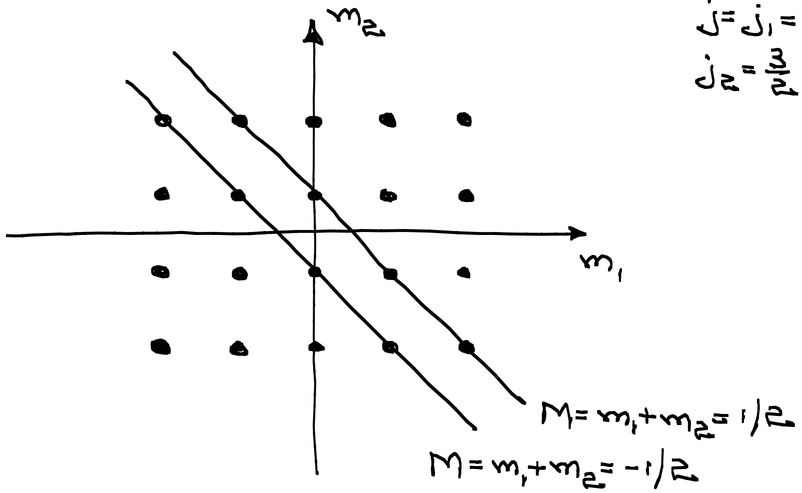
Phys 522

Midterm #1

Solution Set

Problem 1. $j_1 = j,\ j_2 = j - \frac{1}{2}$



$j = j_1 = 2$
$j_2 = \frac{3}{2}$

$$|J = \tfrac{1}{2}, M = \tfrac{1}{2}\rangle = \sum_{m=-j+1}^{j} |j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2}\rangle\langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Recursion relation:

$$C_{\mp}(J, M)\langle j_1 j_2 m_1 m_2 | J, M\mp 1\rangle = C_{\pm}(j_1, m_1)\langle j_1 j_2, m_1 \pm 1, m_2 | JM\rangle$$
$$+ C_{\pm}(j_2, m_2)\langle j_1 j_2 m_1, m_2 \pm 1 | JM\rangle$$

Upper sign:

Lower sign:

$$C_{\pm}(j, m) = \sqrt{j(j+1) - m(m\pm 1)} = \sqrt{(j\mp m)(j\pm m+1)}$$

$M = +\frac{1}{2}$: Use lower sign with $m_1 = m,\ m_2 = -m + \frac{3}{2}$

$$0 = \overbrace{C_-(j, m)}^{\sqrt{(j+m)(j-m+1)}}\langle j, j-\tfrac{1}{2}, m-1, -m+\tfrac{3}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle$$
$$+ \underbrace{C_-(j-\tfrac{1}{2}, -m+\tfrac{3}{2})}_{\sqrt{(j-m+1)(j+m-1)}}\cdot\langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Steps down by 1 unit in m

$$\Rightarrow \sqrt{j+m}\,\langle j, j-\tfrac{1}{2}, m-1, -m+\tfrac{3}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle = -\sqrt{j+m-1}\,\langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Solution: $$\langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle = (-1)^{j-m}\sqrt{\frac{j+m}{2j}}\,\underbrace{\langle j, j-\tfrac{1}{2}, j, -j+\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle}_{\equiv A}$$

A is real and positive by convention.

A is determined by normalization:

$$1 = \sum_{m=-j+1}^{j} \langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2} | \tfrac{1}{2} \tfrac{1}{2} \rangle^2 = A^2 \sum_{m=-j+1}^{j} \frac{j+m}{2j}$$

$$= A^2 \sum_{m=-j+1}^{j} \frac{1}{2} + \frac{m}{2j}$$

all terms cancel except $m=j$

$2j$ terms in sum

$$= A^2 \left(\tfrac{1}{2} 2j + \tfrac{1}{2} \right)$$

$$= A^2 (j + \tfrac{1}{2})$$

$$\Longrightarrow \quad A = \frac{1}{\sqrt{j+1/2}}$$

Putting it all together, we have

$$\boxed{\begin{aligned} &\langle j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2} | \tfrac{1}{2} \tfrac{1}{2} \rangle = (-1)^{j-m} \sqrt{\frac{j+m}{j(2j+1)}} \,, \quad m = -j+1, \ldots, j \\ &|J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{j(2j+1)}} \sum_{m=-j+1}^{j} (-1)^{j-m} \sqrt{j+m}\, | j, j-\tfrac{1}{2}, m, -m+\tfrac{1}{2} \rangle \end{aligned}}$$

Problem 2. $$|JM\rangle = \sqrt{\frac{(J+M)!\,(J-M)!}{(2J)!}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_N \\ M=\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\ldots,\epsilon_N\rangle$$

(a) You could argue—effectively—that $|JM\rangle$ is symmetric among the $N=2J$ particles and has $n_+ = J-M$ particles with spin up, so the probability of finding any particle ℓ with spin up is n_+/N.

$$\boxed{p(+,\ell) = \frac{n_+}{N} = \frac{J+M}{2J}\,.}$$

If you want to do it formally, you could do the following. The projector onto spin-up for the first particle is

$$P_+^{(1)} = |+\rangle\langle+| \otimes \underset{\substack{\uparrow \\ \text{identity on the other } N-1 \text{ particles}}}{I_{N-1}}\,,$$

So the probability for spin up on the first particle is

$$\begin{aligned}
p(+,1) &= \langle JM|P_+^{(1)}|JM\rangle \\
&= \frac{(J+M)!\,(J-M)!}{(2J)!} \sum_{\substack{\epsilon_1,\ldots,\epsilon_N \\ M=\frac{1}{2}\sum_\ell \epsilon_\ell}} \sum_{\substack{\epsilon'_1,\ldots,\epsilon'_N \\ M=\frac{1}{2}\sum_\ell \epsilon'_\ell}} \underbrace{\langle \epsilon'_1,\ldots,\epsilon'_N|P_+^{(1)}|\epsilon_1,\ldots,\epsilon_N\rangle}_{\delta_{\epsilon_1\epsilon'_1}\delta_{\epsilon_1,+1}\delta_{\epsilon'_2\epsilon_2}\cdots\delta_{\epsilon'_N\epsilon_N}} \\
&= \frac{(J+M)!\,(J-M)!}{(2J)!} \underbrace{\sum_{\substack{\epsilon_2,\ldots,\epsilon_N \\ M-\frac{1}{2}=\sum_\ell \epsilon_\ell}} 1}_{\begin{pmatrix}\#\text{ of strings } \epsilon_2,\ldots,\epsilon_N \\ \text{with } n_+ = J+M-1\end{pmatrix} = \frac{(2J-1)!}{(J+M-1)!\,(J-M)!}} \\
&= \frac{(J+M)!\,(J-M)!}{(2J)!} \frac{(2J-1)!}{(J+M-1)!\,(J-M)!} \\
&= \frac{J+M}{2J}\,, \quad \text{as promised}
\end{aligned}$$

This calculation is really better stated in words. The number of strings $\epsilon_1,\ldots,\epsilon_N$ with $n_+ = J+M$ is $(2J)!/(J+M)!\,(J-M)!$, and all of these strings are equally likely. The number of these strings for which the ℓ'th particle is up is $(2J-1)!/(J+M-1)!\,(J-M)!$. The probability for the ℓ'th particle to be up is the fraction of strings with the ℓ'th particle having spin up:

$$P(+,\ell) = \frac{(2J-1)!/(J+M-1)!\,(J-M)!}{(2J)!/(J+M)!\,(J-M)!} = \frac{J+M}{2J}$$

(b) Each time we add a particle, the total angular momentum can go up by $\frac{1}{2}$ or down by $\frac{1}{2}$. To get to the maximal angular momentum $J=N/2$, we have to go up at each step, so there is only one subspace of maximal angular momentum. To get to angular momentum $J-1$, we have go down exactly one time in adding the $N-1$ particles to the first. This gives $N-1 = 2J-1$ subspaces of total angular momentum $J-1$.

(c) We add a final (Nth) spin-$\frac{1}{2}$ particle to the symmetric subspace of $N-1$ particles, which has J_z eigenstates

$$|J-\tfrac{1}{2}, M'\rangle = \sqrt{\frac{(J+M'-1/2)!\,(J-M'-1/2)!}{(2J-1)!}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_{N-1} \\ M'=\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\ldots,\epsilon_{N-1}\rangle .$$

The states $|J-\frac{1}{2},\frac{1}{2},J-1,M\rangle \equiv |J-1,M\rangle_N$ are given in terms of Clebsch-Gordon coefficients by

$$|J-1,M\rangle_N = |J-\tfrac{1}{2},\tfrac{1}{2},J-1,M\rangle_N$$

$$= \sum_{\substack{M',m \\ M'+m=M}} |J-\tfrac{1}{2},\tfrac{1}{2},M',m\rangle\langle J-\tfrac{1}{2},\tfrac{1}{2},M',m|J-\tfrac{1}{2},\tfrac{1}{2},J-1,M\rangle$$

$$= |J-\tfrac{1}{2},M-\tfrac{1}{2}\rangle \otimes \overbrace{|\tfrac{1}{2},\tfrac{1}{2}\rangle}^{|+\rangle} \overbrace{\langle J-\tfrac{1}{2},\tfrac{1}{2},M-\tfrac{1}{2},\tfrac{1}{2}|J-1,M\rangle}^{-\sqrt{\frac{J-M}{2J}}} \leftarrow \begin{array}{l} m=1/2 \\ \text{term} \end{array} \left(\begin{array}{c} N\text{th spin} \\ \text{up} \end{array}\right)$$

$j=J-\frac{1}{2},\ m=M$

$$+\, |J-\tfrac{1}{2},M+\tfrac{1}{2}\rangle \otimes \underbrace{|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{|-\rangle} \underbrace{\langle J-\tfrac{1}{2},\tfrac{1}{2},M+\tfrac{1}{2},-\tfrac{1}{2}|J-1,M\rangle}_{\sqrt{\frac{J+M}{2J}}} \leftarrow \begin{array}{l} m=-1/2 \\ \text{term} \end{array} \left(\begin{array}{c} N\text{th spin} \\ \text{down} \end{array}\right)$$

$j=J-\frac{1}{2},\ m=M$

$$|J-1,M\rangle_N = -\sqrt{\frac{J-M}{2J}}\;\underbrace{|J-\tfrac{1}{2},M-\tfrac{1}{2}\rangle\otimes|+\rangle} + \sqrt{\frac{J+M}{2J}}\;\underbrace{|J-\tfrac{1}{2},M+\tfrac{1}{2}\rangle\otimes|-\rangle}$$

$$= \sqrt{\frac{(J+M-1)!\,(J-M)!}{(2J-1)!}}\;\sum_{\substack{\epsilon_1,\dots,\epsilon_{N-1}\\ M-\frac{1}{2}=\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\dots,\epsilon_{N-1}\rangle\otimes|+\rangle$$

$$= \sqrt{\frac{(J+M)!\,(J-M-1)!}{(2J-1)!}}\;\sum_{\substack{\epsilon_1,\dots,\epsilon_{N-1}\\ M+\frac{1}{2}=\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\dots,\epsilon_{N-1}\rangle\otimes|-\rangle$$

$$\boxed{\begin{aligned} |J-1,M\rangle_N &= \sqrt{\frac{(J+M)!\,(J-M)!}{(2J)!}}\Bigg[-\sqrt{\frac{J-M}{J+M}}\sum_{\substack{\epsilon_1,\dots,\epsilon_{N-1}\\ M=\frac{1}{2}+\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\dots,\epsilon_{N-1}\rangle\otimes|+\rangle \\ &\qquad + \sqrt{\frac{J+M}{J-M}}\sum_{\substack{\epsilon_1,\dots,\epsilon_{N-1}\\ M=-\frac{1}{2}+\frac{1}{2}\sum_\ell \epsilon_\ell}} |\epsilon_1,\dots,\epsilon_{N-1}\rangle\otimes|-\rangle\Bigg] \\ &= \sqrt{\frac{(J+M)!\,(J-M)!}{(2J)!}}\sum_{\substack{\epsilon_1,\dots,\epsilon_N\\ M=\frac{1}{2}\sum_\ell \epsilon_\ell}} -\epsilon_N\sqrt{\frac{J-\epsilon_N M}{J+\epsilon_N M}}\;|\epsilon_1,\dots,\epsilon_N\rangle \end{aligned}}$$

Digression: You should think about whether these states are normalized, i.e.,

$${}_N\langle J-1,M|J-1,M\rangle_N = \frac{(J+M)!\,(J-M)!}{(2J)!}\left(\frac{J-M}{J+M}\,\overbrace{\frac{(2J-1)!}{(J+M-1)!\,(J-M)!}}^{\#\text{ of strings with }\epsilon_N=+1} + \frac{J+M}{J-M}\,\underbrace{\frac{(2J-1)!}{(J+M)!\,(J-M-1)!}}_{\#\text{ of strings with }\epsilon_N=-1}\right]$$

$$= \frac{1}{2J}\left(J-M+J+M\right)$$

$$= 1$$

You should also think about how these states manage to be orthogonal to the states $|JM\rangle$. Different values of M are obviously orthogonal, since they involve different strings. For the same M, we have

$$\langle JM \,|\, J-1, M\rangle_N = \frac{(J+M)!(J-M)!}{(2J)!}\left(-\sqrt{\frac{J-M}{J+M}}\,\overbrace{\frac{(2J-1)!}{(J+M-1)!\,(J-M)!}}^{\#\text{ of strings with } \epsilon_N=+1} + \sqrt{\frac{J+M}{J-M}}\,\underbrace{\frac{(2J-1)!}{(J+M)!\,(J-M-1)!}}_{\#\text{ of strings with } \epsilon_N=-1}\right)$$

$$= \frac{1}{2J}\left(-\sqrt{(J-M)(J+M)} + \sqrt{(J+M)(J-M)}\right)$$

$$= 0$$

(d) You can do this many ways, but probably the easiest way is to use form

$$|J-1, M\rangle_N = -\sqrt{\frac{J-M}{2J}}\,|J-\tfrac{1}{2}, M-\tfrac{1}{2}\rangle \otimes |+\rangle + \sqrt{\frac{J+M}{2J}}\,|J-\tfrac{1}{2}, M+\tfrac{1}{2}\rangle \otimes |-\rangle$$

The probability for the Nth particle to be up is $(J-M)/2J$ and to be down is $(J+M)/2J$, i.e.,

$$\boxed{p(\pm, N) = \frac{J\mp M}{2J}}\,.$$

The probability for the first particle to be up, given that the last is $\pm$, is the probability for the first to be up in the state $|J-\frac{1}{2}, M\mp\frac{1}{2}\rangle$, i.e.,

$$p(+,1\,|\,\pm, N) = \begin{cases} (J+M-1)/(2J-1) \\ (J+M)/(2J-1) \end{cases}$$

So the unconditioned probability for the first particle to be spin up is

$$p(+,1) = p(+,1|+,N)\,p(+,N) + p(+,1|-,N)\,p(-,N)$$

$$= \frac{J+M-1}{2J-1}\,\frac{J-M}{2J} + \frac{J+M}{2J-1}\,\frac{J+M}{2J}$$

$$= \frac{2J(J+M)-(J-M)}{2J(2J-1)}$$

$$\boxed{p(+,1) = \frac{J+M}{2J-1} - \frac{J-M}{2J(2J-1)}}$$

The same logic applies to any of the first $N-1$ particles, so this is also $p(+,\ell)$ for $\ell=1,\ldots,N-1$.