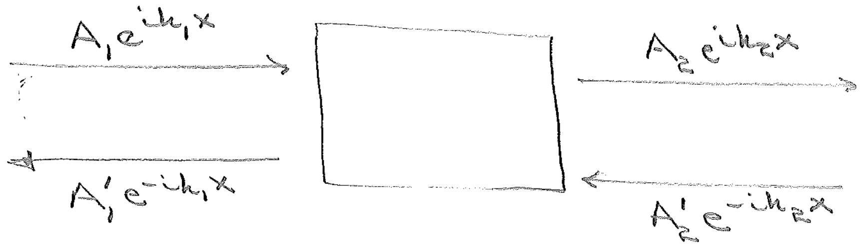
$$\sin k_2 a = -i \sinh p_2 a$$

$$T = \frac{1}{1 + \frac{(k_1^2 - k_2^2)^2}{4k_1^2 k_2^2} \sin^2 k_2 a}$$

$$= \frac{1}{1 + \underbrace{\frac{(k_1^2 + p_2^2)^2}{4k_1^2 p_2^2}}_{\frac{V_0^2}{4E(V_0 - E)}} \sinh^2 p_2 a} \longrightarrow \frac{16E(V_0 - E)}{V_0^2} \exp\left(-\underbrace{\frac{\sqrt{8m(V_0 - E)}\, a}{\hbar}}_{\sim \frac{\text{action}}{\hbar}} \right)$$

A

Theory of M and S



$$\begin{pmatrix} A_2 \\ A_2' \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_1' \end{pmatrix}$$

M's can be multiplied to get overall M.

It is easier to work with $a_1 = \sqrt{k_1} A_1$, $a_1' = \sqrt{k_1} A_1'$, $a_2 = \sqrt{k_2} A_2$, and $a_2' = \sqrt{k_2} A_2'$.

$$\begin{pmatrix} a_2 \\ a_2' \end{pmatrix} = \tilde{M} \begin{pmatrix} a_1 \\ a_1' \end{pmatrix} \iff \begin{pmatrix} a_2 \\ a_1' \end{pmatrix} = S \begin{pmatrix} a_1 \\ a_2' \end{pmatrix}$$

$$\tilde{M} = \sqrt{\frac{k_2}{k_1}}\, M$$

S-matrix

① Conservation of ~~energy~~: probability (or particle number)

$$k_1 |A_1|^2 + k_2 |A_2'|^2 = k_2 |A_2|^2 + k_1 |A_1'|^2$$

$$|a_1|^2 - |a_1'|^2 = |a_2|^2 - |a_2'|^2$$

$$\iff \boxed{\tilde{M}^\dagger \sigma_3 \tilde{M} = \sigma_3}$$

$$|a_1|^2 + |a_2'|^2 = |a_2|^2 + |a_1'|^2$$

$$\iff \boxed{S^\dagger S = I}$$

(2) Time-reversal invariance:

$$\begin{pmatrix} a_2^* \\ a_2'^* \end{pmatrix} = \tilde{M}^* \begin{pmatrix} a_1^* \\ a_1'^* \end{pmatrix}$$

$$\sigma_1 \begin{pmatrix} a_2'^* \\ a_2^* \end{pmatrix} = \tilde{M}^* \sigma_1 \begin{pmatrix} a_1'^* \\ a_1^* \end{pmatrix}$$

$$= \quad \text{t-r invariance}$$

$$\sigma_1 \tilde{M} \begin{pmatrix} a_1'^* \\ a_1^* \end{pmatrix}$$

$$\Longleftrightarrow \boxed{\sigma_1 \tilde{M} = \tilde{M}^* \sigma_1}$$

$$\begin{pmatrix} a_2^* \\ a_1'^* \end{pmatrix} = S^* \begin{pmatrix} a_1^* \\ a_2'^* \end{pmatrix}$$

$$\sigma_1 \begin{pmatrix} a_1'^* \\ a_2^* \end{pmatrix} = S^* \sigma_1 \begin{pmatrix} a_2'^* \\ a_1^* \end{pmatrix} \quad \text{t-r invariance}$$

$$= S^* \sigma_1 S \begin{pmatrix} a_1'^* \\ a_2^* \end{pmatrix}$$

$$\Longleftrightarrow \boxed{S^* \sigma_1 S = \sigma_1}$$

$$\boxed{\sigma_1 S \sigma_1 = S^T}$$

Write $$S = \begin{pmatrix} \sqrt{T} e^{i\alpha} & \sqrt{R'} e^{i\gamma} \\ \sqrt{R} e^{i\beta} & \sqrt{T'} e^{i\delta} \end{pmatrix}$$

$$\begin{aligned} a_2 &= \sqrt{T} e^{i\alpha} a_1 + \sqrt{R'} e^{i\gamma} a_2' \\ a_1' &= \sqrt{R} e^{i\beta} a_1 + \sqrt{T'} e^{i\delta} a_2' \end{aligned} \Longleftrightarrow \begin{aligned} A_2 &= \sqrt{\tfrac{k_1}{k_2}} \sqrt{T} e^{i\alpha} A_1 + \sqrt{R'} e^{i\gamma} A_2' \\ A_1' &= \sqrt{R} e^{i\beta} A_1 + \sqrt{\tfrac{k_2}{k_1}} \sqrt{T'} e^{i\delta} A_2' \end{aligned}$$

(1) $I = S^\dagger S \Longrightarrow$

$$1 = T + R = T' + R'$$

$$0 = \sqrt{RT}\, e^{i(\alpha-\beta)} + \sqrt{R'T'}\, e^{i(\gamma-\delta)}$$

$$\Leftrightarrow e^{i(\alpha-\beta)} = e^{i(\gamma-\delta)}$$

$$+ \; RT = R'T'$$

(C)

(2) $S\sigma_1 S^* = \sigma_1$

$$S\sigma_1 = \begin{pmatrix}\sqrt{T}e^{i\alpha} & \sqrt{R'}e^{i\gamma}\\ \sqrt{R}e^{i\beta} & \sqrt{T'}e^{i\delta}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}\sqrt{R'}e^{i\gamma} & \sqrt{T}e^{i\alpha}\\ \sqrt{T'}e^{i\delta} & \sqrt{R}e^{i\beta}\end{pmatrix}$$

$$I = S\sigma_1(S\sigma_1)^* = \begin{pmatrix}\sqrt{R'}e^{i\gamma} & \sqrt{T}e^{i\alpha}\\ \sqrt{T'}e^{i\delta} & \sqrt{R}e^{i\beta}\end{pmatrix}\begin{pmatrix}\sqrt{R'}e^{-i\gamma} & \sqrt{T}e^{-i\alpha}\\ \sqrt{T'}e^{-i\delta} & \sqrt{R}e^{-i\beta}\end{pmatrix}$$

$$= \begin{pmatrix}R' + \sqrt{TT'}e^{i(\alpha-\delta)} & \sqrt{R'T}e^{-i(\alpha-\gamma)} + \sqrt{RT}e^{i(\alpha-\beta)}\\ \sqrt{R'T'}e^{-i(\gamma-\delta)} + \sqrt{RT'}e^{i(\beta-\delta)} & R + \sqrt{TT'}e^{-i(\alpha-\delta)}\end{pmatrix}$$

$$1 = R' + \sqrt{TT'}e^{i(\alpha-\delta)} = R + \sqrt{TT'}e^{-i(\alpha-\delta)}$$

$$\Longrightarrow R = R',\ T = T',\ \alpha = \delta$$

$$0 = \sqrt{R'T}e^{-i(\alpha-\gamma)} + \sqrt{RT}e^{i(\alpha-\beta)}$$

$$\Longrightarrow e^{i(\alpha-\beta)} = -e^{-i(\alpha-\gamma)}$$

$$0 = \sqrt{R'T'}e^{-i(\gamma-\delta)} + \sqrt{RT'}e^{i(\beta-\delta)}$$

$$\Longrightarrow e^{i(\beta-\delta)} = -e^{-i(\gamma-\delta)}$$

(1) and (2): $R = R',\ T = T',\ R + T = 1$

$\alpha = \delta$

$$e^{i(\beta-\alpha)} = -e^{-i\underbrace{(\gamma-\alpha)}_{\mu}} \Longrightarrow \begin{matrix} e^{i(\gamma-\alpha)} = e^{i\mu} \\ e^{i(\beta-\alpha)} = -e^{-i\mu}\end{matrix}$$

$$S = e^{i\alpha}\begin{pmatrix}\sqrt{T} & \sqrt{R}e^{i\mu}\\ -\sqrt{R}e^{-i\mu} & \sqrt{T}\end{pmatrix} \Longrightarrow \det S = e^{2i\alpha}$$

D

Relation between $\tilde{M}$ and S:

$$a_2 = S_{11} a_1 + S_{12} a_2'$$

$$a_1' = S_{21} a_1 + S_{22} a_2'$$

$$a_1 = 0: \quad \begin{aligned} a_2 &= S_{12} a_2' \\ a_1' &= S_{22} a_2' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_2 &= \frac{S_{12}}{S_{22}} a_1' \\ a_2' &= \frac{1}{S_{22}} a_1' \end{aligned}$$

$$a_1' = 0: \quad \begin{aligned} 0 &= S_{21} a_1 + S_{22} a_2' \\ a_2 &= S_{11} a_1 + S_{12} a_2' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_2' &= -\frac{S_{21}}{S_{22}} a_1 \\ a_2 &= \left(S_{11} - \frac{S_{12} S_{21}}{S_{22}} \right) a_1 \\ &= \frac{\det S}{S_{22}} a_1 \end{aligned}$$

$$\begin{pmatrix} a_2 \\ a_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \det S / S_{22} & S_{12}/S_{22} \\ -S_{21}/S_{22} & 1/S_{22} \end{pmatrix}}_{\tilde{M}} \begin{pmatrix} a_1 \\ a_1' \end{pmatrix}$$

$$\tilde{M} = \frac{1}{S_{22}} \begin{pmatrix} \det S & S_{12} \\ -S_{21} & 1 \end{pmatrix} = \frac{1}{\sqrt{T}} \begin{pmatrix} e^{i\alpha} & +\sqrt{R} e^{+i\mu} \\ \sqrt{R} e^{-i\mu} & e^{-i\alpha} \end{pmatrix}$$

$$\det \tilde{M} = \frac{1}{S_{22}^2} \left(\det S + S_{12} S_{21} \right) = \frac{S_{11}}{S_{22}} = 1$$

$$\tilde{M}_{11} = \tilde{M}_{22}^*, \quad \tilde{M}_{12} = \tilde{M}_{21}^*$$

$\textcircled{E}$

Check of $\tilde{M}$:

① $$\tilde{M}^{\dagger}\sigma_3\tilde{M} = \frac{1}{T}\begin{pmatrix} e^{-i\alpha} & \sqrt{R}e^{+i\mu} \\ \sqrt{R}e^{-i\mu} & e^{i\alpha} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{i\alpha} & \sqrt{R}e^{i\mu} \\ \sqrt{R}e^{-i\mu} & e^{-i\alpha} \end{pmatrix}}_{\begin{pmatrix} e^{i\alpha} & \sqrt{R}e^{i\mu} \\ -\sqrt{R}e^{-i\mu} & -e^{-i\alpha} \end{pmatrix}}$$

$$= \frac{1}{T}\begin{pmatrix} 1-R & \sqrt{R}e^{i(\mu-\alpha)} - \sqrt{R}e^{i(\mu-\alpha)} \\ \sqrt{R}e^{i(\alpha-\mu)} - \sqrt{R}e^{i(\alpha-\mu)} & R-1 \end{pmatrix}$$

$$= \sigma_3$$

② $$\sigma_1\tilde{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{T}}\begin{pmatrix} e^{i\alpha} & \sqrt{R}e^{i\mu} \\ \sqrt{R}e^{-i\mu} & e^{-i\alpha} \end{pmatrix} = \frac{1}{\sqrt{T}}\begin{pmatrix} \sqrt{R}e^{-i\mu} & e^{-i\alpha} \\ e^{i\alpha} & \sqrt{R}e^{i\mu} \end{pmatrix}$$

$$\parallel$$

$$\tilde{M}^{*}\sigma_1 = \frac{1}{\sqrt{T}}\begin{pmatrix} e^{-i\alpha} & \sqrt{R}e^{-i\mu} \\ \sqrt{R}e^{i\mu} & e^{i\alpha} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{T}}\begin{pmatrix} \sqrt{R}e^{-i\mu} & e^{-i\alpha} \\ e^{i\alpha} & \sqrt{R}e^{i\mu} \end{pmatrix}$$

F

Parity:

$$\begin{pmatrix} a_1' \\ a_2 \end{pmatrix} = \sigma_1 \begin{pmatrix} a_2 \\ a_1' \end{pmatrix} = \sigma_1 S \begin{pmatrix} a_1 \\ a_2' \end{pmatrix} = \sigma_1 S \sigma_1 \begin{pmatrix} a_2' \\ a_1 \end{pmatrix}$$

parity $\longrightarrow$ $\parallel$

$$S \begin{pmatrix} a_2' \\ a_1 \end{pmatrix} \Longrightarrow \boxed{S = \sigma_1 S \sigma_1} \Longleftrightarrow \boxed{S = S^T}$$

S is symmetric

$e^{i\mu} = -e^{-i\mu} \Longrightarrow e^{2i\mu} = -1$

$\mu = \pm \pi/2$

$$S = e^{i\alpha} \begin{pmatrix} \sqrt{T} & \pm i\sqrt{R} \\ \pm i\sqrt{R} & \sqrt{T} \end{pmatrix}$$

$$\tilde{M} = \frac{1}{\sqrt{T}} \begin{pmatrix} e^{i\alpha} & \pm i\sqrt{R} \\ \mp i\sqrt{R} & e^{-i\alpha} \end{pmatrix}$$