Phys 522

Midterm #3

Solution Set

①

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2[1 + g\cos 2\omega t]X^2 = \underbrace{\frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2}_{= H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})} + \underbrace{\frac{1}{2}g\, m\omega^2 X^2 \cos 2\omega t}_{W(t)}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,(X + iP/m\omega) \qquad X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,(X - iP/m\omega) \qquad P = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a - a^\dagger)$$

$$[a, a^\dagger] = 1 \qquad [X, P] = i\hbar$$

Initial state: $|\psi(0)\rangle = |0\rangle$

Evolved state: $|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} b_n(t)\, e^{-in\omega t}|n\rangle$

$$X^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2 = \frac{\hbar}{2m\omega}(a^2 + a^{\dagger 2} + \underbrace{aa^\dagger}_{a^\dagger a + 1} + a^\dagger a) = \frac{\hbar}{2m\omega}(a^2 + a^{\dagger 2} + 2a^\dagger a + 1)$$

$$\langle n|X^2|m\rangle = \frac{\hbar}{2m\omega}\Big(\underbrace{\langle n|a^2|m\rangle}_{\sqrt{m(m-1)}\,\delta_{n,m-2}} + \underbrace{\langle n|a^{\dagger 2}|m\rangle}_{\sqrt{(m+1)(m+2)}\,\delta_{n,m+2}} + \underbrace{\langle n|(2a^\dagger a + 1)|m\rangle}_{(2m+1)\,\delta_{nm}}\Big)$$

$$\langle n|X^2|m\rangle = \frac{\hbar}{2m\omega}\Big(\sqrt{m(m-1)}\,\delta_{n,m-2} + \sqrt{(m+1)(m+2)}\,\delta_{n,m+2} + (2m+1)\,\delta_{nm}\Big)$$

$W(t)$ only connects $|m\rangle$ to $|m\pm 2\rangle$ and $|m\rangle$:

$$\langle n|W(t)|m\rangle = \frac{1}{4}g\hbar\omega\cos 2\omega t\Big(\sqrt{m(m-1)}\,\delta_{n,m-2} + \sqrt{(m+1)(m+2)}\,\delta_{n,m+2} + (2m+1)\,\delta_{nm}\Big)$$

(a) First-order amplitudes:

$$\omega_{nm} = \frac{E_n - E_m}{\hbar} = (n-m)\omega$$

$$b_n^{(1)}(\tau) = -\frac{i}{\hbar}\int_0^\tau dt\, e^{in\omega t}\underbrace{W_{n0}(t)}_{\frac{1}{4}g\hbar\omega\cos 2\omega t(\sqrt{2}\,\delta_{n2} + \delta_{n0})}$$

$$= -\frac{i}{4} g\omega\Big(\sqrt{2}\,\delta_{n2}\underbrace{\int_0^\tau dt\, e^{2i\omega t}\cos 2\omega t} + \delta_{n0}\underbrace{\int_0^\tau dt\,\cos 2\omega t}_{\frac{1}{2\omega}\sin 2\omega\tau}\Big)$$

nonresonant

$$= \int_0^\tau dt\,\frac{1}{2}\left(e^{4i\omega t} + 1\right)$$

nonresonant

$$= \frac{1}{2}\tau + \frac{1}{2}\,\frac{e^{4i\omega\tau}-1}{4i\omega}$$

resonant

$$= \frac{1}{2}\tau - \frac{i}{2}\,\frac{e^{4i\omega\tau}-1}{4\omega}$$

Secular (resonant) terms

$$b_0^{(1)}(\tau) = -\frac{i}{8} g\sin\omega\tau \xrightarrow{\omega\tau\ll 1} -\frac{i}{8} b\,\omega\tau$$

$$b_0^{(1)}(\tau) = -\frac{i}{8} g\sin\omega\tau \longrightarrow 0$$

$$b_2^{(1)}(\tau) = -\frac{i}{4\sqrt{2}} g\omega\tau - \frac{1}{4\sqrt{2}} g\,\frac{e^{4i\omega\tau}-1}{4} \longrightarrow -\frac{i}{4\sqrt{2}} g\omega\tau = -\frac{i\sqrt{2}}{8} g\omega\tau$$

$$\Big\downarrow{\scriptstyle \omega\tau\ll 1}$$

$$-\frac{i\sqrt{2}}{4} g\omega\tau$$

$$b_n^{(1)}(\tau) = 0,\quad n\neq 0,2$$

(b) $$b_n^{(2)}(\tau) = -\frac{i}{\hbar}\sum_m \int_0^\tau dt\, e^{i(n-m)\omega t}\, W_{nm}(t)\, b_m^{(1)}(t)$$

Let's keep only the resonant terms from the start.

$$= -\frac{i}{\hbar}\int_0^\tau dt\, e^{i(n-2)\omega t}\, W_{n2}(t)\,\underbrace{b_2^{(1)}(t)}_{-\frac{i}{4\sqrt{2}} g\omega t}$$

only $b_2^{(1)}(t)$ has a resonant term

$$= -\frac{1}{4\sqrt{2}}\frac{g\omega}{\hbar}\int_0^\tau dt\; t\, e^{i(n-2)\omega t}\,\underbrace{W_{n2}(t)}$$

We can have $n=4$, $n=2$, and $n=0$. The time dependence of the integrand in these cases is the following

$n=4$: $t\cos 2\omega t\, e^{2i\omega t} \longrightarrow \frac{1}{2}t$

$n=2$: $t\cos 2\omega t \longrightarrow 0$

$n=0$: $t\cos 2\omega t\, e^{-2i\omega t} \longrightarrow \frac{1}{2}t$

resonant part

$$b_4^{(2)}(\tau) = -\frac{1}{4\sqrt{2}}\frac{g\omega}{\hbar}\int_0^\tau dt\, t\, e^{2i\omega t}\underbrace{W_{42}(t)}_{\frac{\sqrt{3}}{2}g\hbar\omega\cos 2\omega t}$$

$$= -\frac{\sqrt{3}}{8\sqrt{2}}g^2\omega^2\underbrace{\int_0^\tau dt\,\underbrace{t\,e^{2i\omega t}\cos 2\omega t}_{\frac{1}{2}t \;\leftarrow\;\text{resonant part only}}}_{\frac{1}{4}\tau^2}$$

$$\boxed{b_4^{(2)}(\tau) = -\frac{\sqrt{3}}{32\sqrt{2}}g^2\omega^2\tau^2 = -\frac{\sqrt{6}}{64}g^2\omega^2\tau^2}$$

$$b_0^{(2)}(\tau) = -\frac{1}{4\sqrt{2}}\frac{g\omega}{\hbar}\int_0^\tau dt\, t\, e^{-2i\omega t}\underbrace{W_{02}(t)}_{\frac{\sqrt{2}}{4}g\hbar\omega\cos 2\omega t}$$

$$= -\frac{1}{16}g^2\omega^2\underbrace{\int_0^\tau dt\,\underbrace{t\,e^{-2i\omega t}\cos 2\omega t}_{\frac{1}{2}t \;\leftarrow\;\text{resonant part only}}}_{\frac{1}{4}\tau^2}$$

$$\boxed{b_0^{(2)}(\tau) = -\frac{1}{64}g^2\omega^2\tau^2}$$

$\boxed{\text{$b_2^{(2)}(\tau)$ has no resonant part, and all other $b_n^{(2)}(\tau)$ are strictly zero.}}$

(c) $i\hbar \dfrac{dU_I(t,0)}{dt} = W_I(t)\, U_I(t,0)$

$$\begin{pmatrix}\text{Interaction-picture} \\ \text{Hamiltonian}\end{pmatrix} = W_I(t) = e^{\frac{i}{\hbar}H_0 t}\, W(t)\, e^{-\frac{i}{\hbar}H_0 t}$$

$$= e^{i\omega t a^\dagger a}\, W(t)\, e^{-i\omega t a^\dagger a}$$

$$W(t) = \frac{1}{2} g m\omega^2 \underset{\uparrow}{X^2} \cos 2\omega t = \frac{1}{4} g\hbar\omega\left(a^2 + a^{\dagger 2} + 2a^\dagger a + 1\right)\cos 2\omega t$$

$$\frac{\hbar}{2m\omega}\left(a^2 + a^{\dagger 2} + 2a^\dagger a + 1\right)$$

$$\boxed{\begin{aligned} W_I(t) &= \frac{1}{4} g\hbar\omega\left(a^2 e^{-2i\omega t} + a^{\dagger 2} e^{2i\omega t} + 2a^\dagger a + 1\right)\cos 2\omega t \\ \text{keep only resonant terms} &= \frac{1}{8} g\hbar\omega\left(a^2 + a^{\dagger 2}\right)\end{aligned}}$$

Since $W_I(t)$ has no time dependence, the solution for $U_I(t)$ is easy

$$\boxed{U_I(t,0) = \exp\left(-\frac{i}{8} g\omega t\left(a^2 + a^{\dagger 2}\right)\right)}$$

This is called a <u>squeeze operator</u>.

The Schrödinger picture evolution operator is

$$U(t,0) = e^{-iH_0 t/\hbar}\, U_I(t,0),$$

so the amplitudes at time τ are

$$c_n(\tau) = \langle n|\, U(\tau,0)\,|0\rangle = e^{-i\omega t/2}\, e^{-in\omega t} \underbrace{\langle n|\, U_I(\tau,0)\,|0\rangle}_{b_n(\tau)}$$

The various orders in perturbation theory (powers of g) correspond to the terms in the Taylor expansion of $U_I(\tau,0) = \exp\left(-\frac{i}{8} g\omega\tau\left(a^2 + a^{\dagger 2}\right)\right)$.

$$U_I(\tau,0)|0\rangle = \left(I - \frac{i}{8}g\omega\tau(a^2+a^{\dagger 2}) - \frac{1}{128}g^2\omega^2\tau^2\underbrace{(a^2+a^{\dagger 2})^2}_{a^4+a^{\dagger 4}+a^2a^{\dagger 2}+a^{\dagger 2}a^2} + \dots\right)|0\rangle$$

$$= |0\rangle - \frac{i}{8}g\omega\tau\,\underbrace{a^{\dagger 2}|0\rangle}_{\sqrt{2}|2\rangle} - \frac{1}{128}g^2\omega^2\tau^2\,\underbrace{a^{\dagger 4}|0\rangle}_{\sqrt{4!}|4\rangle = 2\sqrt{6}|4\rangle} - \frac{1}{128}g^2\omega^2\tau^2\,\underbrace{a^2a^{\dagger 2}|0\rangle}_{2|0\rangle} + \dots$$

$$\boxed{U_I(\tau,0)|0\rangle = |0\rangle \underbrace{- \frac{i\sqrt{2}}{8}g\omega\tau}_{b_2^{(1)}(\tau)}|2\rangle \underbrace{- \frac{\sqrt{6}}{64}g^2\omega^2\tau^2}_{b_4^{(2)}(\tau)}|4\rangle \underbrace{- \frac{1}{64}g^2\omega^2\tau^2}_{b_0^{(2)}(\tau)}|0\rangle + \dots}$$