Phys 522

Homework #1

Solution Set

1.1. C-T B_{IX}. 1

(a) $1 = \langle\psi|\psi\rangle = \int d^3r \left(|\psi_+(r)|^2 + |\psi_-(r)|^2\right)$

$= \int r^2 dr\, |R(r)|^2 \int d\Omega \left(\left|Y_0^0 + \frac{1}{\sqrt{3}} Y_1^0\right|^2 \right.$

$\left. \frac{1}{3}\left|Y_1^1 - Y_1^0\right|^2 \right)$

Use orthogonality of Y_0^0 and Y_1^0 and of Y_1^1 and Y_1^1. $\rightarrow$ $= \int d\Omega \left(|Y_0^0|^2 + \frac{1}{3}|Y_1^0|^2 + \frac{1}{3}|Y_1^1|^2 + \frac{1}{3}|Y_1^0|^2 \right)$

$= \underbrace{\frac{4}{3}}_{\text{from } +} + \overbrace{\frac{2}{3}}^{\text{from } -}$

$1 = 2\int dr\, r^2 |R(r)|^2$

$$\boxed{\int_0^\infty dr\, r^2 |R(r)|^2 = \frac{1}{2}}$$

(b)

Measure S_z: possible values $\pm\hbar/2$.

$P(S_z = \pm\hbar/2) = \int d^3r\, |\langle \vec{r}, \pm|\psi\rangle|^2 = \int d^3r\, |\psi_\pm(\vec{r})|^2 = \frac{1}{2}$

$$\boxed{\begin{aligned} P(S_z = +\hbar/2) &= \int d^3r\, |\psi_+(\vec{r})|^2 = \frac{2}{3} \\ P(S_z = -\hbar/2) &= \int d^3r\, |\psi_-(\vec{r})|^2 = \frac{1}{3} \end{aligned}}$$

Measure L_z: $\boxed{\text{possible values} \quad 0, \hbar}$ (with nonzero probability)

$$\mathcal{P}(L_z = m\hbar) = \sum_{\pm} \sum_{l=0}^{\infty} \int \underbrace{dr\ \mathcal{P}(r, l, m, \pm)}_{dr\, r^2 |\langle r, l, m, \pm | \psi\rangle|^2}$$

$$\mathcal{P}(m=0) = \underbrace{\int dr\, r^2 |R(r)|^2}_{l=0,\, m=0,\, +} + \underbrace{\int dr\, r^2 \frac{1}{3} |R(r)|^2}_{l=1,\, m=0,\, +}$$

$$+ \underbrace{\int dr\, r^2 \frac{1}{3} |R(r)|^2}_{l=1,\, m=0,\, -}$$

$$\boxed{\mathcal{P}(L_z = 0) = \frac{5}{6}}$$

$$\mathcal{P}(m=0) = \underbrace{\int dr\, r^2 \frac{1}{3} |R(r)|^2}_{l=1,\, m=1,\, -} = \boxed{\frac{1}{6} = \mathcal{P}(L_z = \hbar)}$$

Measure S_x: possible values $\pm \frac{1}{2}\hbar$

$$|\psi\rangle = \int d^3r \left(|\vec{r}\rangle \otimes |+\rangle_z\, \psi_+(\vec{r}) + |\vec{r}\rangle \otimes |-\rangle_z\, \psi_-(\vec{r}) \right)$$

$$\mathcal{P}(S_x = \pm\hbar/2) = \int d^3r \left| \left(\langle \vec{r} | \otimes {}_x\langle \pm | \right) |\psi\rangle \right|^2$$

$$|+\rangle_x = \frac{1}{\sqrt{2}} \left(|+\rangle_z + |-\rangle_z \right)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}} \left(-|+\rangle_z + |-\rangle_z \right)$$

$$\left(\langle\vec{r}| \otimes {}_x\langle +1|\right)|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle\vec{r},+1| + \langle\vec{r},-1|\right)|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\psi_+(\vec{r}) + \psi_-(\vec{r})\right)$$

$$= \frac{1}{\sqrt{2}}R(r)\left(Y_0^0 + \frac{1}{\sqrt{3}}Y_1^0\right)$$

$$\left(\langle\vec{r}| \otimes {}_x\langle -1|\right)|\psi\rangle = \frac{1}{\sqrt{2}}\left(-\langle\vec{r},+1| + \langle\vec{r},-1|\right)|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(-\psi_+(\vec{r}) + \psi_-(\vec{r})\right)$$

$$= \frac{1}{\sqrt{2}}R(r)\left(-Y_0^0 + \frac{1}{\sqrt{3}}Y_1^1 - \frac{2}{\sqrt{3}}Y_1^0\right)$$

$$\mathcal{P}(S_x = +\hbar/2) = \int r^2 dr\, \frac{1}{2}|R(r)|^2\left(1 + \frac{1}{3}\right) = \boxed{\frac{1}{3} = \mathcal{P}(S_x = +\hbar/2)}$$

$$\mathcal{P}(S_x = -\hbar/2) = \int r^2 dr\, \frac{1}{2}|R(r)|^2\left(1 + \frac{1}{3} + \frac{4}{3}\right)$$

$$= \boxed{\frac{2}{3} = \mathcal{P}(S_x = -\hbar/2)}$$

(b) Measure $\vec{L}^2$

(1) Get result $\vec{L}^2 = 0$ $(l=0)$; state immediately after has

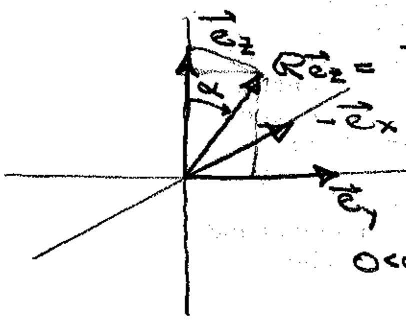
$$\boxed{\begin{aligned} \psi_+'(\vec{r}) &= \underbrace{C}_{\text{normalization}} R(r)\, Y_0^0 \\ \psi_-'(\vec{r}) &= 0 \end{aligned}}$$

(2) Get result $\vec{L}^2 = 2\hbar^2$ $(l=1)$; state immediately after has

$$\boxed{\begin{aligned} \psi_+'(\vec{r}) &= C\, R(r) \frac{1}{\sqrt{3}} Y_1^0 \\ \psi_-'(\vec{r}) &= C\, R(r) \frac{1}{\sqrt{3}} \left(Y_1^1 - Y_1^0\right) \end{aligned}}$$

normalization

1.2.



$$R\vec{e}_z = \vec{e}_z\cos\alpha + \vec{e}_y\sin\alpha \equiv \vec{u} \Longrightarrow$$

	$0<\alpha<\pi$	$\pi<\alpha<2\pi$
	$\theta=\alpha$	$\theta=2\pi-\alpha$
	$\varphi=\pi/2$	$\varphi=3\pi/2$

$0<\alpha<\pi$:

$$|+\rangle_{\vec{u}} = \cos(\alpha/2)|+\rangle_z + i\sin(\alpha/2)|-\rangle_z$$

$$|-\rangle_{\vec{u}} = \sin(\alpha/2)|+\rangle_z - i\cos(\alpha/2)|-\rangle_z$$

$\pi<\alpha<2\pi$:

$$|+\rangle_{\vec{u}} = -\cos(\alpha/2)|+\rangle_z - i\sin(\alpha/2)|-\rangle_z$$

$$|-\rangle_{\vec{u}} = \sin(\alpha/2)|+\rangle_z - i\cos(\alpha/2)|-\rangle_z$$

$$R = e^{i\alpha S_x/\hbar} = e^{i\sigma_x\alpha/2} = \cos(\alpha/2)\mathbb{1} + i\sin(\alpha/2)\sigma_x$$

$$R|\epsilon\rangle_z = \left(\cos(\alpha/2)\mathbb{1} + i\sin(\alpha/2)\sigma_x\right)|\epsilon\rangle_z$$

$$= \cos(\alpha/2)|\epsilon\rangle_z + i\sin(\alpha/2)\sigma_x|\epsilon\rangle_z$$

$$= \cos(\alpha/2)|\epsilon\rangle_z + i\sin(\alpha/2)|-\epsilon\rangle_z$$

$\sigma_x \to \begin{pmatrix}0&1\\1&0\end{pmatrix}$ in $|\epsilon\rangle_z$ basis

$\sigma_x|+\rangle_z = |-\rangle_z$

$\sigma_x|-\rangle_z = |+\rangle_z$

$\Longrightarrow \sigma_x|\epsilon\rangle_z = |-\epsilon\rangle_z$

$$R|+\rangle_z = \cos(\alpha/2)|+\rangle_z + i\sin(\alpha/2)|-\rangle_z$$

$$R|-\rangle_z = \cos(\alpha/2)|-\rangle_z + i\sin(\alpha/2)|+\rangle_z$$

$0<\alpha<\pi$: $R|+\rangle_z = |+\rangle_{\vec{u}}$, $R|-\rangle_z = i|-\rangle_{\vec{u}}$ $\quad \delta_+ = 0$, $\delta_- = \pi/2$

$\pi<\alpha<2\pi$: $R|+\rangle_z = -|+\rangle_{\vec{u}}$, $R|-\rangle_z = i|-\rangle_{\vec{u}}$ $\quad \delta_+ = \pi$, $\delta_- = \pi/2$

The phase discontinues at $\alpha=0$ and $\alpha=\pi$ and the fact that $R(2\pi)$ appears not to be -1 are both expressions of the coördinate singularities at the poles of the Bloch sphere, here appearing as singularities in the phases of the eigenvectors.

1.3. Show

$$R^\dagger_{\vec{u}}(\alpha)\,\vec{J}\,R_{\vec{u}}(\alpha) = \vec{u}(\vec{u}\cdot\vec{J}) + \vec{u}\times(\vec{u}\times\vec{J})\cos\alpha + \vec{u}\times\vec{J}\sin\alpha$$

(a) Use commutators:

$$R^\dagger\vec{J}R = e^{\overbrace{+i\vec{J}\cdot\vec{u}\alpha/\hbar}^{A}}\,\vec{J}\,e^{-i\vec{J}\cdot\vec{u}\alpha/\hbar}$$

Use $e^{A}Be^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \dots$

$$= B + \sum_{k=1}^{\infty}\frac{1}{k!}\underbrace{[A,[A,\dots[A,B]\dots]]}_{\equiv [A,B]^{(k)}}$$

k nested commutators

$$[A,J_j] = \frac{i}{\hbar}\alpha u_k\underbrace{[J_k,J_j]}_{i\hbar\epsilon_{kjl}J_l} = +\alpha\epsilon_{jkl}u_kJ_l = \alpha(\vec{u}\times\vec{J})_j$$

$$\Rightarrow [A,\vec{J}] = \vec{e}_j[A,J_j] = \alpha\,\vec{u}\times\vec{J}$$

$$[A,\vec{J}]^{(2)} = [A,[A,\vec{J}]] = [A,\alpha\vec{u}\times\vec{J}] = \alpha\vec{u}\times\underbrace{[A,\vec{J}]}_{\alpha\vec{u}\times\vec{J}}$$

$$\Rightarrow [A,\vec{J}]^{(2)} = \alpha^2\vec{u}\times(\vec{u}\times\vec{J}) =$$

$$[A,\vec{J}]^{(3)} = [A,[A,\vec{J}]^{(2)}] = \alpha^3\vec{u}\times\big(\underbrace{\vec{u}\times(\vec{u}\times\vec{J})}_{\vec{u}(\vec{u}\cdot\vec{J})-\vec{J}}\big) = -\alpha^3\vec{u}\times\vec{J}$$

The pattern is now established:

$$k>0:\ [A,\vec{J}]^{(k)} = \begin{cases}(-1)^{(k-1)/2}\alpha^k\,\vec{u}\times\vec{J}, & k \text{ odd}\\ (-1)^{(k-2)/2}\alpha^k\,\vec{u}\times(\vec{u}\times\vec{J}), & k \text{ even}\end{cases}$$

$$R^\dagger \vec{J} R = \vec{J} + \sum_{k=1}^{\infty} \frac{1}{k!}\,[A,\vec{J}]^{(k)}$$

$$= \vec{J} + \vec{u}\times\vec{J} \sum_{k\ \text{odd}} \frac{(-1)^{(k-1)/2}}{k!}\,\alpha^k$$

$$+\ \vec{u}\times(\vec{u}\times\vec{J}) \sum_{\substack{k\ \text{even}\\ k\geq 2}} \frac{(-1)^{(k-2)/2}}{k!}\,\alpha^k$$

$$k = 2l+1: \quad \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!}\,\alpha^{2l+1} = \sin\alpha$$

$$k = 2l: \quad \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{(2l)!}\,\alpha^{2l} = 1 - \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\,\alpha^{2l}$$

$$= 1 - \cos\alpha$$

$$\therefore \boxed{\begin{aligned} R^\dagger \vec{J} R &= \vec{J} + \underbrace{\vec{u}\times(\vec{u}\times\vec{J})}_{\vec{u}(\vec{u}\cdot\vec{J}) - \vec{J}}(1-\cos\alpha) + \vec{u}\times\vec{J}\,\sin\alpha \\ &= \vec{u}(\vec{u}\cdot\vec{J}) - \vec{u}\times(\vec{u}\times\vec{J})\cos\alpha + \vec{u}\times\vec{J}\,\sin\alpha \end{aligned}}$$

This method only uses the commutators, $[J_j, J_k] = i\epsilon_{jkl} J_l$, so if we can establish the property for the spin vector $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, which has the angular-momentum commutators, then it will be true for for any operators J_j.

③

(b) Use $R = e^{-i\vec{S}\cdot\vec{u}\alpha/\hbar} = e^{-i\vec{\sigma}\cdot\vec{u}\alpha/2} = 1\cos(\alpha/2) - i\vec{\sigma}\cdot\vec{u}\sin(\alpha/2)$.

$$\sigma_j R = \sigma_j c - i\underbrace{\sigma_j\sigma_k}_{=\delta_{jk}1 + i\epsilon_{jkl}\sigma_l} u_k s = \sigma_j c - iu_j s + \epsilon_{jkl}u_k\sigma_l s$$

$$\vec{\sigma}R = \vec{\sigma}c - i\vec{u}s + \vec{u}\times\vec{\sigma}s$$

$$R^\dagger\sigma_j R = (1c + iu_m\sigma_m s)(\sigma_j c - iu_j s + \epsilon_{jkl}u_k\sigma_l s)$$

$$= \sigma_j c^2 - iu_j cs + \epsilon_{jkl}u_k\sigma_l sc$$

$$+ isc\,u_m\underbrace{\sigma_m\sigma_j}_{=\delta_{mj} + i\epsilon_{mjl}\sigma_l} + s^2 u_j u_m\sigma_m + is^2\epsilon_{jkl}u_k\underbrace{\sigma_m\sigma_l}_{=\delta_{ml} + i\epsilon_{mln}\sigma_n}u_m$$

$$= \sigma_j c^2 - \cancel{iu_j cs} + \epsilon_{jkl}u_k\sigma_l sc$$

$$+ \cancel{iu_j sc} + \epsilon_{jml}u_m\sigma_l sc$$

$$+ u_j u_m\sigma_m s^2$$

$$+ \cancel{is^2\epsilon_{jkl}u_k u_l s^2}^{\,0} + \epsilon_{jkl}\epsilon_{lmn}u_k u_m\sigma_n s^2$$

$$= \frac{1}{2}\sigma_j(1+\cos\theta) + \epsilon_{jkl}u_k\sigma_l\sin\theta$$

$$+ u_j u_k\sigma_k\frac{1}{2}(1-\cos\theta)$$

$$+ \epsilon_{jkl}u_k\epsilon_{lmn}u_m\sigma_n\frac{1}{2}(1-\cos\theta)$$

$$R^\dagger\vec{\sigma}R = \frac{1}{2}\vec{\sigma}(1+\cos\alpha) + \vec{u}\times\vec{\sigma}\sin\alpha$$

$$+ \vec{u}(\vec{u}\cdot\vec{\sigma})\frac{1}{2}(1-\cos\alpha)$$

$$+ \underbrace{\vec{u}\times(\vec{u}\times\vec{\sigma})}\frac{1}{2}(1-\cos\alpha) \rightarrow \vec{u}(\vec{u}\cdot\vec{\sigma}) - \vec{\sigma}$$

$$= \frac{1}{2}\vec{\sigma}(1+\cos\alpha) + \vec{u}\times\vec{\sigma}\sin\alpha$$

$$+ \vec{u}(\vec{u}\cdot\vec{\sigma})(1-\cos\alpha) - \vec{\sigma}\,\frac{1}{2}(1-\cos\alpha)$$

$$= \vec{\sigma}\cos\alpha + \vec{u}(\vec{u}\cdot\vec{\sigma})(1-\cos\alpha) + \vec{u}\times\vec{\sigma}\sin\alpha$$

$$\boxed{R^{\dagger}\vec{\sigma}R = \vec{u}(\vec{u}\cdot\vec{\sigma}) - \vec{u}\times(\vec{u}\times\vec{\sigma})\cos\alpha + \vec{u}\times\vec{\sigma}\sin\alpha}$$

1.4. C-T B_{IX}. 2

$$A = \vec{S}\cdot\vec{P} = S_x P_x + S_y P_y + S_z P_z = \tfrac{1}{2}\hbar\,\vec{\sigma}\cdot\vec{P}$$

(a) $A^\dagger = \vec{P}^\dagger\cdot\vec{S}^\dagger = \vec{P}\cdot\vec{S} = \vec{S}\cdot\vec{P} = A$

$\vec{P}$ and $\vec{S}$ commute

$\vec{P}$ and $\vec{S}$ Hermitian

$\Longrightarrow$ $\boxed{A \text{ is Hermitian.}}$

(b) $[A, P_j] = 0$ because P_j commutes w/ all components of $\vec{P}$ and $\vec{S}$. $\therefore\ \exists$ a simultaneous eigenbasis of P_x, P_y, P_z, and A.

Let $|\vec{p},\epsilon\rangle = |p_x, p_y, p_z, \epsilon\rangle \equiv |\vec{p}\rangle \otimes |\epsilon\rangle_z$.

$$A|\vec{p},\epsilon\rangle = \tfrac{1}{2}\hbar\,\vec{\sigma}\cdot\vec{P}\,|\vec{p},\epsilon\rangle = \tfrac{1}{2}\hbar\,|\vec{p}\rangle\otimes(\vec{p}\cdot\vec{\sigma})|\epsilon\rangle_z$$

In the 2-dimensional subspace spanned by $|\vec{p},+\rangle$ and $|\vec{p},-\rangle$, A acts like

$$\tfrac{1}{2}\hbar\,\vec{p}\cdot\vec{\sigma} = \tfrac{1}{2}\hbar\,(p_x\sigma_x + p_y\sigma_y + p_z\sigma_z),$$

which in the $|\pm\rangle_z$ basis has the matrix representation

$$\tfrac{1}{2}\hbar\left[p_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]$$

$$= \boxed{\tfrac{1}{2}\hbar\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}}$$

(c) The essential task is to diagonalize $\frac{1}{2}\hbar\vec{p}\cdot\vec{\sigma}$ in the 2-dimensional subspace spanned by $|\vec{p},\pm\rangle$. But this is trivial. The eigenvectors are $|\pm\rangle_{\vec{u}}$, where $\vec{p} = |\vec{p}|\vec{u}$, with corresponding eigenvalues $\pm\frac{1}{2}\hbar|\vec{p}|$. Formally we can write

$$A = \frac{1}{2}\hbar\vec{P}\cdot\vec{\sigma}$$

$$= \int d^3p \ \frac{1}{2}\hbar\vec{p}\cdot\vec{\sigma}\ |\vec{p}\rangle\langle\vec{p}|$$

$$= \int d|\vec{p}|\, d\Omega_{\vec{u}} \ \frac{1}{2}\hbar|\vec{p}|\underbrace{\vec{u}\cdot\vec{\sigma}}_{|+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle+| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle-|}|\vec{p}\rangle\langle\vec{p}|$$

$$\boxed{A = \int d|\vec{p}|\,d\Omega_{\vec{u}}\left(\frac{1}{2}\hbar|\vec{p}|\ |\vec{p}\rangle\otimes|+\rangle_{\vec{u}}\ \langle\vec{p}|\otimes{}_{\vec{u}}\langle+| - \frac{1}{2}\hbar|\vec{p}|\ |\vec{p}\rangle\otimes|-\rangle_{\vec{u}}\ \langle\vec{p}|\otimes{}_{\vec{u}}\langle-|\right)}$$

Eigenvectors:	$\lvert\vec{p}\rangle\otimes\lvert+\rangle_{\vec{u}}$	$\lvert\vec{p}\rangle\otimes\lvert-\rangle_{\vec{u}}$
Eigenvalues:	$\frac{1}{2}\hbar\lvert\vec{p}\rvert$	$-\frac{1}{2}\hbar\lvert\vec{p}\rvert$

Each eigenvalue is degenerate over a sphere in momentum space.