Phys 522

Homework #2

Solution Set

$\boxed{\text{Comp. Gx Ex 3}}$ 2.1

3. Two spin 1/2 particles w/ $H = \omega_1 S_{1z} + \omega_2 S_{2z}$

a) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$

at time t S^2 is meas. What results can be found & w/ what prob

$|\psi(t)\rangle = U|\psi(0)\rangle$ Time Indep. Hamiltonian $\Rightarrow$ $U = e^{-iHt/\hbar}$

Since $[S_{1z}, S_{2z}] = 0$, $U = e^{\frac{-i\omega_1 t}{\hbar} S_{1z}} e^{\frac{-i\omega_2 t}{\hbar} S_{2z}}$

$$e^{\alpha S_z}|\pm\rangle = e^{\pm \alpha \hbar/2}|\pm\rangle$$

$$\therefore |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i\omega_1 t}{2}} e^{\frac{i\omega_2 t}{2}}|+-\rangle + e^{\frac{i\omega_1 t}{2}} e^{-\frac{i\omega_2 t}{2}}|-+\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-it/2(\omega_1-\omega_2)}|+-\rangle + e^{it/2(\omega_1-\omega_2)}|-+\rangle\right]$$

$$\boxed{\Delta\omega \equiv \omega_1 - \omega_2}$$

Rewrite $|\psi(t)\rangle$ in $|lm\rangle$ basis $\vec{L} = \vec{S}_1 + \vec{S}_2$, $S^2|lm\rangle = l(l+1)\hbar^2|l

WE have for $\vec{S}_1 = \vec{S}_2 = 1/2$, SINGLET & TRIPLET

$$|11\rangle = |++\rangle$$
$$|10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$|1-1\rangle = |--\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$\Rightarrow \quad |+-\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |00\rangle\right) \quad |-+\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |00\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{2}\left[e^{-it/2\,\Delta\omega}\left(|10\rangle + |00\rangle\right) + e^{it/2\,\Delta\omega}\left(|10\rangle - |00\rangle\right)\right]$$

$$\boxed{|\psi(t)\rangle = \cos\left(\frac{\Delta\omega t}{2}\right)|10\rangle - i\sin\left(\frac{\Delta\omega t}{2}\right)|00\rangle}$$

$$\Rightarrow \boxed{\begin{array}{l} S^2 = 2\hbar^2 \text{ w/ prob } \cos^2\left(\frac{\Delta\omega t}{2}\right) \\ S^2 = 0 \text{ w/ prob } \sin^2\left(\frac{\Delta\omega t}{2}\right) \end{array}}$$

b) IF INITIAL STATE IS ARB. WHAT BOHR FREQ CAN BE FOUND IN THE EVOLUTION OF $\langle S^2 \rangle$

WE CAN WRITE $S^2 = 2\hbar^2 \left(|11\rangle\langle 11| + |10\rangle\langle 10| + |1\,{-1}\rangle\langle 1\,{-1}| \right)$

$$\Rightarrow \langle \psi(t) | S^2 | \psi(t) \rangle = 2\hbar^2 \left(|\langle 11|\psi(t)\rangle|^2 + |\langle 10|\psi(t)\rangle|^2 + |\langle 1\,{-1}|\psi(t)\rangle|^2 \right)$$

$$|\psi(0)\rangle = \sum_{\varepsilon_1 \varepsilon_2} C_{\varepsilon_1 \varepsilon_2} |\varepsilon_1 \varepsilon_2\rangle \qquad \varepsilon_1 = \varepsilon_2 = \pm$$

$$|\psi(t)\rangle = \sum_{\varepsilon_1 \varepsilon_2} C_{\varepsilon_1 \varepsilon_2} e^{\frac{-i\varepsilon_1 \omega_1 t}{2}} e^{\frac{-i\varepsilon_2 \omega_2 t}{2}} |\varepsilon_1 \varepsilon_2\rangle$$

$$\Rightarrow \langle 11 | \psi(t)\rangle = \langle ++ | \psi(t) \rangle = C_{++} e^{-it/2(\omega_1 + \omega_2)}$$

$$\langle 1\,{-1} | \psi(t) \rangle = \langle -- | \psi(t) \rangle = C_{--} e^{it/2(\omega_1 + \omega_2)}$$

$$\langle 10 | \psi(t) \rangle = \frac{1}{\sqrt{2}} \left(\langle +-| + \langle -+| \right) |\psi(t)\rangle = \frac{1}{\sqrt{2}} C_{+-} e^{-it/2(\omega_1 - \omega_2)} + \frac{1}{\sqrt{2}} C_{-+} e^{it/2(\omega_1 - \omega}$$

$$\therefore \frac{\hbar^{-2}}{2} \langle S^2 \rangle = |C_{++}|^2 + |C_{--}|^2 + \frac{1}{2}|C_{+-}|^2 + \frac{1}{2}|C_{-+}|^2 + \frac{1}{2} C_{-+} C_{+-}^* e^{it\Delta\omega}$$
$$+ \frac{1}{2} C_{+-} C_{-+}^* e^{-it\Delta\omega}$$

$\Rightarrow$ BOHR FREQUENCY OF $\Delta\omega = \omega_1 - \omega_2$

SAME PROBLEM w/ $\langle S_x \rangle$

$$S_x = \frac{1}{2}(S_+ + S_-)$$

NOTE : $\langle S_x \rangle = \frac{1}{2}\left[\langle S_+ \rangle + \langle S_- \rangle\right]$

$$\langle S_- \rangle = \langle \psi(t) | S_- | \psi(t) \rangle = \langle \psi(t) | S_-^\dagger | \psi(t) \rangle^* = \langle \psi(t) | S_+ | \psi(t) \rangle^*$$

$$= \langle S_+ \rangle^*$$

$$\therefore \boxed{\langle S_x \rangle = \quad \mathrm{Re}\, \langle S_+ \rangle}$$

$$S_+ = \sqrt{2}\hbar \left(|11\rangle\langle 10| + |10\rangle\langle 1\,-1| \right)$$

$$\Rightarrow \langle S_+ \rangle = \sqrt{2}\hbar \left[\langle \psi(t) | 11 \rangle \langle 10 | \psi(t) \rangle + \langle \psi(t) | 10 \rangle \langle 1\,-1 | \psi(t) \rangle \right]$$

$$= \sqrt{2}\hbar \left[\langle 11 | \psi(t) \rangle^* \langle 10 | \psi(t) \rangle + \langle 10 | \psi(t) \rangle^* \langle 1\,-1 | \psi(t) \rangle \right]$$

$$\frac{\langle S_+ \rangle}{\sqrt{2}\hbar} = C_{++}^* e^{it/2(\omega_1+\omega_2)} \frac{1}{\sqrt{2}} \left[C_{+-} e^{-it/2(\omega_1-\omega_2)} + C_{-+} e^{it/2(\omega_1-\omega_2)} \right]$$

$$+ \frac{1}{\sqrt{2}} \left[C_{+-}^* e^{it/2(\omega_1-\omega_2)} + C_{-+}^* e^{-it/2(\omega_1-\omega_2)} \right] C_{--} e^{it/2(\omega_1+\omega_2)}$$

$$\frac{\langle S_+\rangle}{\sqrt{2}\hbar} = \frac{C_{++}^* C_{+-}}{\sqrt{2}} e^{it\omega_2} + \frac{C_{++}^* C_{-+}}{\sqrt{2}} e^{it\omega_1} + \frac{C_{+-}^* C_{--}}{\sqrt{2}} e^{it\omega_1} + \frac{C_{-+}^* C_{--}}{\sqrt{2}} e^{it\omega}$$

$$\langle S_x\rangle = \mathrm{Re}\,\langle S_+\rangle$$

$$\Rightarrow \langle S_x\rangle = \frac{1}{2\sqrt{2}}\Big[\left(C_{++}^* C_{-+} + C_{+-}^* C_{--}\right) e^{i\omega_1 t} + \left(C_{++} C_{-+}^* + C_{+-} C_{--}^*\right) e^{-i\omega_1 t}$$

$$+ \left(C_{++}^* C_{+-} + C_{-+}^* C_{--}\right) e^{i\omega_2 t} + \left(C_{++} C_{+-}^* + C_{-+} C_{--}^*\right) e^{-i\omega_2 t}\Big]$$

$\Rightarrow$ Bohr FREQUENCIES OF ω_1 & ω_2

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$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

GIVE A BASIS OF EIGENVECTORS COMMON TO S^2 AND S_z

BEGIN BY ADDING $\vec{S}_1$ & $\vec{S}_2$ $\vec{L} = \vec{S}_1 + \vec{S}_2$

$$|1\,1\rangle = |\underset{\text{spin 1}}{+}\underset{\text{spin 2}}{+}\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \qquad |0\,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$|1\,{-1}\rangle = |--\rangle$$

TREAT AS SPIN 1 PARTICLE

AND ADD TO SPIN ½

$$|l \pm ½, M\rangle = \frac{1}{\sqrt{2l+1}}\left[\pm\sqrt{l \pm M + ½}\;|l\,½\,M-½\,+\rangle + \sqrt{l \mp M + ½}\;|l\,½\,M+½\,-\rangle\right]$$

$$|3/2\;3/2\rangle = |1\,½\,1\,+\rangle$$

$$|3/2\;½\rangle = \sqrt{\tfrac{2}{3}}\,|1\,½\,0\,+\rangle + \sqrt{\tfrac{1}{3}}\,|1\,½\,1\,-\rangle$$

$$|3/2\;{-½}\rangle = \sqrt{\tfrac{1}{3}}\,|1\,½\,{-1}\,+\rangle + \sqrt{\tfrac{2}{3}}\,|1\,½\,0\,-\rangle$$

$$|3/2\;{-3/2}\rangle = |\underset{l}{1}\,\underset{s_3}{½}\,\underset{m_l}{-1}\,\underset{m_{s_3}}{-}\rangle$$

$$|½\;½\rangle = -\sqrt{\tfrac{1}{3}}\,|1\,½\,0\,+\rangle + \sqrt{\tfrac{2}{3}}\,|1\,½\,1\,-$$

$$|½\;{-½}\rangle = -\sqrt{\tfrac{2}{3}}\,|1\,½\,{-1}\,+\rangle + \sqrt{\tfrac{1}{3}}\,|1\,½\,0\,-\rangle$$

Now subst. our expressions for $|\ell m_\ell\rangle$

$$|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = |\overset{s_1}{+}\overset{s_2}{+}\overset{s_3}{+}\rangle$$
$$|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\left[|+-+\rangle + |-++\rangle + |++-\rangle\right]$$
$$|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\left[|--+\rangle + |+--\rangle + |-+-\rangle\right]$$
$$|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = |---\rangle$$

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{6}}\left[|+-+\rangle + |-++\rangle\right] + \sqrt{\tfrac{2}{3}}\,|++-\rangle$$
$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{6}}\left[|+--\rangle + |-+-\rangle\right] - \sqrt{\tfrac{2}{3}}\,|--+\rangle$$

Notice we can also construct from $|00\rangle$ which we treat as a spin 0 particle

$$\boxed{\begin{aligned} |\tfrac{1}{2}\,\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}\left[|+-+\rangle - |-++\rangle\right] \\ |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}\left[|+--\rangle - |-+-\rangle\right] \end{aligned}}$$

So $|\tfrac{1}{2}\pm\tfrac{1}{2}\rangle$ are not unique. A way to keep seperate would be to label as follows

$$|1,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{6}}\left[|+-+\rangle + |-++\rangle\right] + \sqrt{\tfrac{2}{3}}\,|++-\rangle$$
$$|0,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\left[|+-+\rangle - |-++\rangle\right]$$
$$|1,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{6}}\left[|+--\rangle + |-+-\rangle\right] - \sqrt{\tfrac{2}{3}}\,|--+\rangle$$

ALSO, NOTE THAT S^2 AND S_z ARE NOT A CSCO BECAUSE THEY CAN NOT DISTINGUISH between $|1, \frac{1}{2}, \pm\frac{1}{2}\rangle$ AND $|0, \frac{1}{2}, \pm\frac{1}{2}\rangle$

2.3.

(a) $|J=j+1, M=j+1\rangle = |j1j1\rangle$

$$\underbrace{C_-(j+1,j+1)}_{\sqrt{2(j+1)}}|J=j+1, M=j\rangle = J_-|J=j+1, M=j+1\rangle$$

$$= J_{1-}|j1j1\rangle + J_{2-}|j1j1\rangle$$

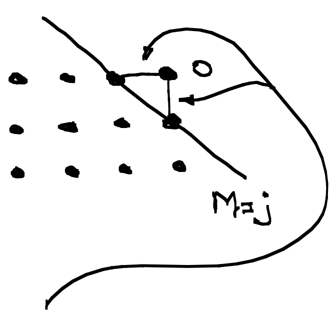
$$= \underbrace{C_-(j,j)}_{\sqrt{2j}}|j1,j-1,1\rangle + \underbrace{C_-(1,1)}_{\sqrt{2}}|j1j0\rangle$$

$$\Rightarrow \boxed{|J=j+1, M=j\rangle = \frac{1}{\sqrt{j+1}}\left(\sqrt{j}\,|j1,j-1,1\rangle + |j1j0\rangle\right)}$$

The state $|J=j, M=j\rangle$ is the orthogonal linear combination with real, positive amplitude for $|j1j0\rangle$:

$$\boxed{|J=j, M=j\rangle = \frac{1}{\sqrt{j+1}}\left(-|j1,j-1,1\rangle + \sqrt{j}\,|j1j0\rangle\right)}$$

We can also get these from the recursion relations. Let's do that for the latter case



$$0 = \underbrace{C_+(j,j)}_{0}\langle j1j1|j,j+1\rangle = \underbrace{C_-(j,j)}_{\sqrt{2j}}\langle j1,j-1,1|jj\rangle + \underbrace{C_-(1,1)}_{\sqrt{2}}\langle j1j0|jj\rangle$$

$$\Rightarrow \underbrace{\langle j1,j-1,1|jj\rangle}_{-\frac{1}{\sqrt{j+1}}} = -\frac{1}{\sqrt{j}}\underbrace{\langle j1j0|jj\rangle}_{\sqrt{\frac{j}{j+1}}} \longleftarrow$$

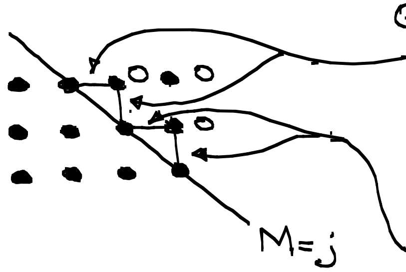
Normalization + $\langle j1j0|jj\rangle$ real and positive

$$|jj\rangle = \sum_{\substack{m_1, m_2 \\ m_1+m_2=j}} |j\,1\,m_1 m_2\rangle\langle j\,1\,m_1 m_2|jj\rangle$$

$$= |j\,1, j-1, 1\rangle\langle j\,1, j-1, 1|jj\rangle + |j\,1\,j\,0\rangle\langle j\,1\,j\,0|jj\rangle$$

$$= \frac{1}{\sqrt{j+1}}\left(-|j1, j-1, 1\rangle + \sqrt{j}\,|j1j0\rangle\right) \quad \text{as above}$$

(b) In this case it's definitely a good idea to use the recursion relations so that we don't have to find other states in addition to those required.



① $$0 = \underbrace{C_+(j,j)}_{0}\langle j2, j-1, 2|j, j+1\rangle = \overbrace{C_-(j, j-1)}^{\sqrt{2(2j-1)}}\langle j2, j-2, 2|jj\rangle + \underbrace{C_-(2,2)}_{2}\langle j2, j-1, 1|jj\rangle$$

② $$0 = \overbrace{C_+(j,j)}^{0}\langle j2j1|j, j+1\rangle = \overbrace{C_-(j,j)}^{\sqrt{2j}}\langle j2, j-1, 1|jj\rangle + \underbrace{C_-(2,1)}_{\sqrt{6}}\langle j2j0|jj\rangle$$

② $\Rightarrow$ $\langle j2, j-1, 1|jj\rangle = -\sqrt{\frac{3}{j}}\langle j2j0|jj\rangle$

① $\Rightarrow$ $\langle j2, j-2, 2|jj\rangle = -\sqrt{\frac{2}{2j-1}}\langle j2, j-1, 1|jj\rangle = \sqrt{\frac{6}{j(2j-1)}}\langle j2j0|jj\rangle$

Normalization $\Rightarrow$ $1 = \langle j2, j-2, 2|jj\rangle^2 + \langle j2, j-1, 1|jj\rangle^2 + \langle j2j0|jj\rangle^2$

$$= \langle j2j0|jj\rangle^2 \underbrace{\left(1 + \frac{3}{j} + \frac{6}{j(2j-1)}\right)}_{\frac{j(2j-1)+3(2j-1)+6}{j(2j-1)} = \frac{2j^2+5j+3}{j(2j-1)} = \frac{(j+1)(2j+3)}{j(2j-1)}}$$

$\Rightarrow$ $\langle j2j0|jj\rangle = \sqrt{\frac{j(2j-1)}{(j+1)(2j+3)}}$ $\longleftarrow$ real and positive

$$\langle j\,2, j-1, 1 | j\,j \rangle = -\sqrt{\frac{3(2j-1)}{(j+1)(2j+3)}}$$

$$\langle j\,2, j-2, 2 | j\,j \rangle = \sqrt{\frac{6}{(j+1)(2j+3)}}$$

$$\Longrightarrow \boxed{|j\,j\rangle = \frac{1}{\sqrt{(j+1)(2j+3)}}\left(\sqrt{j(2j-1)}\,|j\,2\,j\,0\rangle - \sqrt{3(2j-1)}\,|j\,2, j-1, 1\rangle + \sqrt{6}\,|j\,2, j-2, 2\rangle\right)}$$

2.4. $\mathcal{R} = \mathcal{R}_{\vec{e}_y}(\pi)$ $\quad R = R_{\vec{e}_y}(\pi) = e^{-iJ_y\pi/\hbar}$

(a) Since $\vec{J}$ is a vector operator, $R^\dagger \vec{J} R = \mathcal{R}\vec{J}$.

$$\Longrightarrow R^\dagger J_z R = \mathcal{R}\vec{J}\cdot\vec{e}_z = \vec{J}\cdot\underbrace{\mathcal{R}^{-1}\vec{e}_z}_{-\vec{e}_z} = -J_z$$

So $J_z R|jm\rangle = -R\underbrace{J_z|jm\rangle}_{m\hbar|jm\rangle} = -m\hbar R|jm\rangle$, i.e., $R|jm\rangle$ is an eigenstate of J_z with eigenvalue $-m\hbar$. This implies that

$$\boxed{R|jm\rangle = e^{i\delta_m}|j,-m\rangle.}$$

(b)

$$|jm\rangle = \sqrt{\frac{n_+!n_-!}{N!}}\sum_{\substack{\epsilon_1,\dots,\epsilon_N \\ m=(n_+-n_-)/2}} |\epsilon_1,\dots,\epsilon_N\rangle$$

$N = 2j = n_+ + n_-$ spin-1/2 particles

$m = (n_+ - n_-)/2$

n_+: # of $\epsilon_j = +1$; n_-: # of $\epsilon_j = -1$

$n_\pm = N/2 \pm m = j \pm m$

$$\binom{N}{n_+} = \frac{N!}{n_+!n_-!} = \frac{(2j)!}{(N/2+m)!(N/2-m)!}$$ is the # of sequences of length N with n_+ $+1$'s and n_- -1's

$$R = e^{-iJ_y\pi/\hbar} = \exp\left(-\frac{i}{\hbar}\pi\sum_{l=1}^N S_{ly}\right) = e^{-iS_{1y}\pi/\hbar}\cdots e^{-iS_{Ny}\pi/\hbar}$$

$$= \underbrace{e^{-i\sigma_{1y}\pi/2}}_{-i\sigma_{1y}}\cdots\underbrace{e^{-i\sigma_{Ny}\pi/2}}_{-i\sigma_{Ny}}$$

$$= -i\sigma_{1y}\cdots -i\sigma_{Ny}$$

$$\sigma_y = -i|+\rangle\langle-| + i|-\rangle\langle+| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$$

$$-i\sigma_y = -|+\rangle\langle-| + |-\rangle\langle+| \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$$

$$-i\sigma_y|\epsilon\rangle = \epsilon|-\epsilon\rangle$$

$$\Rightarrow \quad R|\epsilon_1,\dots,\epsilon_N\rangle = \epsilon_1\cdots\epsilon_N|\epsilon_1,\dots,\epsilon_N\rangle = (-1)^{n_-}|-\epsilon_1,\dots,-\epsilon_N\rangle = (-1)^{j-m}|-\epsilon_1,\dots,-\epsilon_N\rangle$$

$$\boxed{R|jm\rangle = (-1)^{j-m}|j,-m\rangle \text{ so } \delta_{jm} = \pi(j-m) \text{ and } D^{(j)}_{m'm} = \langle jm'|R|jm\rangle = (-1)^{j-m}\delta_{m',-m}}$$

(c) We will need to relate $\langle j_1j_2m_1m_2|R_{\vec{e}_y}(\pi)|JM\rangle$ to $\langle j_1j_2m_1m_2|JM\rangle$.

We use

$$R = R_{\vec{e}_y}(\pi) = e^{-iJ_y\pi/\hbar} = e^{-iJ_{1y}\pi/\hbar}\,e^{-iJ_{2y}\pi/\hbar} = R_1R_2.$$

We will need

$$R|jm\rangle = (-1)^{j-m}|j,-m\rangle \quad \text{and}$$

$$R^\dagger|jm\rangle = (-1)^{j+m}|j,-m\rangle \longleftarrow$$

trace through above with $\pi \to -\pi$ OR use

$$|j,-m\rangle = R^\dagger R|j,-m\rangle = (-1)^{j+m}R^\dagger|jm\rangle$$

$$\Rightarrow \langle jm|R = (-1)^{j+m}\langle j,-m|$$

So we have

$$\langle j_1j_2m_1m_2|R_{\vec{e}_y}(\pi)|JM\rangle = (-1)^{J-M}\langle j_1j_2m_1m_2|J,-M\rangle$$

$$\langle j_1j_2m_1m_2|R_1R_2|JM\rangle = \Big(\underbrace{\langle j_1m_1|R_1\otimes\langle j_2m_2|R_2}\Big)|JM\rangle$$

$M = -m_1 - m_2$

$$= (-1)^{j_1+m_1}(-1)^{j_2+m_2}\langle j_1,-m_1|\otimes\langle j_2,-m_2|$$

$$= (-1)^{j_1+j_2+m_1+m_2}\langle j_1j_2,-m_1,-m_2|$$

$$= (-1)^{j_1+j_2+m_1+m_2}\langle j_1j_2,-m_1,-m_2|JM\rangle$$

$$\Rightarrow \langle j_1j_2m_1m_2|J,-M\rangle = (-1)^{j_1+j_2-J+\underbrace{m_1+m_2+M}_{0}}\langle j_1j_2,-m_1,-m_2|JM\rangle$$

So $\langle j_1j_2m_1m_2|J,-M\rangle = (-1)^{j_1+j_2-J}\langle j_1j_2,-m_1,-m_2|JM\rangle$

<u>OR</u> $\langle j_1j_2,-m_1,-m_2|JM\rangle = (-1)^{j_1+j_2-J}\langle j_1j_2m_1m_2|JM\rangle$

<u>Note</u>: In (c) we often use $(-1)^{-a} = (-1)^a$, which is only true for integers, not half-integers, but if you think about it — and you should — we only use this for integers.

2.5. $D^{(j)}_{m'm} = \langle jm' | e^{-i\alpha \vec{u}\cdot\vec{J}/\hbar} | jm \rangle$

$J_\pm = J_x \pm iJ_y$

$J_x = \frac{1}{2}(J_+ + J_-)$

$J_y = -\frac{i}{2}(J_+ - J_-)$

$\vec{u}\cdot\vec{J} = u_z J_z + u_x J_x + u_y J_y$

$= u_z J_z + \frac{1}{2}(u_x - iu_y) J_+ + \frac{1}{2}(u_x + iu_y) J_-$

(a) $\alpha = d\alpha$: $\quad e^{-id\alpha \vec{u}\cdot\vec{J}/\hbar} = 1 - id\alpha \vec{u}\cdot\vec{J}/\hbar$

$$\Rightarrow D^{(j)}_{m'm} = \delta_{m'm} - id\alpha \underbrace{\langle jm' | \vec{u}\cdot\vec{J}/\hbar | jm \rangle}$$

$$u_z m\, \delta_{m'm} + \frac{1}{2}(u_x - iu_y) C_+(j,m)\, \delta_{m',m+1} + \frac{1}{2}(u_x + iu_y) C_-(j,m)\, \delta_{m',m-1}$$

$$\boxed{D^{(j)}_{m'm} = \delta_{m'm}(1 - id\alpha\, u_z m) - \frac{i}{2} d\alpha (u_x - iu_y) C_+(j,m)\, \delta_{m',m+1} - \frac{i}{2} d\alpha (u_x + iu_y) C_-(j,m)\, \delta_{m',m-1}}$$

(b) We first specialize to $j=1$:

$$D^{(1)}_{m'm} = \delta_{m'm}(1 - id\alpha\, u_z m) - \frac{i}{2} d\alpha (u_x - iu_y) C_+(1,m)\, \delta_{m',m+1} - \frac{i}{2} d\alpha (u_x + iu_y) C_-(1,m)\, \delta_{m',m-1}$$

$$C_+(1,m) = C_-(1,-m)$$

$$C_+(1,1) = C_-(1,-1) = 0$$

$$C_+(1,0) = C_-(1,0) = \sqrt{2}$$

$$C_+(1,-1) = C_-(1,1) = \sqrt{2}$$

Translating to matrix form

$$\| D^{(1)}_{m'm} \| = \begin{array}{c|ccc} & m=1 & 0 & -1 \\ \hline m'=1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array} - id\alpha \underbrace{\begin{pmatrix} u_z & (u_x - iu_y)/\sqrt{2} & 0 \\ (u_x + iu_y)/\sqrt{2} & 0 & (u_x - iu_y)/\sqrt{2} \\ 0 & (u_x + iu_y)/\sqrt{2} & -u_z \end{pmatrix}}_{\equiv A}$$

Now we're going to change basis:

$$D_{ab} = \langle a | R | b \rangle = \sum_{m',m} \underbrace{\langle a | m' \rangle}_{\langle m' | a \rangle^* = V^*_{m'a} = V^\dagger_{am'}} \langle m' | R | m \rangle \underbrace{\langle m | b \rangle}_{\equiv V_{mb}} = \sum_{m,m'} V^\dagger_{am'} D^{(1)}_{m'm} V_{mb}$$

$$D_{ab} = \left(V^\dagger D^{(1)} V\right)_{ab} \longleftarrow$$ All basis changes are described by a unitary transformation.

$$\|V_{mb}\| = \begin{array}{c|c} & b \\ m & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix} \end{array}$$

$$V^\dagger A V = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} u_z & (u_x - iu_y)/\sqrt{2} & 0 \\ (u_x+iu_y)/\sqrt{2} & 0 & (u_x-iu_y)/\sqrt{2} \\ 0 & (u_x+iu_y)/\sqrt{2} & -u_z \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix}}$$

$$\begin{pmatrix} u_z/\sqrt{2} & -iu_z/\sqrt{2} & -(u_x-iu_y)/\sqrt{2} \\ iu_y & -iu_x & 0 \\ u_z/\sqrt{2} & iu_z/\sqrt{2} & -(u_x+iu_y)/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -iu_z & iu_y \\ iu_z & 0 & -iu_x \\ -iu_y & iu_x & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \|D_{ab}\| = 1 - i\,d\alpha\, V^\dagger A V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + d\alpha \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \\ \Leftrightarrow \quad D_{ab} = \delta_{ab} - d\alpha\, \epsilon_{abc} u_c = \vec{e}_a \cdot \underbrace{\left(\vec{e}_b + d\alpha\, \vec{u} \times \vec{e}_b\right)}_{\mathcal{R}\vec{e}_b \,\leftarrow\, \text{infinitesimal rotation by } d\alpha \text{ about } \vec{u}} = \vec{e}_a \cdot \mathcal{R}\vec{e}_b \end{array}}$$

2.6. Schwinger representation

$N = 2J$ spin-$\frac{1}{2}$ particles

$$\vec{J} = \sum_{\ell=1}^{N} \vec{S}_\ell = \frac{1}{2}\hbar \sum_{\ell=1}^{N} \vec{\sigma}_\ell \qquad J_\pm = J_x \pm i J_y = \frac{1}{2}\hbar \sum_{\ell=1} \sigma_{\ell x} \pm i\sigma_{\ell y} = \hbar \sum_\ell \sigma_{\ell\pm}$$

(a) The state $|JJ\rangle$ corresponds to all spins up, i.e.,

$$|JJ\rangle = |+,+,\dots,+\rangle \longleftarrow$$ There is an arbitrary phase here, which we choose to be 1.

It is clear that $|JM\rangle$ is a superposition of states with n_+ spins up and n_- spins down, with $M=(n_+-n_-)/2$. We can obtain the amplitudes in the superposition by successive lowering of $|JJ\rangle$. But it is clear that the lowering treats all the spins symmetrically and only gives real, positive amplitudes, so all the amplitudes are equal and positive. There being $\binom{N}{n_+}$ states in the superposition, normalization gives

$$|JM\rangle = \sqrt{\frac{n_+!\,n_-!}{N!}} \sum_{\substack{\epsilon_1,\dots,\epsilon_N \\ M=(n_+-n_-)/2}} |\epsilon_1 \dots \epsilon_N\rangle \equiv |n_+,n_-\rangle$$

(b) $$D_{\epsilon'\epsilon} = \langle \epsilon' | \underbrace{e^{-i\vec{u}\cdot\vec{\sigma}\alpha/2}}_{= 1\cos(\alpha/2) - i\vec{u}\cdot\vec{\sigma}\sin(\alpha/2)} |\epsilon\rangle$$

$$= 1\cos(\alpha/2) - i(u_x\sigma_x + u_y\sigma_y + u_z\sigma_z)\sin(\alpha/2)$$

$$\boxed{\|D_{\epsilon'\epsilon}\| = \begin{array}{c|cc} & \epsilon=+1 & \epsilon=-1 \\ \epsilon'=+1 & c - iu_z s & (-iu_x - u_y)s \\ \epsilon'=-1 & (-iu_x + u_y)s & c + iu_z s \end{array}} \qquad \boxed{\begin{array}{l} c = \cos(\alpha/2) \\ s = \sin(\alpha/2) \end{array}}$$

(c) $\mathcal{R}_{\vec{e}_y}(\pi)$:

same rotation on all N spins

$$\mathcal{R}_{\vec{e}_y}(\pi) = e^{-iJ_y\pi/\hbar} = \overbrace{\underbrace{e^{-i\sigma_{1y}\pi/2}}_{-i\sigma_{1y}} \otimes \dots \otimes \underbrace{e^{-i\sigma_{Ny}\pi/2}}_{-i\sigma_{Ny}}}$$

$$-i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -|+\rangle\langle -| + |-\rangle\langle +|$$

$$R_{\vec{e}_y}(\pi)\,|\epsilon_1 \ldots \epsilon_N\rangle = (-1)^{n_-}\,|-\epsilon_1, \ldots, -\epsilon_N\rangle$$

$\uparrow$ flips each spin and puts in a minus sign for each spin down

$$\Rightarrow \quad R_{\vec{e}_y}(\pi)\,|JM\rangle = (-1)^{J-M}\,|J,-M\rangle$$

$$n_- = N/2 - M = J - M$$

$$\Rightarrow \quad \boxed{D^{(J)}_{M'M} = \langle JM'|\,R_{\vec{e}_y}(\pi)\,|JM\rangle = (-1)^{J-M}\,\delta_{M',-M}}$$

(d) $D^{(1)}_{M'M} = \langle 1M'|\,\underbrace{R_{\vec{u}}(\alpha)}_{R_1 \otimes R_2}\,|1M\rangle$

$R_1 \otimes R_2 \leftarrow$ identical rotations on two spins

Use
$$|11\rangle = |++\rangle$$
$$|10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$|1,-1\rangle = \frac{1}{\sqrt{2}}|--\rangle$$
spin triplet

$$D^{(1)}_{11} = \langle 11|R|11\rangle = \langle ++|R_1 \otimes R_2|++\rangle = \langle +|R_1|+\rangle\langle +|R_2|+\rangle = D^2_{++}$$

$$D^{(1)}_{10} = \langle 11|R|10\rangle = \frac{1}{\sqrt{2}}\left(\langle ++|R_1 \otimes R_2|+-\rangle + \langle ++|R_1 \otimes R_2|-+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\langle +|R_1|+\rangle\langle +|R_2|-\rangle + \langle +|R_1|-\rangle\langle +|R_2|+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(D_{++}D_{+-} + D_{+-}D_{++}\right)$$

$$= \sqrt{2}\,D_{++}\,D_{+-}$$

Now it should be obvious how to proceed, so we'll just write the answer:

$$
\| D^{(1)}_{M'M} \| = \begin{array}{c} \\ M'=+1 \\ M'=0 \\ M'=-1 \end{array}
\begin{array}{c}
\begin{array}{ccc} M=+1 & M=0 & M=-1 \end{array} \\
\begin{pmatrix}
D_{++}^2 & \sqrt{2}D_{++}D_{+-} & D_{+-}^2 \\
\sqrt{2}D_{++}D_{-+} & \frac{1}{2}(D_{++}D_{--}+D_{+-}D_{-+}) & \sqrt{2}D_{--}D_{+-} \\
D_{-+}^2 & \sqrt{2}D_{--}D_{-+} & D_{--}^2
\end{pmatrix}
\end{array}
$$

<u>Digression:</u> Since the rotation matrix for the N spins factors into a product of rotations for each spin, it is clear that the overall rotation matrix factors into a <u>tensor</u> product, i.e.,

$$\langle \epsilon_1', \dots, \epsilon_N' | R | \epsilon_1, \dots, \epsilon_N \rangle = \langle \epsilon_1' | R_1 | \epsilon_1 \rangle \cdots \langle \epsilon_N' | R_N | \epsilon_N \rangle = D_{\epsilon_1'\epsilon_1} \cdots D_{\epsilon_N'\epsilon_N}.$$

This representation is reducible, however, into all the angular-momentum subspaces mentioned in the introduction to the problem. When we project into the subspace of maximal angular momentum J, as we do here, we are finding the irrep in that subspace.

(e) $J_z = \frac{1}{2}\hbar(a_+^\dagger a_+ - a_-^\dagger a_-)$

$J_x = \frac{1}{2}\hbar(a_+^\dagger a_- + a_-^\dagger a_+)$

$J_y = -\frac{i}{2}\hbar(a_+^\dagger a_- - a_-^\dagger a_+)$

$$\vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2 = \frac{1}{4}\hbar^2\left[(a_+^\dagger a_- + a_-^\dagger a_+)^2 - (a_+^\dagger a_- - a_-^\dagger a_+)^2 + (a_+^\dagger a_+ - a_-^\dagger a_-)^2\right]$$

$$= \frac{1}{4}\hbar^2\Big(2\underbrace{a_+^\dagger a_- a_-^\dagger a_+}_{a_+^\dagger a_+(a_-^\dagger a_- + 1) \,=\, a_+^\dagger a_+ a_-^\dagger a_- + a_+^\dagger a_+} + 2\underbrace{a_-^\dagger a_+ a_+^\dagger a_-}_{a_-^\dagger a_-(a_+^\dagger a_+ + 1) \,=\, a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_-} + a_+^\dagger a_+ a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_- - 2a_+^\dagger a_+ a_-^\dagger a_-\Big)$$

$$= \frac{1}{4}\hbar^2\left[(a_+^\dagger a_+)^2 + (a_-^\dagger a_-)^2 + 2a_+^\dagger a_+ a_-^\dagger a_- + 2a_+^\dagger a_+ + 2a_-^\dagger a_-\right]$$

$$= \frac{1}{4}\hbar^2\left(a_+^\dagger a_+ + a_-^\dagger a_- + 2\right)\left(a_+^\dagger a_+ + a_-^\dagger a_-\right)]$$

$$\boxed{\vec{J}\cdot\vec{J} = \hbar^2(N/2+1)N/2}$$

$$[J_x, J_y] = -\frac{i}{4}\hbar^2\left[a_+^\dagger a_- + a_-^\dagger a_+,\ a_+^\dagger a_- - a_-^\dagger a_+\right]$$

$$= -\frac{i}{2}\hbar^2 \left[a_-^\dagger a_+, a_+^\dagger a_-\right]$$

$$= a_-^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_- a_-^\dagger a_+$$

$$= a_-^\dagger a_-(a_+^\dagger a_+ + 1) - a_+^\dagger a_+(a_-^\dagger a_- + 1)$$

$$= a_-^\dagger a_- - a_+^\dagger a_+$$

$$= i\hbar J_z$$

$$[J_x, J_z] = \frac{1}{4}\hbar^2\left[a_+^\dagger a_- + a_-^\dagger a_+,\ a_+^\dagger a_+ - a_-^\dagger a_-\right]$$

$$= \frac{1}{4}\hbar^2\left(a_-\left[a_+^\dagger, a_+^\dagger a_+\right] + a_-^\dagger\left[a_+, a_+^\dagger a_+\right] - a_+^\dagger\overbrace{\left[a_-, a_-^\dagger a_-\right]}^{a_-} - a_+\overbrace{\left[a_-^\dagger, a_-^\dagger a_-\right]}^{-a_-^\dagger}\right)$$

$$\left[a_+^\dagger, a_+^\dagger a_+\right] = a_+^\dagger a_+^\dagger a_+ - a_+^\dagger a_+ a_+^\dagger$$

$$= a_+^\dagger a_+^\dagger a_+ - a_+^\dagger(a_+^\dagger a_+ + 1)$$

$$= -a_+^\dagger$$

$$\left[a_+, a_+^\dagger a_+\right] = \left[a_+^\dagger a_+, a_+^\dagger\right]^\dagger$$

$$= a_+$$

$$= \frac{1}{2}\hbar^2\left(-a_+^\dagger a_- + a_-^\dagger a_+\right)$$

$$= -i\hbar J_y$$

Same commutators give $[J_y, J_z] = i\hbar J_x$

So $\boxed{[J_j, J_k] = i\hbar\,\epsilon_{jkl} J_l}$

(f) We just need to check that $|n_+, n_-\rangle$ is an eigenstate of $\vec{J}^2$ and J_z:

$$\vec{J}^2|n_+, n_-\rangle = \hbar^2(N/2+1)\overbrace{(N/2)}^{J}|n_+, n_-\rangle$$

$$= \hbar^2\left(\frac{n_+ + n_-}{2} + 1\right)\frac{n_+ + n_-}{2}$$

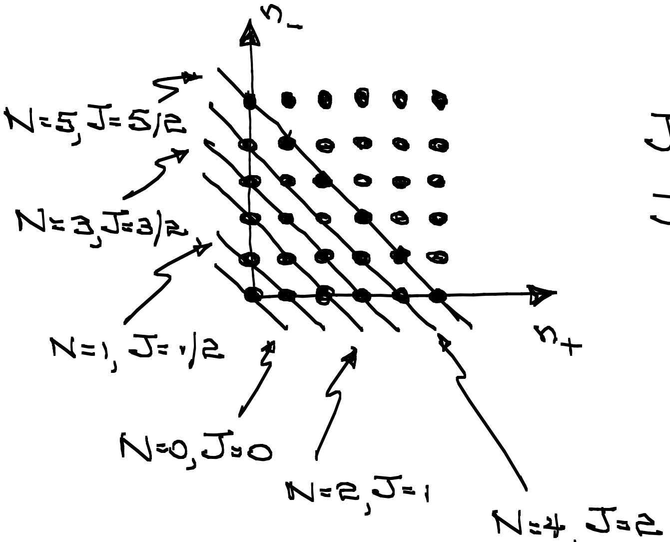
$$J_z|n_+, n_-\rangle = \frac{1}{2}\hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)|n_+, n_-\rangle$$

$$= \hbar\underbrace{\frac{n_+ - n_-}{2}}_{M}|n_+, n_-\rangle$$

$$\Longrightarrow \boxed{|n_+, n_-\rangle = |J = (n_+ + n_-)/2,\ M = (n_+ - n_-)/2\rangle}$$

The eigenvalue equations allow a phase in this relation. But, suppose we choose $|n_+ = N, n_- = 0\rangle = |J = N/2, M = N/2\rangle$ and then lower using $J_- = J_x - iJ_y = \hbar a_-^\dagger a_+$. Each lowering decreases n_+ by 1 and increases n_- by 1, i.e., decreases M by 1, without putting in any phase. So things are as stated.

Digression: The states $|n_+, n_-\rangle = |JM\rangle$ lie on a grid, with the angular-momentum subspaces corresponding to the antidiagonals.



$J_+ = \hbar a_+^\dagger a_-$ ⟵ moves down one unit and to the right one unit

$J_- = \hbar a_-^\dagger a_+$ ⟵ moves to the left one unit and up one unit

(g) $R^\dagger a_\pm^\dagger R$: We could evaluate this directly using the nested-commutator formula – and that's not hard – but just thinking about the nested commutators already shows that the answer is a linear combination of $a_+^\dagger$ and $a_-^\dagger$:

$$R^\dagger a_\epsilon^\dagger R = \sum_{\epsilon'} M_{\epsilon\epsilon'} a_{\epsilon'}^\dagger .$$

We can find the matrix M easily by applying this equation to the vacuum state. This amounts to working in the one-photon ($J = 1/2$) sector of Hilbert space, where things are very simple:

$$\sum_{\epsilon'} M_{\epsilon\epsilon'} |\epsilon'\rangle = \sum_{\epsilon'} M_{\epsilon\epsilon'} a^\dagger_{\epsilon'} |0,0\rangle$$

$$= R^\dagger a^\dagger_\epsilon \underbrace{R|0,0\rangle}_{|0,0\rangle}$$

$$= R^\dagger \underbrace{a^\dagger_\epsilon |0,0\rangle}_{|\epsilon\rangle}$$

$$= \sum_{\epsilon'} |\epsilon'\rangle \underbrace{\langle\epsilon'| R^\dagger |\epsilon\rangle}_{= \langle\epsilon|R|\epsilon'\rangle^* = D^*_{\epsilon\epsilon'}}$$

$$= \sum_{\epsilon'} D^*_{\epsilon\epsilon'} |\epsilon'\rangle$$

$$\Longrightarrow \boxed{M_{\epsilon\epsilon'} = D^*_{\epsilon\epsilon'}}$$

$$\boxed{\begin{gathered} R^\dagger a^\dagger_\epsilon R = \sum_{\epsilon'} D^*_{\epsilon\epsilon'} a^\dagger_{\epsilon'} \\ R^\dagger \begin{pmatrix} a^\dagger_+ \\ a^\dagger_- \end{pmatrix} R = D^* \begin{pmatrix} a^\dagger_+ \\ a^\dagger_- \end{pmatrix} \end{gathered}}$$

Notice that

$$R^\dagger a_\epsilon R = \sum_{\epsilon'} D_{\epsilon\epsilon'} a_{\epsilon'}$$

$$R^\dagger \begin{pmatrix} a_+ \\ a_- \end{pmatrix} R = D \begin{pmatrix} a^\dagger_+ \\ a^\dagger_- \end{pmatrix}$$

(h) A quarter-wave plate transforms $a^\dagger_{+x}$ and $a^\dagger_{-x}$ according to

$$Q a^\dagger_{+x} Q^\dagger = e^{i\alpha} a^\dagger_{+x} \quad , \quad Q a^\dagger_{-x} Q^\dagger = e^{i\alpha} e^{i\pi/2} a^\dagger_{-x}$$

overall phase across plate, which is irrelevant, but can be chosen for convenience

relative $\pi/2$ phase shift

These mean that

$$Q a^\dagger_{+x} Q^\dagger |0,0\rangle = Q a^\dagger_{+x} |0,0\rangle = Q |1,0\rangle_x = e^{i\alpha} |1,0\rangle_x$$

$$Q a^\dagger_{-x} Q^\dagger |0,0\rangle = Q a^\dagger_{-x} |0,0\rangle = Q |0,1\rangle_x = e^{i\alpha} i |1,0\rangle_x$$

It might be clear that this is a 90° rotation about the x axis, but let's write it in the $+z$ basis to drive the point home:

$$Q\, a^\dagger_{\pm z} Q^\dagger = \frac{1}{\sqrt{2}} Q\left(a^\dagger_{+x} \pm i a^\dagger_{-x}\right) Q^\dagger$$

$$= \frac{1}{\sqrt{2}} e^{i\alpha}\left(a^\dagger_{+x} \pm i(i)\, a^\dagger_{-x}\right)$$

$$= \frac{1}{\sqrt{2}} e^{i\alpha}\left(a^\dagger_{+x} \mp a^\dagger_{-x}\right)$$

$$= \mp e^{i\alpha} a^\dagger_{\mp y}$$

This is clearly a 90° rotation about the x axis, which takes $\pm$ helicity to $\mp y$ linear polarization and takes $\pm y$ polarization to to $\pm$ helicity.

$$Q\begin{pmatrix} a^\dagger_{+z} \\ a^\dagger_{-z}\end{pmatrix} Q^\dagger = e^{i\alpha}\begin{pmatrix} -a^\dagger_{-y} \\ a^\dagger_{+y}\end{pmatrix} = e^{i\alpha}\frac{1}{\sqrt{2}}\begin{pmatrix} i e^{-i\pi/4} & -i e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4}\end{pmatrix}\begin{pmatrix} a^\dagger_{+z} \\ a^\dagger_{-z}\end{pmatrix}$$

$$= \underset{\substack{\uparrow \\ \text{choose} \\ \alpha = -\pi/4}}{e^{i(\alpha+\pi/4)}} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix}}_{D^T}\begin{pmatrix} a^\dagger_{+z} \\ a^\dagger_{-z}\end{pmatrix}$$

$$\Rightarrow D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix} = \mathbb{1}\cos(\pi/4) - i\sigma_x \sin(\pi/4) = e^{-i\sigma_x \pi/4}$$

rotation by $+\pi/2$ about $\vec{e}_x$

$$\Rightarrow \boxed{Q = R_{\vec{e}_x}(\pi/2) = e^{-iJ_x\pi/2\hbar} = e^{-i(a_+^\dagger a_- - a_-^\dagger a_+)\pi/4}}$$

A half-wave plate is two quarter-wave plates in succession, so is described by

$$\boxed{Q^2 = R_{\vec{e}_x}(\pi) = e^{-iJ_x\pi/\hbar} = e^{-i(a_+^\dagger a_- - a_-^\dagger a_+)\pi/2}}$$