Phys 521

Lectures 7-11

Mathematical structure of quantum mechanics

Mathematical Structure of Quantum Mechanics

Particle of mass m in a potential $V(x)$

Position x

Momentum $p = \frac{\hbar}{i}\frac{d}{dx}$

Hamiltonian $H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

Stationary states $\left(\begin{matrix}\text{energy eigenfunctions} \\ \text{and eigenvalues}\end{matrix}\right)$

$$H\varphi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2\varphi_n}{dx^2} + V(x)\varphi_n = E_n\varphi_n$$

time-independent Schrödinger eqn

$E_n = \left(\begin{matrix}\text{energy} \\ \text{eigenvalues}\end{matrix}\right)$

$\varphi_n(x) = \left(\begin{matrix}\text{energy} \\ \text{eigenfunctions}\end{matrix}\right)$

Introduce examples

The stationary states are a complete, orthonormal set:

Orthonormality: $\int dx\, \varphi_n^*(x)\varphi_m(x) = \delta_{nm}$

↑ limits

Completeness: $\sum_n \varphi_n(x)\varphi_n^*(x') = \delta(x-x')$ Closure relation

↑ Dirac δ-function

Example: Particle in 1-dim. box:

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{otherwise} \end{cases}$$

$0 < x < a$: $\quad -\dfrac{\hbar^2}{2m}\dfrac{d^2\varphi}{dx^2} = E\varphi \iff \dfrac{d^2\varphi}{dx^2} + k^2\varphi = 0$

$$k^2 = \frac{2mE}{\hbar}$$

B.C.: $\varphi(x=0) = 0 = \varphi(x=a)$

Solution: $\quad \varphi_n(x) = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right) \longleftarrow$ Normalized Fourier functions on the interval $0 < x < a$

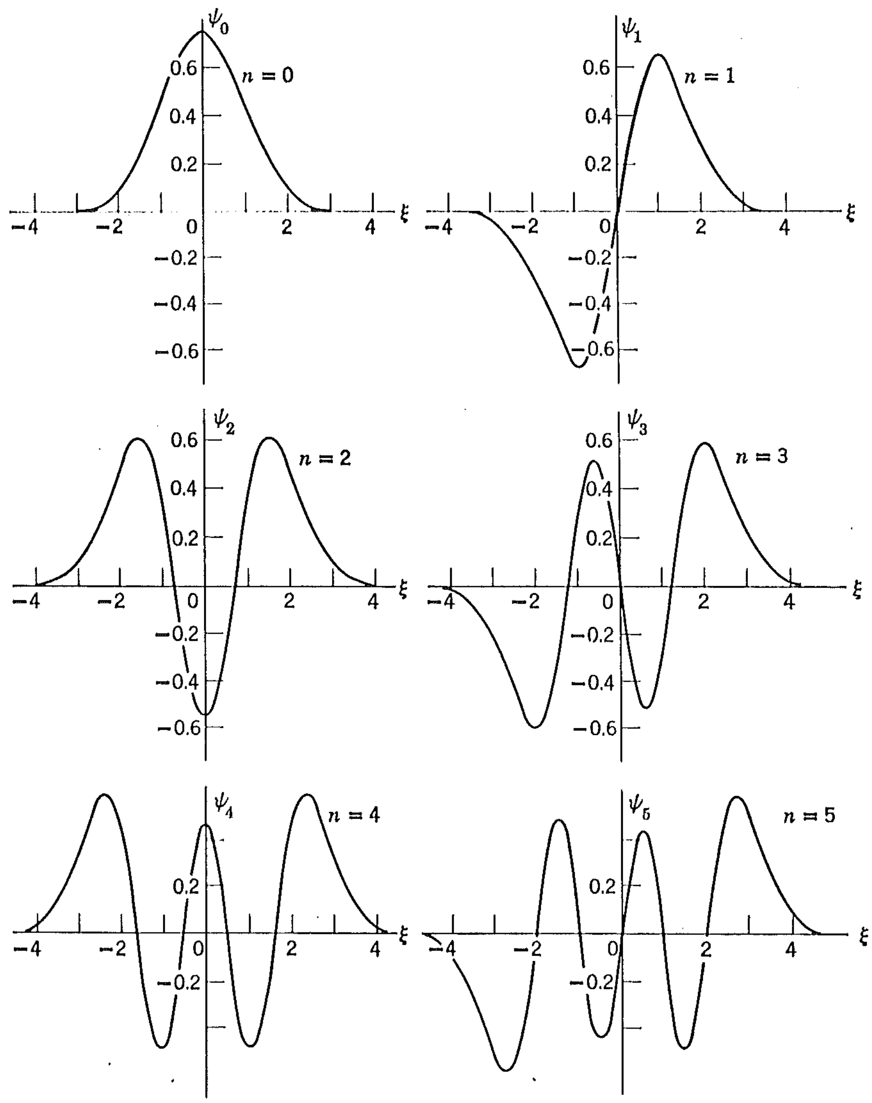
Orthonormal and complete

$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$ (1b)

Energy eigenfunctions for: SHO:

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)$$

nth Hermite polynomial



From *Quantum Mechanics*, by E. Merzbacher

Figure 5.1. The eigenfunctions of the linear harmonic oscillator for the quantum numbers $n = 0$ to 5. ψ_n is plotted versus $\xi = \sqrt{m\omega/\hbar}\,x$, and all eigenfunctions are normalized as $\int_{-\infty}^{+\infty} |\psi_n(\xi)|^2\, d\xi = 1$.

Dirac δ-function: $\delta(x)$

Not really a function. To do things rigorously requires the theory of generalized functions. For us, however, we shall think of $\delta(x)$ as a "function" [so] sharply peaked at $x=0$ that it has a unit integral. We do not make mistakes as long as we always (in the end) put the δ-function under an integral.

$$f(x) = \int_{-\infty}^{\infty} dx' \, \delta(x-x') f(x') \qquad \text{for any function } f(x')$$

$$\Rightarrow \quad \int_{x_1}^{x_2} \delta(x)\,dx = \begin{cases} 1, & \text{if } x_1 < 0 < x_2 \\ 0, & \text{otherwise} \end{cases}$$

Representations:

① $$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \longleftarrow \text{inverting a Fourier transform}$$

② $$\delta(x) = \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon}$$

③ $$\delta(x) = \lim_{\epsilon\to 0^+} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$$

④ $$\delta(x) = \lim_{N\to\infty} \frac{\sin Nx}{\pi x}$$

⑤ $$\delta(x) = \frac{1}{2} \frac{d^2}{dx^2} |x|$$

$\delta^2(x)$ is nonsense.

Properties:

① $\delta(ax) = \frac{1}{|a|}\delta(x)$

② $f(x')\,\delta(x-x') = f(x)\,\delta(x-x')$

③ $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i)\,, \quad g(x_i)=0$

Expand $g(x)$ about its zeroes.

Arbitrary wave function: $\psi(x) = \sum_n c_n \varphi_n(x)$

Orthonormality $\Rightarrow$ $c_n = \int dx\, \varphi_n^*(x)\, \psi(x)$ Derive

$\uparrow$ probability amplitude

When does completeness come in? Ability to expand <u>any</u> function.

① $\psi(x) = \sum_n c_n \varphi_n(x) = \int dx' \underbrace{\left(\sum_n \varphi_n^*(x') \varphi_n(x) \right)}_{\delta(x-x')} \psi(x')$

Use this backwards to derive the expansion

② $\delta(x-x') = \sum_n c_n \varphi_n(x)$, $c_n = \int dx\, \varphi_n^*(x)\, \delta(x-x') = \varphi_n^*(x')$

$= \sum_n \varphi_n^*(x') \varphi_n(x)$

$|\psi(x)|^2 dx = \begin{pmatrix} \text{probability that particle} \\ \text{lies between } x \text{ and } x+dx \end{pmatrix}$ $\psi(x)$ is a probability amplitude

Normalization: $1 = \int dx\, |\psi(x)|^2$

$|c_n|^2 = \begin{pmatrix} \text{probability that} \\ \text{energy is } E_n \end{pmatrix}$ $\Updownarrow$ show in both directions c_n is a probability amplitude

Normalization: $1 = \sum_n |c_n|^2$

Inner (scalar) product of two wave functions:

$$\psi(x) = \sum_n c_n \varphi_n(x), \qquad \phi(x) = \sum_n d_n \varphi_n(x)$$

$$\begin{pmatrix}\text{inner}\\ \text{product}\end{pmatrix} = \int dx\, \phi^*(x)\psi(x) = \sum_n d_n^* c_n = (d_1^*\ d_2^* \ldots)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Abstract to a complex vector space

Why? Notational ease
Calculational convenience
Better understanding

Dirac bra-ket formulation:

Complex vector space

Vector (ket) $|\psi\rangle$

Vector space rules

Examples:

① $V(x)$, $-\infty \le x < \infty$ (SHO)

$|\psi\rangle \leftrightarrow \psi(x)$ $\quad L^2$

← no x-dependence

rules

$|\varphi_n\rangle$ are basis vectors

② Particle in a box

$|\psi\rangle \leftrightarrow \psi(x)$ $\quad L^2$ on $[0, a]$

rules

$|\varphi_n\rangle$ are a basis

③ Spin-$\frac{1}{2}$

$|\uparrow\rangle$, $|\downarrow\rangle$, $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

Inner product: $(|\phi\rangle, |\psi\rangle) =$ (complex number)

① Complex bilinear

② Complex symmetric: $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$

(3) $(|\psi\rangle, |\psi\rangle) \geq 0$, $(|\psi\rangle, |\psi\rangle) = 0 \Rightarrow |\psi\rangle = 0$.

State vectors (quantum states) are normalized vectors:

$(|\psi\rangle, |\psi\rangle) = 1$.

Dual vectors (bras): $\langle\psi|$

Linear functions on vectors: $\langle\psi|\phi\rangle = \begin{pmatrix}\text{complex}\\ \text{number}\end{pmatrix}$

Make up a vector space

The inner product induces a map from kets to bras:

$$|\psi\rangle \longleftrightarrow \langle\psi| \quad : \quad \langle\psi|\phi\rangle \equiv (|\psi\rangle, |\phi\rangle)$$

(1), (2), + (3) above

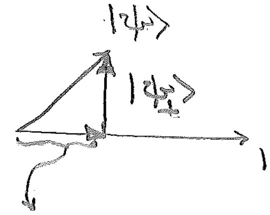
This map is antilinear

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \longleftrightarrow \langle\psi| = c_1^*\langle\psi_1| + c_2^*\langle\psi_2|$$

Generally drop (,) notation for inner product, but caution!

This map is 1-1 for finite-dimensional vector spaces.

Schwarz inequality: $|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle\langle\psi|\psi\rangle$



$$\frac{|\phi\rangle\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$$

$$|\psi_\perp\rangle = |\psi\rangle - \frac{|\phi\rangle\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$$

$$\langle\psi_\perp| = \langle\psi| - \frac{\langle\psi|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}$$

$$\langle\psi_\perp|\psi_\perp\rangle = \langle\psi|\psi\rangle - \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0$$

↑ Pythagorus

Representation of a vector:

Complete, orthonormal basis $|\varphi_n\rangle$

Example: energy eigenstates

Orthonormality: $\langle\varphi_n|\varphi_m\rangle = \delta_{nm}$

Completeness: $\mathbb{1} = \sum_n |\varphi_n\rangle\langle\varphi_n| = \hat{1}$

Expansion of arbitrary vector:

$$|\psi\rangle = \sum_n c_n|\varphi_n\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle$$

Meaning of completeness

$$c_n = \langle\varphi_n|\psi\rangle = \begin{pmatrix}\text{components in} \\ |\varphi_n\rangle\text{-representation}\end{pmatrix}$$

$$\langle\psi| = \sum_n \langle\varphi_n|c_n^* = \sum_n \langle\psi|\varphi_n\rangle\langle\varphi_n|$$

Inner product & normalization

Linear operators: $\hat{O} \leftarrow$ hat distinguishes operators

C-T uses caps for operators

$\hat{O}|\psi\rangle$ is a ket

$$\hat{O}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1(\hat{O}|\psi_1\rangle) + c_2(\hat{O}|\psi_2\rangle)$$

Unit operator: $\hat{1}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$

Outer product: $|\phi\rangle\langle\psi|$ is an operator

Projection operator:

1-d: $\hat{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|$

Subspace S, spanned by $|\varphi_n\rangle$: $\hat{P}_S = \sum_n |\varphi_n\rangle\langle\varphi_n|$

$\hat{P}_S^2 = \hat{P}_S$

Unit operator $\hat{1} = \sum_n |\varphi_n\rangle\langle\varphi_n|$

Matrix representation

$$\hat{O}|\psi\rangle = \sum_n |\varphi_n\rangle\underbrace{\langle\varphi_n|\hat{O}|\psi\rangle}_{d_n} = \sum_{nm} |\varphi_n\rangle\underbrace{\langle\varphi_n|\hat{O}|\varphi_m\rangle}_{O_{nm}}\underbrace{\langle\varphi_m|\psi\rangle}_{c_m}$$

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix} = O \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

matrix elements of $\hat{O}$ in $|\varphi_n\rangle$-rep

$$\hat{O} = \hat{1}\hat{O}\hat{1} = \sum_{n,m} |\varphi_n\rangle\underbrace{\langle\varphi_n|\hat{O}|\varphi_m\rangle}_{O_{nm}}\langle\varphi_m|$$

Operator products: $(\hat{N}\hat{O})|\psi\rangle = \hat{N}(\hat{O}|\psi\rangle)$ matrix rep?

Acting to left: $(\langle\phi|\hat{O})|\psi\rangle \equiv \langle\phi|(\hat{O}|\psi\rangle) = \langle\phi|\hat{O}|\psi\rangle$

Adjoint operator (Hermitian conjugate):

$$\hat{O}|\psi\rangle = |\psi'\rangle \longleftrightarrow \langle\psi'| = \langle\psi|\hat{O}^\dagger$$

$$\langle\phi|\hat{O}|\psi\rangle = \langle\phi|\psi'\rangle = \langle\psi'|\phi\rangle^* = \langle\psi|\hat{O}^\dagger|\phi\rangle^*$$

$$\langle\phi|\hat{O}|\psi\rangle = (|\phi\rangle, \hat{O}|\psi\rangle)$$

$$\langle\psi|\hat{O}^\dagger|\phi\rangle^* = (|\psi\rangle, \hat{O}^\dagger|\phi\rangle)^* = (\hat{O}^\dagger|\phi\rangle, |\psi\rangle)$$

$$(|\phi\rangle, \hat{O}|\psi\rangle) = (\hat{O}^\dagger|\phi\rangle, |\psi\rangle)$$

No way to write this in Dirac notation except as $\langle\phi|\hat{O}|\psi\rangle$

The map $\hat{O}^\dagger$ is antilinear

$$(\alpha\hat{O} + \beta\hat{N})|\psi\rangle \longleftrightarrow \langle\psi|(\alpha\hat{O} + \beta\hat{N})^\dagger$$

$$(\alpha\hat{O} + \beta\hat{N})|\psi\rangle = \alpha(\hat{O}|\psi\rangle) + \beta(\hat{N}|\psi\rangle) \longleftrightarrow \alpha^*\langle\psi|\hat{O}^\dagger + \beta^*\langle\psi|\hat{N}^\dagger = \langle\psi|(\alpha^*\hat{O}^\dagger + \beta^*\hat{N}^\dagger)$$

$$\Rightarrow (\alpha\hat{O} + \beta\hat{N})^\dagger = \alpha^*\hat{O}^\dagger + \beta^*\hat{N}^\dagger$$

$$|\psi\rangle\langle\phi|\chi\rangle \longleftrightarrow \langle\chi|\phi\rangle\langle\psi| \Rightarrow \left(|\psi\rangle\langle\phi|\right)^\dagger = |\phi\rangle\langle\psi|$$

Matrix representation

$$(\hat{O}^\dagger)_{nm} = \langle\varphi_n|\hat{O}^\dagger|\varphi_m\rangle = \langle\varphi_m|\hat{O}|\varphi_n\rangle^* = O^*_{mn}$$

or

$$\hat{O}^\dagger = \left(\sum_{nm} O_{nm}|\varphi_n\rangle\langle\varphi_m|\right)^\dagger = \sum_{nm} O^*_{nm}|\varphi_m\rangle\langle\varphi_n| = \sum_{nm} \underbrace{O^*_{mn}}_{(\hat{O}^\dagger)_{nm}}|\varphi_n\rangle\langle\varphi_m|$$

Properties ① $(\hat{O}^\dagger)^\dagger = \hat{O}$, ② $(\hat{N}\hat{O})^\dagger = \hat{O}^\dagger\hat{N}^\dagger$

Hermitian (self-adjoint) operators (observables)

Hermitian operator: $\hat{A} = \hat{A}^\dagger \Leftrightarrow A_{nm} = A^*_{mn}$

Eigenvalue problem: $\hat{A}|\psi_k\rangle = \lambda_k|\psi_k\rangle$

$$\langle\psi_\ell|\hat{A}|\psi_k\rangle = \langle\psi_k|\hat{A}^\dagger|\psi_\ell\rangle^* = \langle\psi_k|\hat{A}|\psi_\ell\rangle^*$$

$$\langle\psi_\ell|\hat{A}|\psi_k\rangle = A_k\langle\psi_\ell|\psi_k\rangle \qquad \langle\psi_k|\hat{A}|\psi_\ell\rangle^* = A^*_\ell\langle\psi_\ell|\psi_k\rangle$$

$$\Rightarrow (A_k - A^*_\ell)\langle\psi_\ell|\psi_k\rangle = 0$$

$\Longrightarrow$ ① $A_k = A^*_k$ (eigenvalues are real)

② If $A_k \neq A_\ell$, then $\langle\psi_\ell|\psi_k\rangle = 0$ (degeneracy vs. nondegeneracy)

How to determine the eigenvalues:

characteristic equation

$$(\hat{A}-\lambda_n\hat{1})|\psi_n\rangle = 0 \Rightarrow \det(\hat{A}-\lambda_n\hat{1}) = 0$$

What does this mean?
Rep in a particular orthonormal basis.
Basis-dependence

D dimensions $\Rightarrow$ D roots, including multiple roots
Dealing with degeneracies

$\boxed{\text{Eigenvectors of a Hermitian operator } \hat{A} \text{ form a complete orthonormal basis, the A-representation.}}$

A-representation: $\hat{A}|\psi_n\rangle = \lambda_n|\psi_n\rangle$

Orthonormality: $\langle\psi_n|\psi_l\rangle = \delta_{nl}$

Completeness: $\hat{1} = \sum_n |\psi_n\rangle\langle\psi_n|$

Spectral decomposition: $\hat{A} = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n| = \sum_\lambda \lambda\hat{P}_\lambda$

Describe situation for nonHermitian operators. For QM Hermitian is enough.

Unitary operators: Operators that preserve inner products

$$\langle\phi|\psi\rangle = \langle\phi|\hat{U}^\dagger\hat{U}|\psi\rangle \quad \text{for all } |\phi\rangle, |\psi\rangle$$

$$\Longleftrightarrow \hat{U}^\dagger\hat{U} = \hat{1} = \hat{U}\hat{U}^\dagger$$

Eigenvalues of $\hat{U}$ are phases

$\hat{U} = e^{i\hat{A}}$ for some Hermitian $\hat{A}$

Normal operators: diagonalized in a complete orthonormal basis $\Leftrightarrow [\hat{A}, \hat{A}^\dagger] = 0$

Polar decomposition

Change of representation:

$$\hat{1} = \sum_n |\varphi_n\rangle\langle\varphi_n| = \sum_m |\chi_m\rangle\langle\chi_m|$$

Kets: $|\psi\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle = \sum_m |\chi_m\rangle\langle\chi_m|\psi\rangle$

$$\langle\chi_m|\psi\rangle = \sum_n \underbrace{\langle\chi_m|\varphi_n\rangle}_{u_{mn}}\langle\varphi_n|\psi\rangle$$

Matrix form

u unitary matrix

Bras:

Operators:

Commutators and simultaneous eigenstates:

Commutator: $[\hat{A}, \hat{B}] = 0$

<u>Theorem:</u> $\hat{A}$ and $\hat{B}$ normal. $[\hat{A}, \hat{B}] = 0 \Leftrightarrow \hat{A}$ and $\hat{B}$ have a complete, orthonormal set of simultaneous eigenstates.

<u>Proof:</u>

① Suppose $\hat{A}$ and $\hat{B}$ have simultaneous eigenstates $|\psi_k\rangle$. Then $\hat{A} = \sum_k A_k |\psi_k\rangle\langle\psi_k|$ and $\hat{B} = \sum_k B_k |\psi_k\rangle\langle\psi_k|$.

It is trivial to show that $[\hat{A}, \hat{B}] = 0$.

(2) Suppose $[\hat{A}, \hat{B}] = 0$. Let $|\psi_k^\nu\rangle$ be an eigenstate of $\hat{A}$ with eigenvalue A_k, i.e., $\hat{A}|\psi_k^\nu\rangle = A_k|\psi_k^\nu\rangle$. Consider the vector $\hat{B}|\psi_k^\nu\rangle$:

$$\hat{A}(\hat{B}|\psi_k^\nu\rangle) = \hat{B}(\hat{A}|\psi_k^\nu\rangle) = A_k(\hat{B}|\psi_k^\nu\rangle.$$

$\therefore$ $\hat{B}|\psi_k^\nu\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue A_k. If A_k is nondegenerate, this implies that $\hat{B}|\psi_k^\nu\rangle = B_k|\psi_k^\nu\rangle$. If A_k is degenerate, then it implies that $\hat{B}|\psi_k^\nu\rangle$ lies in the subspace spanned by the degenerate eigenvectors. Diagonalizing $\hat{B}$ in all degenerate subspaces of $\hat{A}$ yields simultaneous eigenstates.

General – C-T

Continuous spectra: Example – position $\hat{x}$ and momentum $\hat{p}$

Position eigenstates: $|x\rangle$, $-\infty < x < \infty$

Eigenvalue equation: $\hat{x}|x\rangle = x|x\rangle$

x-representation for arbitrary ket $|\psi\rangle$: $|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle \langle x|\psi\rangle$

$$\psi(x) = \begin{pmatrix}\text{wave}\\ \text{function}\end{pmatrix} = \begin{pmatrix}\text{amplitude to}\\ \text{be at } x\end{pmatrix}$$

If this is true, we must have

$$\underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int_{-\infty}^{\infty} dx' \langle x|x'\rangle \underbrace{\langle x'|\psi\rangle}_{\psi(x')}$$

$$\Longrightarrow \quad \delta(x-x')$$

Orthonormality: $\langle x|x'\rangle = \delta(x-x')$ δ-fcn normalization

Completeness: $\int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \hat{1}$

Eigenstates $|x\rangle$ not normalizable $\Longleftrightarrow$ $\langle x|x'\rangle = \delta(x-x')$, the wave function for position eigenstate $|x'\rangle$, is not square-integrable.

$|x\rangle$ not in Hilbert space

$\langle x|$ is in dual space

Position representation:

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)}$$

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle\phi|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, \phi^*(x)\psi(x)$$

$$\hat{O} = \int dx\, dx'\, |x\rangle \underbrace{\langle x|\hat{O}|x'\rangle}_{O(x,x')} \langle x'|$$

Momentum representation: $\hat{p}$, $|p\rangle$, $\tilde{\psi}(p)$

Unitary transformation: $\langle x|p\rangle \equiv \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$

$$|p\rangle = \int \frac{dx}{\sqrt{2\pi\hbar}} |x\rangle e^{\frac{i}{\hbar}px}$$

Changing this changes ... momentum integration measure

Orthonormality:

$$\langle p|p'\rangle = \int dx\, \langle p|x\rangle\langle x|p'\rangle = \int \frac{dx}{2\pi\hbar} e^{\frac{i}{\hbar}(p'-p)x} = \delta(p-p')$$

Completeness:

$$\int dp\, |p\rangle\langle p| = \int dx\, dx'\, |x\rangle\langle x'| \underbrace{\int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p(x-x')}}_{\delta(x-x')}$$

$$= \int dx\, |x\rangle\langle x|$$

$$= \hat{1}$$

$\hat{p}|p\rangle = p|p\rangle \longleftarrow$ defn of $\hat{p}$

$$\langle x|\hat{p}|\psi\rangle = \int dp\, dx' \underbrace{\langle x|\hat{p}|p\rangle}_{p\langle x|p\rangle} \langle p|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \langle x'|\psi\rangle \underbrace{\int \frac{dp}{2\pi\hbar}\, p\, e^{\frac{i}{\hbar}p(x-x')}}_{= \frac{\hbar}{i}\frac{d}{dx} \underbrace{\int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p(x-x')}}_{\delta(x-x')}}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\int dx' \langle x'|\psi\rangle\, \delta(x-x')$$

$$= \underbrace{\frac{\hbar}{i}\frac{d}{dx}}_{\text{position representation of } \hat{p}} \langle x|\psi\rangle$$

$$\langle p|\hat{x}|\psi\rangle = \underbrace{-\frac{\hbar}{i}\frac{d}{dp}}_{\text{momentum representation of } \hat{x}} \langle p|\psi\rangle$$

$$\langle x|\hat{x}\hat{p}|\psi\rangle = x\langle x|\hat{p}|\psi\rangle = x\frac{\hbar}{i}\frac{d\psi(x)}{dx}$$

$$\langle x|\hat{p}\hat{x}|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}(x\psi(x)) = x\frac{\hbar}{i}\frac{d\psi}{dx} + \frac{\hbar}{i}\psi(x)$$

$$\Longrightarrow \quad [\hat{x},\hat{p}] = i\hbar$$

Operator algebra