Phys 522

Homework #4

Solution Set

$$a_0 = \frac{\hbar^2}{m_e e^2}$$

$$\alpha \equiv \frac{e^2}{\hbar c}$$

(a)

4.1. $$E_n = -\frac{e^2}{2a_0 n^2} = -\frac{1}{2} m_e c^2 \alpha^2 \frac{1}{n^2}$$

$$\Longrightarrow \langle R \rangle \sim a_0 n^2$$

$$\frac{\langle \vec{P}^2 \rangle}{2m_e} \sim \frac{e^2}{a_0 n^2} = m_e c^2 \alpha^2 \frac{1}{n^2}$$

① $$W_{mv} = -\frac{\vec{P}^4}{8 m_e^3 c^2}$$

$$\langle W_{mv} \rangle \sim -\underbrace{\frac{\langle \vec{P}^4 \rangle}{4 m_e^2}}_{\sim m_e^2 c^4 \alpha^4 \frac{1}{n^4}} \frac{1}{m_e c^2} \sim m_e c^2 \alpha^4 \frac{1}{n^4}$$

$$\boxed{\langle W_{mv} \rangle \sim m_e c^2 \alpha^4 \frac{1}{n^4}}$$

② $$W_{SO} = \frac{e^2}{2 m_e^2 c^2} \frac{1}{R^3} \vec{L} \cdot \vec{S}$$

$$\langle W_{SO} \rangle \sim \frac{e^2}{m_e^2 c^2} \frac{1}{a_0^3} \frac{1}{n^6} \ell \hbar^2$$

$= m_e c^2 \alpha^4$

$$\boxed{\langle W_{SO} \rangle \sim m_e c^2 \alpha^4 \frac{\ell}{n^6} \lesssim m_e c^2 \alpha^4 \frac{1}{n^5}}$$

③ $$W_D = \frac{\pi e^2 \hbar^2}{2 m_e^2 c^2} \delta(\vec{R})$$

$$\langle W_D \rangle \sim \frac{e^2 \hbar^2}{m_e^2 c^2} \frac{1}{a_0^3 n^6} = m_e c^2 \alpha^4 \frac{1}{n^6}$$

s- states only

$$\boxed{\langle W_D \rangle \sim m_0 c^2 \alpha^4 \frac{1}{n^6}}$$

Relativistic mass increase dominates for large n.

(b) For a Rydberg atom with $n = 50$, the typical size of the fine structure in Hz is

$$\frac{\langle W_{mv} \rangle}{h} \sim \underbrace{\frac{m_0 c^2 \alpha^4}{h}}_{350.37\ \text{GHz}} \frac{1}{n^4} \sim 56\ \text{kHz}$$

4.2 $\vec{B} = B_0 \vec{e}_z$

$$\vec{M} = g_\ell \mu_B \vec{L}/\hbar + g_s \mu_B \vec{S}/\hbar = \frac{q}{2m_e}(\vec{L} + 2\vec{S})$$

$$g_\ell = 1, \quad \mu_B = \frac{q\hbar}{2m_e}, \qquad g_\ell \mu_B/\hbar = q/2m_e$$

$$g_s = 2 \qquad\qquad g_s \mu_B/\hbar = q/m_e$$

$$W_Z = -\vec{M}\cdot\vec{B} = \underbrace{-\frac{qB_0}{2m_e}}_{\equiv\, \omega_0}(L_z + 2S_z) = \omega_0(L_z + 2S_z)$$

Typical Zeeman shift $\hbar\omega_0$

Notice that W_Z is diagonal in the $|n\ell m_\ell m_s\rangle$ basis; without the fine structure one can calculate exact Zeeman splittings.

(a) The relative size of Zeeman and hyperfine effects is determined by

$$\xi \equiv \frac{|\hbar\omega_0|}{m_e c^2 \alpha^4} = \frac{\omega_0/2\pi}{m_e c^2 \alpha^4/h} = \frac{-qB_0/4\pi m_e}{m_e c^2 \alpha^4/h}$$

$$m_e c^2 \alpha^4/h = 350.37 \text{ GHz} \qquad -q = +1.6022 \times 10^{-19}\ \text{C}, \quad m_e = 9.1095 \times 10^{-31}\ \text{kg}$$

$$\xi = 1 \quad \text{when} \quad B_0 = \frac{m_e c^2 \alpha^4/h}{-q/4\pi m_e} \simeq \frac{(3.5\times10^{11})(4\pi \times 9.1\times10^{-31})}{1.6\times10^{-19}}$$

$$\boxed{B_0 \simeq 25\ \text{T}}$$

If you prefer CGS Gaussian, this becomes

$$\omega_0 = -qB_0/2m_e c$$

$$\xi = \frac{-qB_0/4\pi m_e c}{m_e c^2 \alpha^4/h}$$

$$\xi = 1 \quad \text{when} \quad B_0 = \frac{m_e c^2 \alpha^4/h}{-q/4\pi m_e c} \simeq \frac{(3.5\times10^{11})(4\pi)\times(9\times10^{-28})(3\times10^{10})}{4\times10^{-10}}$$

$$\boxed{B_0 \simeq 3\times10^5 \text{ gauss}}$$

(b) $n=1$ is easy

$$|1\,0\,0\,m_s\rangle = |1\,0\,\tfrac{1}{2}, m_J = m_s\rangle$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ l & m_L & J \end{array}$$

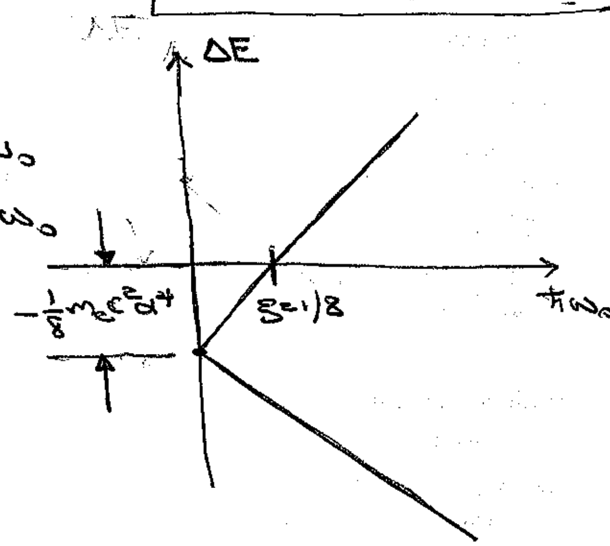
All the W's are diagonal in this basis

$$\langle 100 m_s' | W_{\substack{mv \\ D \\ SO}} | 100 m_s\rangle = \delta_{m_s' m_s} \langle W_{\substack{mv \\ D \\ SO}} \rangle_{1S}$$

$$\left.\begin{aligned} \langle W_{mv}\rangle_{1S} &= -\frac{5}{8} m_e c^2 \alpha^4 \\ \langle W_D\rangle_{1S} &= \frac{1}{2} m_e c^2 \alpha^4 \\ \langle W_{SO}\rangle_{1S} &= 0 \end{aligned}\right\} \quad \langle W_{FS}\rangle = -\frac{1}{8} m_e c^2 \alpha^4$$

$$\langle 100 m_s' | W_Z | 100 m_s\rangle = \delta_{m_s m_s'} \hbar\omega_0 \begin{cases} +1, & m_s = 1/2 \\ -1, & m_s = -1/2 \end{cases}$$

Eigenstates	Eigenvalues
$\lvert 100\tfrac{1}{2}\rangle = \lvert 10\tfrac{1}{2}\tfrac{1}{2}\rangle$	$E_1 - \frac{1}{8} m_e c^2 \alpha^4 + \hbar\omega_0$
$\lvert 100, -\tfrac{1}{2}\rangle = \lvert 10\tfrac{1}{2}, -\tfrac{1}{2}\rangle$	$E_1 - \underbrace{\frac{1}{8} m_e c^2 \alpha^4}_{\langle W_{FS}\rangle_{1S,1/2}} - \hbar\omega_0$



(b) ① W_{mv} and W_D are diagonal in the $|2 l m_L m_S\rangle$ basis; indeed, they are multiples of the unit operator in the 2s and 2p subspaces

$$\langle 2 l' m_L' m_S' | W_{\substack{mv\\D}} | n l m_L m_S \rangle = \delta_{l'l}\,\delta_{m_L' m_L}\,\delta_{m_S' m_S} \langle W_{\substack{mv\\D}} \rangle_{2l}$$

$$\langle W_{mv} \rangle_{2s} = -\frac{13}{128} m_e c^2 \alpha^4$$

$$\langle W_{mv} \rangle_{2p} = -\frac{7}{384} m_e c^2 \alpha^4$$

$$\langle W_D \rangle_{2s} = \frac{1}{16} m_e c^2 \alpha^4$$

$$\langle W_D \rangle_{2p} = 0$$

② W_{SO} is diagonal in the $|2 l J m_J\rangle$ basis; indeed, it is a multiple of the unit operator in the $2s_{1/2}$, $2p_{1/2}$, and $2p_{3/2}$ subspaces.

$$\langle 2 l' J' m_J' | W_{SO} | 2 l J m_J \rangle = \delta_{l'l}\,\delta_{J'J}\,\delta_{m_J' m_J} \langle W_{SO} \rangle_{nlJ}$$

$$\langle W_{SO} \rangle_{2s_{1/2}} = 0$$

$$\langle W_{SO} \rangle_{2p_{1/2}} = -\frac{1}{48} m_e c^2 \alpha^4$$

$$\langle W_{SO} \rangle_{2p_{3/2}} = \frac{1}{96} m_e c^2 \alpha^4$$

(3) W_Z is diagonal in the $|2 l m_L m_S\rangle$ basis:

$$\langle 2 l' m_L' m_S' | W_Z | 2 l m_L m_S\rangle = \hbar\omega_0 (m_L + 2m_S)\,\delta_{l'l}\,\delta_{m_L' m_L}\,\delta_{m_S' m_S}$$

Our job is to diagonalize W in the $n=2$ subspace.

We already know 4 eigenstates:

$$\boxed{\begin{array}{ll}
|200\tfrac{1}{2}\rangle = |20\tfrac{1}{2}\tfrac{1}{2}\rangle & E_2 - \underbrace{\tfrac{5}{128} m_e c^2 \alpha^4}_{\langle W_{FS}\rangle_{2S1/2}} + \hbar\omega_0 \\
|200,-\tfrac{1}{2}\rangle = |20\tfrac{1}{2},-\tfrac{1}{2}\rangle & E_2 - \tfrac{5}{128} m_e c^2 \alpha^4 - \hbar\omega_0 \\
|211\tfrac{1}{2}\rangle = |21\tfrac{3}{2}\tfrac{3}{2}\rangle & E_2 - \tfrac{1}{128} m_e c^2 \alpha^4 + 2\hbar\omega_0 \\
|21,-1,-\tfrac{1}{2}\rangle = |21\tfrac{3}{2},-\tfrac{3}{2}\rangle & E_2 - \underbrace{\tfrac{1}{128} m_e c^2 \alpha^4}_{\langle W_{FS}\rangle_{2P3/2}} - 2\hbar\omega_0
\end{array}}$$

We need to diagonalize in the subspace spanned by

$j_1=1, m=1/2$ $\quad |21\tfrac{3}{2}\tfrac{1}{2}\rangle = |210\tfrac{1}{2}\rangle\underbrace{\langle 210\tfrac{1}{2}|\tfrac{3}{2}\tfrac{1}{2}\rangle}_{\sqrt{2/3}} + |21 1,-\tfrac{1}{2}\rangle\underbrace{\langle 211,-\tfrac{1}{2}|\tfrac{3}{2}\tfrac{1}{2}\rangle}_{\sqrt{1/3}}$

$j_1=1, m=-1/2$ $\quad |21\tfrac{3}{2},-\tfrac{1}{2}\rangle = |21,-1,\tfrac{1}{2}\rangle\underbrace{\langle 21,-1,\tfrac{1}{2}|\tfrac{3}{2},-\tfrac{1}{2}\rangle}_{\sqrt{1/3}} + |210,-\tfrac{1}{2}\rangle\underbrace{\langle 210,-\tfrac{1}{2}|\tfrac{3}{2},-\tfrac{1}{2}\rangle}_{\sqrt{2/3}}$

$j_1=1, m=1/2$ $\quad |21\tfrac{1}{2},\tfrac{1}{2}\rangle = |210\tfrac{1}{2}\rangle\underbrace{\langle 210\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle}_{-\sqrt{1/3}} + |211,-\tfrac{1}{2}\rangle\underbrace{\langle 211,-\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2}\rangle}_{\sqrt{2/3}}$

$j_1=1, m=-1/2$ $\quad |21\tfrac{1}{2},-\tfrac{1}{2}\rangle = |21,-1,\tfrac{1}{2}\rangle\underbrace{\langle 21,-1,\tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{-\sqrt{2/3}} + |210,-\tfrac{1}{2}\rangle\underbrace{\langle 210,-\tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{\sqrt{1/3}}$

We don't have to worry about W_{mv} and W_D because they're multiples of the unit operator in this subspace.

Work in the total angular momentum basis

$$\langle 211,-\tfrac{1}{2}|W_z|211,-\tfrac{1}{2}\rangle = 0 = \langle 21,-1,\tfrac{1}{2}|W_z|21,-1,\tfrac{1}{2}\rangle$$

$$\langle 210\tfrac{1}{2}|W_z|210\tfrac{1}{2}\rangle = \hbar\omega_0$$

$$\langle 210,-\tfrac{1}{2}|W_z|210,-\tfrac{1}{2}\rangle = -\hbar\omega_0$$

$\Longrightarrow$ Nonzero matrix elements in the total angular momentum basis are

$$\langle 21\tfrac{3}{2}\tfrac{1}{2}|W_z|21\tfrac{3}{2}\tfrac{1}{2}\rangle = \tfrac{2}{3}\hbar\omega_0 = -\langle 21\tfrac{3}{2},-\tfrac{1}{2}|W_z|21\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$\langle 21\tfrac{3}{2}\tfrac{1}{2}|W_z|21\tfrac{1}{2}\tfrac{1}{2}\rangle = -\tfrac{\sqrt{2}}{3}\hbar\omega_0 = +\langle 21\tfrac{3}{2}-\tfrac{1}{2}|W_z|21\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\langle 21\tfrac{1}{2}\tfrac{1}{2}|W_z|21\tfrac{3}{2}\tfrac{1}{2}\rangle = '' = \langle 21\tfrac{1}{2},-\tfrac{1}{2}|W_z|21\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$\langle 21\tfrac{1}{2}\tfrac{1}{2}|W_z|21\tfrac{1}{2}\tfrac{1}{2}\rangle = \tfrac{1}{3}\hbar\omega_0 = -\langle 21\tfrac{1}{2},-\tfrac{1}{2}|W_z|21\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$W_z = \begin{array}{c} \\ |\tfrac{3}{2}\tfrac{1}{2}\rangle \\ |\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle \\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{array} \frac{1}{3}\hbar\omega_0 \begin{array}{c} \begin{array}{cccc} |\tfrac{3}{2}\tfrac{1}{2}\rangle & |\tfrac{1}{2}\tfrac{1}{2}\rangle & |\tfrac{3}{2},-\tfrac{1}{2}\rangle & |\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{array} \\ \begin{pmatrix} 2 & -\sqrt{2} & 0 & 0 \\ -\sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & -2 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & -1 \end{pmatrix} \end{array}$$

$$W_{SO} = \frac{1}{96} m_e c^2 \alpha^4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The 4x4 problem separates into 2 2x2 problems one for the $m_J = +1/2$ subspace and one for the $m_J = -1/2$ subspace. We can do both cases simultaneously

$$W_Z = \begin{array}{c c} & \begin{array}{cc} |\tfrac{3}{2},\pm\tfrac{1}{2}\rangle & |\tfrac{1}{2},\pm\tfrac{1}{2}\rangle \end{array} \\ \begin{array}{c} |\tfrac{3}{2},\pm\tfrac{1}{2}\rangle \\ |\tfrac{1}{2},\pm\tfrac{1}{2}\rangle \end{array} & \frac{1}{3}\hbar\omega_0 \begin{pmatrix} \pm 2 & -\sqrt{2} \\ -\sqrt{2} & \pm 1 \end{pmatrix} \end{array} = \frac{1}{3}\hbar\omega_0 \left[\pm\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} \pm \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} - \begin{pmatrix} 0 & +\sqrt{2} \\ +\sqrt{2} & 0 \end{pmatrix} \right]$$

$$W_Z = \pm\frac{1}{2}\hbar\omega_0 \mathbb{1} \pm \frac{1}{6}\hbar\omega_0 \sigma_3 - \frac{\sqrt{2}}{3}\hbar\omega_0 \sigma_1$$

$$W_{SO} = \frac{1}{96} m_e c^2 \alpha^4 \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \frac{1}{96} m_e c^2 \alpha^4 \left[\begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \begin{pmatrix} 3/2 & 0 \\ 0 & -3/2 \end{pmatrix} \right]$$

$$= -\frac{1}{192} m_e c^2 \alpha^4 \mathbb{1} + \frac{1}{64} m_e c^2 \alpha^4 \sigma_3$$

$$W_Z + W_{SO} = \left(-\frac{1}{192} m_e c^2 \alpha^4 \pm \frac{1}{2}\hbar\omega_0 \right) \mathbb{1} + \left(\frac{1}{64} m_e c^2 \alpha^4 \pm \frac{1}{6}\hbar\omega_0 \right) \sigma_3 - \frac{\sqrt{2}}{3}\hbar\omega_0 \sigma_1$$

$$= \frac{1}{2} m_e c^2 \alpha^4 \left(\left(-\frac{1}{96} \pm \xi \right) \mathbb{1} + \left(\frac{1}{32} \pm \frac{1}{3}\xi \right) \sigma_3 - \frac{2\sqrt{2}}{3} \xi \sigma_1 \right)$$

Let $A_1 = -\frac{2\sqrt{2}}{3}\xi$ and $A_3 = \frac{1}{32} \pm \frac{1}{3}\xi$

$\tan\theta$, $\tan\theta' = \frac{A_1}{A_3} = -\frac{2\sqrt{2}}{3}\frac{\xi}{1/32 \pm \xi/3}$

$$A_\pm = \sqrt{A_1^2 + A_3^2} = \sqrt{\frac{8}{9}\xi^2 + \left(\frac{1}{32} \pm \frac{1}{3}\xi\right)^2}$$

$$n_1 = A_1/A = -\frac{2\sqrt{2}}{3}\frac{\xi}{\sqrt{8\xi^2/9 + (1/32 \pm \xi/3)^2}} = \begin{matrix}\sin\theta \\ \sin\theta'\end{matrix}$$

$$n_3 = A_3/A = \frac{1/32 \pm \xi/3}{\sqrt{8\xi^2/9 + (1/32 \pm \xi/3)^2}} = \begin{matrix}\cos\theta \\ \cos\theta'\end{matrix}$$

$$A_1\sigma_1 + A_3\sigma_3 = A_\pm \vec{\sigma}\cdot\vec{n}$$
$$= A_\pm\left(|+\rangle_{\vec{n}}\,{}_{\vec{n}}\langle +| - |-\rangle_{\vec{n}}\,{}_{\vec{n}}\langle -|\right)$$

$$|+\rangle_{\vec{n}} = \begin{matrix}\cos(\theta/2) \\ \cos(\theta'/2)\end{matrix}\left|\tfrac{3}{2}, \pm\tfrac{1}{2}\right\rangle + \begin{matrix}\sin(\theta/2) \\ \sin(\theta'/2)\end{matrix}\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

$$|-\rangle_{\vec{n}} = -\begin{matrix}\sin(\theta/2) \\ \sin(\theta'/2)\end{matrix}\left|\tfrac{3}{2}, \pm\tfrac{1}{2}\right\rangle + \begin{matrix}\cos(\theta/2) \\ \cos(\theta'/2)\end{matrix}\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

Eigenvectors and eigenvalues for $n=2$, $l=1$, $m_J = \pm 1/2$:

$$|\psi_{1/2}\rangle \equiv \cos(\theta/2)\left|2 1 \tfrac{3}{2} \tfrac{1}{2}\right\rangle + \sin(\theta/2)\left|2 1 \tfrac{1}{2} \tfrac{1}{2}\right\rangle$$

$W_z + W_{so}$ $\quad \frac{1}{2}m_e c^2\alpha^4\left(-\frac{1}{96} + \xi + A_+\right)$

Add $\langle W_{mv}\rangle_{2p} + \langle W_D\rangle_{2p} = \langle W_{mv}\rangle_{2p} = -\frac{7}{384}m_e c^2\alpha^4$

$$E_2 + \frac{1}{2}m_e c^2\alpha^4\left(-\frac{3}{64} + \xi + \sqrt{8\xi^2/9 + (1/32 + \xi/3)^2}\right)$$

$$|\phi_{1/2}\rangle = -\sin(\theta/2)\left|2 1 \tfrac{3}{2} \tfrac{1}{2}\right\rangle + \cos(\theta/2)\left|2 1 \tfrac{1}{2} \tfrac{1}{2}\right\rangle$$

$W_z + W_{so}$ $\quad \frac{1}{2}m_e c^2\alpha^4\left(-\frac{1}{96} + \xi - A_+\right)$

$$E_2 + \frac{1}{2}m_e c^2\alpha^4\left(-\frac{3}{64} + \xi - \sqrt{8\xi^2/9 + (1/32 + \xi/3)^2}\right)$$

$$|\psi'_{-1/2}\rangle = \cos(\theta'/2)\,|2\,1\,\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sin(\theta'/2)\,|2\,1\,\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$W_Z + W_{SO} \qquad \tfrac{1}{2} m_e c^2 \alpha^4 \left(-\tfrac{1}{96} - \xi + A_-\right)$$

$$E_2 + \tfrac{1}{2} m_e c^2 \alpha^4 \left(-\tfrac{3}{64} - \xi + \sqrt{8\xi^2/9 + (1/32 - \xi/3)^2}\right)$$

$$|\phi_{-1/2}\rangle = -\sin(\theta'/2)\,|2\,1\,\tfrac{3}{2},-\tfrac{1}{2}\rangle + \cos(\theta'/2)\,|2\,1\,\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$W_Z + W_{SO} \qquad \tfrac{1}{2} m_e c^2 \alpha^4 \left(-\tfrac{1}{96} - \xi - A_-\right)$$

$$E_2 + \tfrac{1}{2} m_e c^2 \alpha^4 \left(-\tfrac{3}{64} - \xi - \sqrt{8\xi^2/9 + (1/32 - \xi/3)^2}\right)$$

(c)

Weak-field limit: $\xi \ll 1$

$$A_\pm = \tfrac{1}{32} \pm \tfrac{1}{3}\xi$$

$$-\tfrac{3}{64} + \xi + A_+ = +\tfrac{1}{64} + \tfrac{4}{3}\xi$$

$$-\tfrac{3}{64} + \xi - A_+ = -\tfrac{5}{64} + \tfrac{2}{3}\xi$$

$$-\tfrac{3}{64} - \xi + A_- = +\tfrac{1}{64} - \tfrac{4}{3}\xi$$

$$n_1 = 0,\ n_3 = 1 \Rightarrow \theta = \theta' = 0 \qquad -\tfrac{3}{64} - \xi - A_- = -\tfrac{5}{64} - \tfrac{2}{3}\xi \leftarrow$$

$|2\,l\,J\,m_J\rangle$ states

Eigenstate	ΔE
$\lvert 2\,0\,0\,\tfrac{1}{2}\rangle = \lvert 2\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$	$-\tfrac{5}{128} m_e c^2 \alpha^4 + \hbar\omega_0$
$\lvert 2\,0\,0,-\tfrac{1}{2}\rangle = \lvert 2\,0\,\tfrac{1}{2},-\tfrac{1}{2}\rangle$	$-\tfrac{5}{128} m_e c^2 \alpha^4 - \hbar\omega_0$
$\lvert \phi_{1/2}\rangle = \lvert 2\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$	$-\tfrac{5}{128} m_e c^2 \alpha^4 + \tfrac{1}{3}\hbar\omega_0$
$\lvert \phi_{-1/2}\rangle = \lvert 2\,1\,\tfrac{1}{2},-\tfrac{1}{2}\rangle$	$-\tfrac{5}{128} m_e c^2 \alpha^4 - \tfrac{1}{3}\hbar\omega_0$
$\lvert 2\,1\,1\,\tfrac{1}{2}\rangle = \lvert 2\,1\,\tfrac{3}{2}\,\tfrac{3}{2}\rangle$	$-\tfrac{1}{128} m_e c^2 \alpha^4 + 2\hbar\omega_0$
$\lvert \psi'_{1/2}\rangle = \lvert 2\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle$	$-\tfrac{1}{128} m_e c^2 \alpha^4 + \tfrac{2}{3}\hbar\omega_0$
$\lvert \psi'_{-1/2}\rangle = \lvert 2\,1\,\tfrac{3}{2},-\tfrac{1}{2}\rangle$	$-\tfrac{1}{128} m_e c^2 \alpha^4 - \tfrac{2}{3}\hbar\omega_0$
$\lvert 2\,1,-1,-\tfrac{1}{2}\rangle = \lvert 2\,1\,\tfrac{3}{2},-\tfrac{3}{2}\rangle$	$-\tfrac{1}{128} m_e c^2 \alpha^4 - 2\hbar\omega_0$

$-\frac{3}{64}+\xi+A_+ = -\frac{7}{192}+2\xi$

Strong-field limit: $\xi \gg 1$ $\quad -\frac{3}{64}+\xi-A_+ = -\frac{11}{192}$ $\qquad A_\pm = \sqrt{\frac{8\xi^2}{9}+\frac{1}{9}\xi^2 \pm 2\frac{1}{3}\frac{1}{32}\xi}$

$-\frac{3}{64}-\xi+A_- = -\frac{11}{192}$ $\qquad = \left(\xi^2 \pm \frac{1}{48}\xi\right)^{1/2}$

$A_\pm = \xi \pm \frac{1}{96}$

$-\frac{3}{64}-\xi-A_- = -\frac{7}{192}-2\xi$ $\qquad = \xi\left(1 \pm 1/48\xi\right)^{1/2}$

$n_1 = -\frac{2\sqrt{2}}{3} \qquad n_3 = \pm\frac{1}{3}$

$\cos^2(\theta/2) = \frac{1}{2}(1+\cos\theta) = \frac{2}{3}$ $\qquad = \xi\left(1 \pm \frac{1}{96\xi}\right)$

$\cos^2(\theta'/2) = \frac{1}{2}(1+\cos\theta') = \frac{1}{3}$ $\qquad = \xi \pm \frac{1}{96}$

$\cos(\theta/2) = \sqrt{2/3}, \quad \sin(\theta/2) = -\sqrt{1/3}$

$\cos(\theta'/2) = \sqrt{1/3}, \quad \sin(\theta'/2) = -\sqrt{2/3}$

$|\psi_{1/2}\rangle = \sqrt{\frac{2}{3}}\,|21\tfrac{3}{2}\tfrac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|21\tfrac{1}{2}\tfrac{1}{2}\rangle = |210\tfrac{1}{2}\rangle$

$|\phi_{1/2}\rangle = \sqrt{\frac{1}{3}}\,|21\tfrac{3}{2}\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|21\tfrac{1}{2}\tfrac{1}{2}\rangle = |211,-\tfrac{1}{2}\rangle$

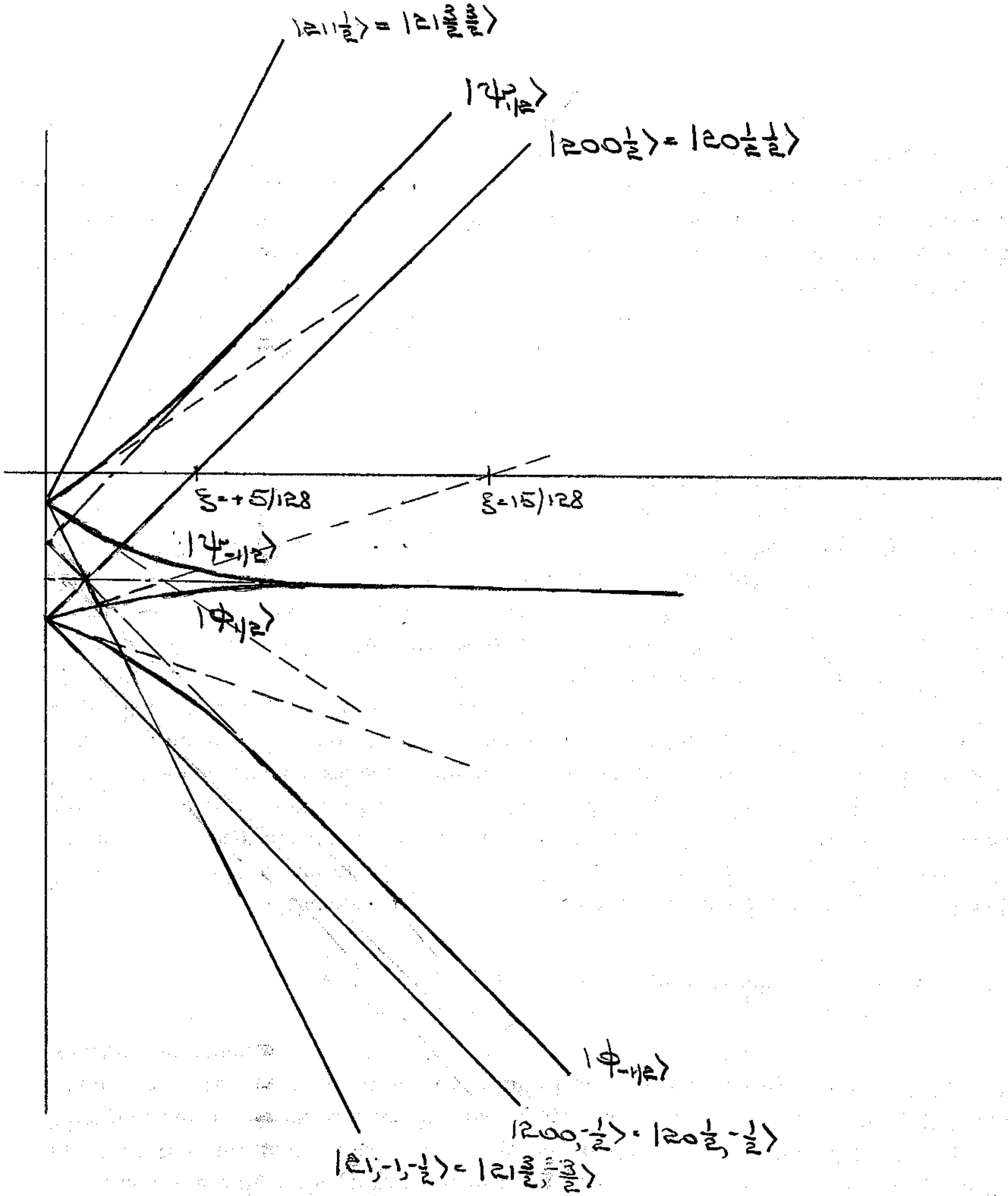
$|\psi_{-1/2}\rangle = \sqrt{\frac{1}{3}}\,|21\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|21\tfrac{1}{2},-\tfrac{1}{2}\rangle = |21,-1,\tfrac{1}{2}\rangle$

$|\phi_{-1/2}\rangle = \sqrt{\frac{2}{3}}\,|21\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|21\tfrac{1}{2},-\tfrac{1}{2}\rangle = |210,-\tfrac{1}{2}\rangle$

Eigenstate

$|n l m_L m_s\rangle$ states

$|200\tfrac{1}{2}\rangle = |20\tfrac{1}{2}\tfrac{1}{2}\rangle$	$-\frac{5}{128}m_ec^2\alpha^4 + \hbar\omega_0$
$|200,-\tfrac{1}{2}\rangle = |20\tfrac{1}{2},-\tfrac{1}{2}\rangle$	$-\frac{5}{128}m_ec^2\alpha^4 - \hbar\omega_0$
$|\phi_{1/2}\rangle = |211,-\tfrac{1}{2}\rangle$	$-\frac{11}{384}m_ec^2\alpha^4$
$|\phi_{-1/2}\rangle = |21,0,-\tfrac{1}{2}\rangle$	$-\frac{7}{384}m_ec^2\alpha^4 - \hbar\omega_0$
$|211\tfrac{1}{2}\rangle = |21\tfrac{3}{2}\tfrac{3}{2}\rangle$	$-\frac{1}{128}m_ec^2\alpha^4 + 2\hbar\omega_0$
$|\psi_{1/2}\rangle = |210\tfrac{1}{2}\rangle$	$-\frac{7}{384}m_ec^2\alpha^4 + \hbar\omega_0$
$|\psi_{-1/2}\rangle = |21,-1,+\tfrac{1}{2}\rangle$	$-\frac{11}{384}m_ec^2\alpha^4$
$|21,-1,-\tfrac{1}{2}\rangle = |21\tfrac{3}{2},-\tfrac{3}{2}\rangle$	$-\frac{1}{128}m_ec^2\alpha^4 - 2\hbar\omega_0$



Notice that the weak to strong field transition occurs at $\xi \sim \frac{1}{25}$, corresponding to a field of ~ 1 Tesla.