Phys 522

Midterm #2

Solution Set

Background

Two-d, isotropic harmonic oscillator:

$$H_{osc} = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2} m\omega^2 (X^2 + Y^2) - \hbar\omega = \hbar\omega (a_x^\dagger a_x + a_y^\dagger a_y)$$

subtracts the zero-point energy of the two modes

$$a_x = \sqrt{\frac{m\omega}{2\hbar}} \left(X + i \frac{P_x}{m\omega} \right) \qquad a_y = \sqrt{\frac{m\omega}{2\hbar}} \left(X + i \frac{P_y}{m\omega} \right)$$

$$X = \sqrt{\frac{\hbar}{2m\omega}} (a_x + a_x^\dagger) \qquad Y = \sqrt{\frac{\hbar}{2m\omega}} (a_y + a_y^\dagger)$$

$$P_x = -i \sqrt{\frac{\hbar m\omega}{2}} (a_x - a_x^\dagger) \qquad P_y = -i \sqrt{\frac{\hbar m\omega}{2}} (a_y - a_y^\dagger)$$

$$L_z = XP_y - YP_x = -i\frac{\hbar}{2} \Big[\underbrace{(a_x + a_x^\dagger)(a_y - a_y^\dagger) - (a_y + a_y^\dagger)(a_x - a_x^\dagger)}_{-2a_x a_y^\dagger + 2a_x^\dagger a_y} \Big]$$

$$= i\hbar \left(a_y^\dagger a_x - a_x^\dagger a_y \right)$$

$$= \frac{1}{2}\hbar \Big[\underbrace{(a_+^\dagger - a_-^\dagger)(a_+ + a_-) + (a_+^\dagger + a_-^\dagger)(a_+ - a_-)}_{2a_+^\dagger a_+ - 2a_-^\dagger a_-} \Big]$$

$$= \hbar (a_+^\dagger a_+ - a_-^\dagger a_-)$$

$$a_\pm = \frac{1}{\sqrt{2}} (a_x \mp i a_y) \qquad a_x = \frac{1}{\sqrt{2}} (a_+ + a_-)$$

$$a_y = \frac{i}{\sqrt{2}} (a_+ - a_-)$$

$$H_{osc} = \hbar\omega (a_+^\dagger a_+ + a_-^\dagger a_-)$$

Electric field $\mathcal{E}$ along y direction: $W_e = -q\mathcal{E}Y$

Magnetic field $\mathcal{B}$ along z direction: $W_m = -\frac{q\mathcal{B}}{2m} L_z = -\Omega L_z$

Larmor frequency

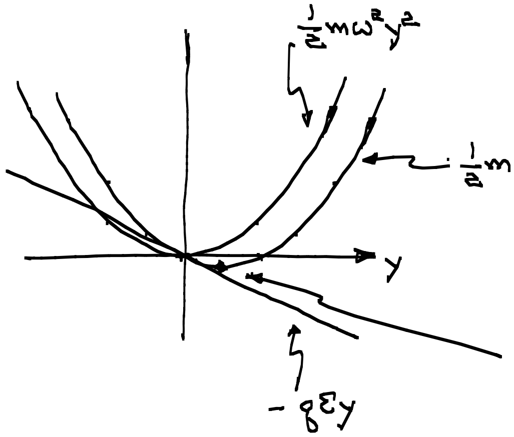
(a) $H = H_{osc} + W_e = \dfrac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(X^2+Y^2) - q\mathcal{E}Y - \hbar\omega$

The electric field dispaces the equilibrium position from the origin:

$$\frac{1}{2}m\omega^2Y^2 - q\mathcal{E}Y = \frac{1}{2}m\omega^2\Big(Y^2 - 2\underbrace{\frac{q\mathcal{E}}{m\omega^2}}_{\equiv y_0}Y\Big)$$

$$= \frac{1}{2}m\omega^2(Y-y_0)^2 - \underbrace{\frac{1}{2}m\omega^2y_0^2}_{= q^2\mathcal{E}^2/2m\omega^2}$$

new equilibrium position



The new potential is the same parabola, but displaced so that the energy at the new equilibrium position y_0 is lower by $q^2\mathcal{E}^2/2m\omega^2$.

So $H = \dfrac{P_x + P_y^2}{2m} + \frac{1}{2}m\omega^2[X^2 + (Y-y_0)^2] - \hbar\omega - \dfrac{q^2\mathcal{E}^2}{2m\omega^2}$

Let $T_{y_0} \equiv \exp(-\frac{i}{\hbar}P_y y_0)$ be the unitary translation operator that displaces by y_0 in the y direction. Since $T_{y_0} Y T_{y_0}^\dagger = Y - y_0$, we can write H as

$$H = T_{y_0}\left(\frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(X^2+Y^2) - \hbar\omega - \frac{q^2\mathcal{E}^2}{2m\omega^2}\right)T_{y_0}^\dagger$$

$$= T_{y_0}(H_{osc} - q^2\mathcal{E}^2/2m\omega^2)T_{y_0}^\dagger$$

Thus the eigenstates of H are $T_{y_0}|n_x, n_y\rangle$, with corresponding eigenvalues $\hbar\omega(n_x+n_y) - q^2\mathcal{E}^2/2m\omega^2$. The electric field shifts all the eigenvalues by the same amount, leaving all the degeneracies. Thus the states $T_{y_0}|n_+, n_-\rangle$ are also eigenstates.

(b) $H = H_{osc} + W_m = H_{osc} - \Omega L_z$

Since both H_0 and L_z are diagonal in the $|n_+, n_-\rangle$ basis (i.e., they commute), the eigenstates of H are $|n_+, n_-\rangle$, with corresponding eigenvalues

$$\hbar\omega(n_+ + n_-) - \hbar\Omega(n_+ - n_-) = \hbar(\omega - \Omega)n_+ + \hbar(\omega + \Omega)n_-$$

The only remaining degeneracies are accidental. Assuming $q\mathcal{B}/2m\hbar\omega$ is irrational eliminates such accidental degeneracies, although there will always be near degeneracies.

(c) $H = \underbrace{H_{osc} + W_m}_{H_0} + \underbrace{W_e}_{\text{perturbation}}$

H_0: eigenstates $|n_+, n_-\rangle$

eigenvalues $E^{(0)}_{n_+,n_-} = \hbar\omega(n_+ + n_-) - \hbar\Omega(n_+ - n_-) = \hbar(\omega - \Omega)n_+ + \hbar(\omega + \Omega)n_-$

Since W_m lifts all the degeneracies, we should use 2nd-order nondegenerate perturbation theory. The relevant matrix elements are

$$\langle n'_+, n'_-|W_e|n_+, n_-\rangle = -\frac{i}{2} q\mathcal{E}\sqrt{\frac{\hbar}{m\omega}}\Big(\sqrt{n_+}\,\delta_{n'_+,n_+-1}\delta_{n'_-,n_-} - \sqrt{n_-}\,\delta_{n'_+,n_+}\delta_{n'_-,n_--1}$$
$$- \sqrt{n_++1}\,\delta_{n'_+,n_++1}\delta_{n'_-,n_-} + \sqrt{n_-+1}\,\delta_{n'_+,n_+}\delta_{n'_-,n_-+1}\Big)$$

$$W_e = -q\mathcal{E}Y = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega}}(a_y + a_y^\dagger) = -\frac{i}{2} q\mathcal{E}\sqrt{\frac{\hbar}{m\omega}}(a_+ - a_- - a_+^\dagger + a_-^\dagger)$$

The general 2nd-order expression for E_{n_+,n_-} is

$$E_{n_+,n_-} = E^{(0)}_{n_+,n_-} + \underbrace{\langle n_+,n_-|W_e|n_+,n_-\rangle}_{=0} + \sum_{\substack{(n'_+,n'_-)\\ \neq (n_+,n_-)}} \frac{|\langle n'_+,n'_-|W_e|n_+,n_-\rangle|^2}{E^{(0)}_{n_+,n_-} - E^{(0)}_{n'_+,n'_-}}$$

W_e has no diagonal terms

only 4 terms survive

$$= \frac{\hbar}{4}\frac{g^2\mathcal{E}^2}{m\omega}\left(\frac{n_+}{E^{(0)}_{n_+,n_-}-E^{(0)}_{n_+-1,n_-}} + \frac{n_-}{E^{(0)}_{n_+,n_-}-E^{(0)}_{n_+,n_--1}} + \frac{n_++1}{E^{(0)}_{n_+,n_-}-E^{(0)}_{n_++1,n_-}} + \frac{n_-+1}{E^{(0)}_{n_+,n_-}-E^{(0)}_{n_+,n_-+1}}\right)$$

$$= \frac{\hbar}{4}\frac{g^2\mathcal{E}^2}{m\omega}\left(\frac{n_+}{\hbar(\omega-\Omega)} + \frac{n_-}{\hbar(\omega+\Omega)} - \frac{n_++1}{\hbar(\omega-\Omega)} - \frac{n_-+1}{\hbar(\omega+\Omega)}\right)$$

$$= -\frac{g^2\mathcal{E}^2}{4m\omega}\underbrace{\left(\frac{1}{\omega-\Omega}+\frac{1}{\omega+\Omega}\right)}_{=\frac{2\omega}{\omega^2-\Omega^2}}$$

$$= -\frac{g^2\mathcal{E}^2}{2m(\omega^2-\Omega^2)}$$

$$\boxed{E_{n_+,n_-} = E^{(0)}_{n_+,n_-} - \frac{g^2\mathcal{E}^2}{2m(\omega^2-\Omega^2)} = \hbar(\omega-\Omega)n_+ + \hbar(\omega+\Omega)n_- - \frac{g^2\mathcal{E}^2}{2m(\omega^2-\Omega^2)}}$$

Agrees with (a) when $\Omega=0$

(d) $H = \underbrace{H_{osc} + W_e}_{H_0} + W_m$

W_m: perturbation

H_0:
eigenstates $T_{y_0}|n_x,n_-\rangle$
eigenvalues $E^{(0)}_{n_x,n_y} = \hbar\omega(n_x+n_y) - \frac{g^2\mathcal{E}^2}{2m\omega^2}$

We need to do degenerate perturbation theory in each subspace of fixed total number of quanta, $N = n_x+n_y = n_++n_-$, spanned by displaced states $T_{y_0}|n_x,n_y\rangle$ or $T_{y_0}|n_+,n_-\rangle$. In each such subspace, we need to diagonalize $W_m = -\Omega L_z$, i.e., diagonalize L_z.

So diagonalize $\langle n_x',n_y'|T^\dagger_{y_0} L_z T_{y_0}|n_x,n_y\rangle$ for fixed N. But this means we diagonalize $T^\dagger_{y_0} L_z T_{y_0}$ in the original basis $|n_x,n_y\rangle$ for fixed N.

$$T_{y_0}^\dagger L_z T_{y_0} = T_{y_0}^\dagger (XP_y - YP_x) T_{y_0}$$
$$= XP_y - (Y + y_0) P_x$$
$$= L_z - y_0 P_x$$

Since P_x creates or annihilates an x-quantum, it vanishes in a subspace of fixed N. Thus the problem reduces to diagonalizing L_z, and that is already done by the $|n_+, n_-\rangle$ basis. In other words, we have

$$\langle n_+', n_-' | T_{y_0}^\dagger W_m T_{y_0} | n_+, n_- \rangle = -\Omega \langle n_+', n_-' | T_{y_0}^\dagger L_z T_{y_0} | n_+, n_- \rangle$$
$$= -\Omega \underbrace{\langle n_+', n_-' | L_z | n_+, n_- \rangle}_{= \hbar (n_+ - n_-) \delta_{n_+', n_+} \delta_{n_-', n_-}} + \Omega y_0 \underbrace{\langle n_+', n_-' | P_x | n_+, n_- \rangle}_{= 0 \text{ for } n_+' + n_-' = n_+ + n_- = N}$$
$$= \underbrace{-\hbar \Omega (n_+ - n_-)}_{\text{These are the 1st-order corrections}} \delta_{n_+', n_+} \delta_{n_-', n_-}$$

The 1st-order energy eigenvalues are

$$\boxed{\begin{aligned} E_{n_+, n_-} &= E^{(0)}_{n_+, n_-} - \hbar \Omega (n_+ - n_-) \\ &= \hbar \omega (n_+ + n_-) - \frac{g^2 \mathcal{E}^2}{2m\omega^2} - \hbar \Omega (n_+ - n_-) \\ &= \hbar (\omega - \Omega) n_+ + \hbar (\omega + \Omega) n_- - \frac{g^2 \mathcal{E}^2}{2m\omega^2} \end{aligned}}$$

Agrees with (b)
when $\mathcal{E} = 0$

Exact solution: A little thought shows that conjugating the Hamiltonian $H_{osc}+W_m$ with displacements in Y and P_x will give linear terms in Y and P_x plus constants. Adjusting the displacements properly can eliminate the P_x term and make the Y term equal to W_e. Then the problem is solved. Let's do it.

$$T_y = \exp\left(-\frac{i}{\hbar}P_y y\right) \longleftarrow \text{displacement of } Y \text{ by } y$$

$$T_p = \exp\left(\frac{i}{\hbar}X p\right) \longleftarrow \text{displacement of } P_x \text{ by } p$$

$$T_y(H_{osc}+W_m)T_y^\dagger = T_y\left(\frac{P_x^2+P_y^2}{2m}+\frac{1}{2}m\omega^2(X^2+Y^2)-\hbar\omega-\Omega L_z\right)T_y^\dagger$$

$$= \frac{P_x^2+P_y^2}{2m}+\frac{1}{2}m\omega^2\left[X^2+(Y-y)^2\right]-\hbar\omega-\Omega L_z-\Omega y P_x$$

$$= \frac{P_x^2+P_y^2}{2m}+\frac{1}{2}m\omega^2(X^2+Y^2)-\hbar\omega-\Omega L_z$$

$$-m\omega^2 y Y-\Omega y P_x+\frac{1}{2}m\omega^2y^2$$

$$T_pT_y(H_{osc}+W_m)T_y^\dagger T_p^\dagger$$

$$= \frac{(P_x-p)^2+P_y^2}{2m}+\frac{1}{2}m\omega^2(X^2+Y^2)-\hbar\omega-\Omega L_z-\Omega pY$$

$$-m\omega^2yY-\Omega yP_x+\Omega yp+\frac{1}{2}m\omega^2y^2$$

$$= \frac{P_x^2+P_y^2}{2m}+\frac{1}{2}m\omega^2(X^2+Y^2)-\hbar\omega-\Omega L_z-(\Omega p+m\omega^2y)Y$$

$$-(\Omega y+p/m)P_x+\frac{p^2}{2m}+\Omega yp+\frac{1}{2}m\omega^2y^2$$

Choose $\Omega y+\dfrac{p}{m}=0$ and $\Omega p+m\omega^2 y = q\mathcal{E}$ $\Longleftrightarrow$ $y=\dfrac{q\mathcal{E}}{m(\omega^2-\Omega^2)}$, $p=-m\Omega y=-\dfrac{q\mathcal{E}\Omega}{\omega^2-\Omega^2}$

$$\Longrightarrow \frac{p^2}{2m}+\Omega yp+\frac{1}{2}m\omega^2y^2=\frac{1}{2}m(\omega^2-\Omega^2)y^2=\frac{q^2\mathcal{E}^2}{2m(\omega^2-\Omega^2)}$$

Thus we have

$$T_p T_y \left(H_{osc} + W_m - \frac{g^2 \mathcal{E}^2}{2m(\omega^2 - \Omega^2)} \right) T_y^\dagger T_p^\dagger = H_{osc} + W_m + W_e = H$$

So the eigenstates of H are

$$T_p T_y |n_+, n_-\rangle,$$

with eigenvalues

$$E_{n_+ n_-} = \hbar(\omega - \Omega) n_+ + \hbar(\omega + \Omega) n_- - \frac{g^2 \mathcal{E}^2}{2m(\omega^2 - \Omega^2)} .$$

Part (c) got it exactly right because 2nd-order perturbation theory is exact for linear/quadratic Hamiltonians. Part (d) needed to assume that $\Omega \ll \omega$, so missed out on the exact energy shift.