Phys 522

Homework #5

Solution Set

5.1 C-T Ex III-1

$$W(t) = \begin{cases} -q\mathcal{E}X & 0 \le t \le \tau \\ 0 & \text{else} \end{cases}$$

a) Calc P_{01} using 1st Order TDPT

$$P_{01} = \left| \int_0^{\tau} \frac{1}{i\hbar} \langle 1 | W | 0 \rangle e^{i\omega_{10} t'} dt' \right|^2$$

$$\omega_{10} = \frac{E_1 - E_0}{\hbar} = \omega_0$$

$$X = \sqrt{\frac{\hbar}{2m\omega_0}} (a + a^+) \quad \therefore \langle 1 | W | 0 \rangle = -q\mathcal{E} \sqrt{\frac{\hbar}{2m\omega_0}}$$

$$\Rightarrow P_{01} = \frac{q^2\mathcal{E}^2}{\hbar^2} \frac{\hbar}{2m\omega_0} \left| \int_0^{\tau} e^{i\omega_0 t'} dt' \right|^2 = \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0} \left| \frac{e^{i\omega_0\tau} - 1}{i\omega_0} \right|^2$$

$$= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0} \left| e^{i\omega_0\tau/2} \frac{\left(e^{i\omega_0\tau/2} - e^{-i\omega_0\tau/2}\right)}{i\omega_0} \right|^2 = \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0} \frac{4\sin^2 \omega_0\tau/2}{\omega_0^2}$$

$$\boxed{P_{01} = \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0} \left(\frac{\sin\frac{\omega_0\tau}{2}}{\omega_0/2} \right)^2}$$

$$\boxed{\text{if } \tau = \frac{2\pi n}{\omega_0} \quad P_{01} = 0}$$

b) Calculate P_{02}

$\langle 2|W|0\rangle = 0 \Rightarrow$ 2ND Order TDPT

$P_{02} = |b_2^{(2)}(t)|^2 \qquad b_2^{(2)} = \frac{1}{\hbar^2} \sum_K \int_0^t dt' \int_0^{t'} dt'' \, e^{i\omega_{2,K}t'} W_{2,K}(t') \, e^{i\omega_{K,0}t''} W_{K,0}(t''$

$\langle K|W(t)|0\rangle = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\langle K|(a+a^+)|0\rangle = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\,\delta_{K,1}$

$\langle 2|W(t)|1\rangle = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\sqrt{2} = -q\mathcal{E}\sqrt{\frac{\hbar}{m\omega_0}}$

$$b_2^{(2)} = \frac{1}{\hbar^2}\int_0^t dt' \int_0^{t'} dt'' \, e^{i\omega_{2,1}t'}\left(-q\mathcal{E}\sqrt{\frac{\hbar}{m\omega_0}}\right) e^{i\omega_{1,0}t''}\left(-q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\right)$$

$\omega_{21} = \omega_0 \quad \omega_{10} = \omega_0$

$$= \frac{q^2\mathcal{E}^2}{m\hbar\omega_0}\frac{1}{\sqrt{2}}\int_0^t e^{i\omega_0 t'}dt' \int_0^{t'} dt'' \, e^{i\omega_0 t''}$$

$$= \frac{q^2\mathcal{E}^2}{m\hbar\omega_0}\frac{1}{\sqrt{2}}\int_0^t e^{i\omega_0 t'}\left(\frac{e^{i\omega_0 t'}-1}{i\omega_0}\right) = \frac{q^2\mathcal{E}^2}{m\hbar\omega_0^2}\frac{1}{\sqrt{2}i}\int_0^t \left(e^{i2\omega_0 t'} - e^{i\omega_0 t'}\right)$$

$$\boxed{= \frac{q^2\mathcal{E}^2}{m\hbar\omega_0^2}\frac{1}{\sqrt{2}i}\left(\frac{e^{i2\omega_0 t}-1}{i2\omega_0} - \frac{e^{i\omega_0 t}-1}{i\omega_0}\right)}$$ → THIS will BE IMPORTANT LATER.

$$= \frac{q^2\mathcal{E}^2}{m\hbar\omega_0^2}\frac{1}{\sqrt{2}i}\left(\frac{e^{i\omega_0 t}}{\omega_0}(\sin\omega_0 t) - \frac{e^{i\omega_0 t/2}}{\omega_0}2\sin\omega_0 t/2\right)$$

$$b_2^{(2)}(t) = \frac{g^2\varepsilon^2}{m\hbar\omega_0^3}\frac{1}{\sqrt{2}i}\left(e^{i\omega_0\tau}\sin\omega_0\tau - 2e^{i\omega_0\tau/2}\sin\omega_0\tau/2\right)$$

$$P_{2,0} = \frac{g^4\varepsilon^4}{2m^2\hbar^2\omega_0^6}\left(\sin^2\omega_0\tau + 4\sin^2\frac{\omega_0\tau}{2} - 2\sin\omega_0\tau\sin\frac{\omega_0\tau}{2}\left(e^{-i\omega_0\tau/2}+e^{i\omega_0\tau/2}\right)\right.$$

$$\boxed{P_{2,0} \doteq \frac{g^4\varepsilon^4}{2m^2\hbar^2\omega_0^6}\left(\sin^2\omega_0\tau + 4\sin^2\frac{\omega_0\tau}{2} - 4\sin\omega_0\tau\sin\frac{\omega_0\tau}{2}\cos\frac{\omega_0\tau}{2}\right)}$$

c. Solve the Problem Exactly and Find $P_{1,0}$ & $P_{2,0}$

$$H' = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2X^2 - g\varepsilon X = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2\left(X^2 - \frac{2g\varepsilon X}{m\omega_0^2}\right)$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2\left(X - \frac{g\varepsilon}{m\omega_0^2}\right)^2 - \frac{g^2\varepsilon^2}{2m\omega_0^2}$$

$$\therefore \boxed{E_n' = \hbar\omega_0(n+1/2) - \frac{g^2\varepsilon^2}{2m\omega_0^2}}$$

Let $X' = X - \frac{g\varepsilon}{m\omega_0^2}$ $\quad H' = \frac{p'^2}{2m} + \frac{1}{2}m\omega_0^2X'^2 - \frac{g^2\varepsilon^2}{2m\omega_0^2}$

$p' = p$

$$\Rightarrow H'|n'\rangle = E_n'|n'\rangle \qquad \boxed{|n'\rangle = e^{-\frac{ig\varepsilon p}{\hbar m\omega_0^2}}|n\rangle}$$

$\hookrightarrow$ Translation operator

$P_{10} = |\langle 1 | \psi(t) \rangle|^2 \qquad |\psi(t)\rangle = e^{-iH't/\hbar}|\psi(0)\rangle = e^{-iH't/\hbar}|0\rangle$

$e^{-iH't/\hbar}|n'\rangle = e^{-iE_n't/\hbar}|n'\rangle \quad \therefore$ switch to $|n'\rangle$ basis

$\langle 1|\psi(\tau)\rangle = \langle 1| e^{-iH'\tau/\hbar}|0\rangle = \langle 1'| e^{\frac{-ig\varepsilon p}{m\omega_0^2}} e^{-iH'\tau/\hbar} e^{\frac{ig\varepsilon p}{\hbar m\omega_0^2}}|0'\rangle$

$e^{\frac{ig\varepsilon p}{\hbar m\omega_0^2}}|0\rangle = e^{\frac{ig\varepsilon p'}{\hbar m\omega_0^2}}|0'\rangle = \left(1 + \frac{ig\varepsilon p'}{\hbar m\omega_0^2} + \dots\right)|0'\rangle = |0'\rangle + \frac{ig\varepsilon}{\hbar m\omega_0^2}\frac{1}{i}\sqrt{\frac{\hbar m\omega_0}{2}}(a'-a'^{\dagger})|0'\rangle$

$p = p'$

1st ORDER EXPANSION

$= |0'\rangle + \frac{-g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}|1'\rangle$

$\therefore e^{-iH\tau/\hbar}|0\rangle = e^{-\frac{i\omega_0\tau}{2}} e^{\frac{-ig^2\varepsilon^2\tau}{2m\omega_0^2}}|0'\rangle - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0} e^{-\frac{3i\omega_0\tau}{2}} e^{\frac{-ig^2\varepsilon^2\tau}{2m\omega_0^2}}|1'\rangle$

$\langle 1'| e^{\frac{-ig\varepsilon p'}{m\omega_0^2}} = \langle 1'| \left(1 - \frac{ig\varepsilon}{m\omega_0^2}\frac{1}{i}\sqrt{\frac{\hbar m\omega_0}{2}}(a' - a'^{\dagger}\right)$

$= \langle 1'| - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}\left(\sqrt{2}\langle 2'| - \langle 0'|\right)$

$\therefore \langle 1|\psi(\tau)\rangle = \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}\left(e^{-\frac{i\omega_0\tau}{2}} - e^{-\frac{3i\omega_0 t}{2}}\right) e^{\frac{-ig^2\varepsilon^2}{2m\omega_0^2}\tau}$

$= \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0} e^{-i\omega_0\tau}\left(e^{\frac{i\omega_0\tau}{2}} - e^{-\frac{i\omega_0\tau}{2}}\right) e^{-i\frac{g^2\varepsilon^2}{2m\omega_0^2}\tau}$

$$\langle 1|\psi(t)\rangle = \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}\; e^{-i\omega_0\tau}\; 2i\sin\frac{\omega_0\tau}{2}\; e^{\frac{-ig\varepsilon^2}{2m\omega_0^2}\tau}$$

$$\therefore \boxed{P_{10} = \frac{g^2\varepsilon^2}{2\hbar m\omega_0}\;\frac{\sin\frac{\omega_0\tau}{2}}{\omega_0/2}}$$

To find P_{20} a little thought shows, we need to calculate to 2^{nd} order

$$e^{\frac{ig\varepsilon p'}{\hbar m\omega_0^2}}|0'\rangle = \left(1 + \frac{ig\varepsilon p'}{\hbar m\omega_0^2} - \frac{g^2\varepsilon^2 p'^2}{2\hbar^2 m^2\omega_0^4}\right)|0'\rangle$$

$$= |0'\rangle - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}|1'\rangle + \frac{g^2\varepsilon^2}{2\hbar^2 m^2\omega_0^4}\,\frac{\hbar m\omega_0}{2}\,(a' - a'^{\dagger})^2\,|0\rangle$$

$$= |0'\rangle - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}|1'\rangle + \frac{g^2\varepsilon^2}{4\hbar m\omega_0^3}\left(+\sqrt{2}\,|2'\rangle - |0'\rangle\right)$$

$$= \left(1 - \frac{g^2\varepsilon^2}{4\hbar m\omega_0^3}\right)|0'\rangle - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}|1'\rangle + \frac{g^2\varepsilon^2}{\sqrt{2^3}\,\hbar m\omega_0^3}|2'\rangle$$

$$e^{-iH't/\hbar}|0\rangle = \left[\left(1 - \frac{g^2\varepsilon^2}{4\hbar m\omega_0^3}\right)e^{-i\omega_0\tau/2}|0'\rangle - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}e^{-\frac{3i\omega_0\tau}{2}}|1'\rangle + \frac{g^2\varepsilon^2}{\sqrt{2^3}\,\hbar m\omega_0^3}e^{-\frac{5i\omega_0\tau}{2}}|2'\rangle\right]\cdot e^{\frac{-ig\varepsilon^2\tau}{2m\omega_0^2}}$$

$$\langle 2'|e^{\frac{-ig\varepsilon p'}{\hbar m\omega_0^2}} = \langle 2'|\left(1 - \frac{ig\varepsilon p'}{\hbar m\omega_0^2} - \frac{g^2\varepsilon^2 p^2}{2\hbar^2 m^2\omega_0^4}\right)$$

$$= \langle 2'| - \frac{g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}\left(\sqrt{3}\,\langle 3'| - \sqrt{2}\,\langle 1'|\right) + \frac{g^2\varepsilon^2}{4\hbar m\omega_0^3}\left(2\sqrt{3}\,\langle 4'| + 5\langle 2'| - \sqrt{2}\,\langle 0'|\right)$$

$$= \frac{-g^2\varepsilon^2}{\sqrt{2^3}\,\hbar m\omega_0^3}\langle 0'| + \frac{g\varepsilon}{\sqrt{\hbar m\omega_0}\,\omega_0}\langle 1'| + \left(1 + \frac{5g^2\varepsilon^2}{4\hbar m\omega_0^3}\right)\langle 2'| - \frac{\sqrt{3}\,g\varepsilon}{\sqrt{2\hbar m\omega_0}\,\omega_0}\langle 3'|$$

$$+ \frac{\sqrt{3}\,g^2\varepsilon^2}{2\hbar m\omega_0^3}\langle 4'|$$

$$\therefore \langle 2|\psi(t)\rangle = \left(1 - \frac{g^2\varepsilon^2}{4\hbar m\omega_0^3}\right)\left(\frac{-g^2\varepsilon^2}{\sqrt{2^3}\,\hbar m\omega_0^3}\right)e^{-i\omega_0 t/2} - \frac{g^2\varepsilon^2}{\sqrt{2}\,\hbar m\omega_0^3}e^{-\frac{3i\omega_0 t}{2}}$$

$$+ \left(1 + \frac{5g^2\varepsilon^2}{4\hbar m\omega_0^3}\right)\frac{g^2\varepsilon^2}{\sqrt{2^3}\,\hbar m\omega_0^3}e^{-\frac{5i\omega_0 t}{2}}$$

(Dropping unimportant $e^{\frac{-ig^2\varepsilon^2 t}{2m\omega_0^2}}$)

Keeping terms to order $\frac{g^2\varepsilon^2}{m\hbar\omega_0^2}$, we get

$$\frac{-g^2\varepsilon^2}{2\sqrt{2}\,\hbar m\omega_0^3}e^{-i\omega_0 t/2} - \frac{g^2\varepsilon^2}{\sqrt{2}\,\hbar m\omega_0^3}e^{-\frac{3i\omega_0 t}{2}} + \frac{g^2\varepsilon^2}{2\sqrt{2}\,\hbar m\omega_0^3}e^{-5i\omega_0 t/2}$$

$$= \frac{-g^2\varepsilon^2}{\sqrt{2}\,\hbar m\omega_0^3}e^{-i\omega_0 t/2}\left(\frac{1}{2} + e^{-i\omega_0 t} - \frac{e^{-2i\omega_0 t}}{2}\right)$$

$$= \frac{-g^2\varepsilon^2}{m\hbar\omega_0^3}\frac{1}{\sqrt{2}}\left(\frac{-e^{-2i\omega_0 t}}{2} + e^{-i\omega_0 t} + \frac{1}{2}\right)$$

$$= \boxed{\frac{g^2\varepsilon^2}{m\hbar\omega_0^3}\frac{1}{\sqrt{2}}\left(\frac{e^{-2i\omega_0 t}-1}{2} - \left(e^{-i\omega_0 t}-1\right)\right)}$$

Comparing this expression with the one on 1-2, we see this ~~is~~ is ~~to~~ EQUIV. TO THE

5.2 C-T E_{XIII}.2

$$H = a(t)\,\vec{S}_1 \cdot \vec{S}_2$$

a) $|\psi(-\infty)\rangle = |+-\rangle$ Calculate w/out Approx $P(+- \to -+)$

SINCE H FACTORS INTO AN EXPLICIT TIME DEPENDENCE & OPERATOR, WE CAN SOLVE DIRECTLY FOR $U(t,-\infty)$

$$i\hbar \frac{dU(t,-\infty)}{dt} = H\,U(t,-\infty) \Rightarrow U(t,-\infty) = \exp\left(-\frac{i}{\hbar}\vec{S}_1\cdot\vec{S}_2 \int_{-\infty}^{t} a(t')dt'\right)$$

$$|\psi(t)\rangle = e^{-i/\hbar \int_{-\infty}^{t} a(t')dt'\, \vec{S}_1\cdot\vec{S}_2} |+-\rangle$$

Let $\vec{S} = \vec{S}_1 + \vec{S}_2 \therefore \vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$

$$|\psi(t)\rangle = e^{-i/\hbar \int_{-\infty}^{t} a(t')dt'\, \frac{1}{2}(S^2 - S_1^2 - S_2^2)} |+-\rangle$$

$$S_1^2|+-\rangle = S_2^2|+-\rangle = 3/4\hbar^2 |+-\rangle$$

$$S^2|+-\rangle$$

LOOK IN THE $|SM_s\rangle$ basis

$$|10\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$\therefore |+-\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle + |00\rangle\right]$$

$$|\psi(t)\rangle = e^{-i/\hbar \int_{-\infty}^{t} a(t')dt' \cdot \frac{1}{2}(S^2 - \frac{3}{2}\hbar^2)} |+-\rangle$$

$$e^{S^2}|+-\rangle = e^{S^2}\frac{1}{\sqrt{2}}\left[|10\rangle+|00\rangle\right]=\frac{1}{\sqrt{2}}\left[e^{2\hbar^2}|10\rangle + |00\rangle\right]$$

$$|\psi(t)\rangle = \left[e^{-i/\hbar\int_{-\infty}^{t} a(t')\,dt' \cdot \hbar^2/4}\,|10\rangle + e^{-i/\hbar\int_{-\infty}^{t} a(t')\,dt'\,\hbar^2(-3/4)}\,|00\rangle\right]\frac{1}{\sqrt{2}}$$

$$= e^{-\frac{i\hbar}{4}\int_{-\infty}^{t} a(t')\,dt'}\left(\frac{1}{2}\left(|+-\rangle+|-+\rangle\right)\right) + e^{\frac{3i\hbar}{4}\int_{-\infty}^{t} a(t')\,dt'}\left(\frac{1}{2}\left(|+-\rangle-|-+\rangle\right)\right)$$

$$\Rightarrow P(+- \to -+) \text{ at } t=\infty = \frac{1}{4}\left| e^{-i\hbar/4\int_{-\infty}^{\infty} a(t')dt} - e^{3i\hbar/4\int_{-\infty}^{\infty} a(t')\,dt}\right|^2$$

$$= \frac{1}{4}\left|e^{i\hbar/4\int_{-\infty}^{\infty} a(t')dt'}\right|^2 \left|e^{-i\hbar/2\int_{-\infty}^{\infty} a(t')\,dt'} - e^{i\hbar/2\int_{-\infty}^{\infty} a(t')dt'}\right|^2$$

$$\boxed{P(+- \to -+) \quad = \sin^2\left(\frac{\hbar}{2}\int_{-\infty}^{\infty} a(t')\,dt'\right)}$$

b. Calculate $P(+- \to -+)$ using 1st order TDPT

$$P(+- \to -+) = |b_f^{(1)}|^2 = \left|\frac{1}{i\hbar}\int_{\infty}^{\infty} W_{fi}\, e^{i\omega_{fi}t'}\right|^2$$

$\omega_{fi} = 0$ (assuming $H = H_0 + a(t)\vec{S}_1\cdot\vec{S}_2$ $H_0 = 0$)

$$W_{fi} = \langle -+|\,a(t)\vec{S}_1\cdot\vec{S}_2\,|+-\rangle = \langle -+|\frac{a(t)}{2}\left(\vec{S}^2-\vec{S}_1^2-\vec{S}_2^2\right)|+-\rangle$$

$$= \underbrace{\langle -+|\frac{a(t)}{2}\hbar^2\left(S^2/\hbar^2\right)|+-\rangle}_{\frac{a(t)\hbar^2}{2\sqrt{2}}\langle -+|S^2/\hbar^2\,|00\rangle+|10\rangle) = \frac{a(t)\hbar^2}{2\sqrt{2}}\,2\,\langle -+|10\rangle = \frac{a(t)\hbar^2}{2}} + \underbrace{\langle -+|\frac{a(t)}{2}|-\tfrac{3}{2}\hbar^2\,|+-\rangle}_{0}$$

$$\boxed{P(+- \rightarrow -+) = \left| \frac{1}{i\hbar} \frac{\hbar^2}{2} \int_{-\infty}^{\infty} a(t')\, dt' \right|^2 = \left(\frac{\hbar}{2} \int_{-\infty}^{\infty} a(t')\, dt' \right)^2}$$

WE SEE PT holds only when $\frac{\hbar}{2} \int_{-\infty}^{\infty} a(t')\, dt'$ is small.

C. $H_0 = -B_0(\gamma_1 S_{1z} + \gamma_2 S_{2z})$ $\quad a(t) = a_0 e^{-t^2/\tau^2}$

Calc. $P(+- \rightarrow -+)$

Now $\omega_{fi} = \hbar^{-1}(E_f - E_i)$ $\quad E_i = \langle +- | H_0 | +- \rangle = -B_0(\gamma_1 \frac{\hbar}{2} - \gamma_2 \hbar/2)$

$$= \frac{B_0 \hbar}{2}(\gamma_2 - \gamma_1)$$

$$E_f = \langle -+ | H_0 | -+ \rangle = -B_0(-\gamma_1 \hbar/2 + \gamma_2 \hbar/2)$$

$$= -\frac{B_0 \hbar}{2}(\gamma_2 - \gamma_1)$$

$\therefore$ $\omega_{fi} = B_0(\gamma_1 - \gamma_2)$

$$\therefore P(+- \rightarrow -+) = \frac{\hbar^2}{4}\left(\int_{-\infty}^{\infty} a_0 e^{-t^2/\tau^2} e^{i\omega_{fi} t}\, dt \right)^2 = \frac{\hbar^2 a_0^2}{4}\left(\int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega_{fi} t}\, dt \right)^2$$

$$= \frac{\hbar^2 a_0^2}{4}\left(\int_{-\infty}^{\infty} e^{-(t/\tau - \frac{i\omega_{fi}\tau}{2})^2} dt \right)^2 = \frac{\hbar^2 a_0^2}{4} e^{\frac{\omega_{fi}^2 \tau^2}{4}} \pi \tau^2$$

$$\boxed{= \frac{\hbar^2 a_0^2 \pi \tau^2}{4} e^{\frac{B_0^2 (\gamma_1 - \gamma_2)^2 \tau^2}{4}}}$$

exp. DEPENDENCE ON B_0

5.3 CT $E_{XIII}.3$

$$W(t) = \frac{\omega_1}{2}\left(J_+ e^{-i\omega t} + J_- e^{i\omega t}\right)$$

a. $$|\psi(t)\rangle = \sum_{m=-1}^{1} b_m(t)\, e^{-iE_m t/\hbar}\, |\phi_m\rangle$$

write system of Diff. EQNs

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H\,|\psi(t)\rangle \qquad H = H_0 + W(t)$$

write as a vector $\begin{pmatrix} b_{-1}(t) \\ b_0(t) \\ b_1(t) \end{pmatrix} \Rightarrow |\psi(t)\rangle = \begin{pmatrix} b_{-1}(t)\, e^{-iE_{-1}t/\hbar} \\ b_0(t)\, e^{-iE_0 t/\hbar} \\ b_1(t)\, e^{-iE_1 t/\hbar} \end{pmatrix}$

$$W(t) = \frac{\hbar\omega_1}{2}\begin{array}{c} \begin{array}{ccc} |-1\rangle & |0\rangle & |+1\rangle \end{array} \\ \begin{pmatrix} 0 & \sqrt{2}e^{+i\omega t} & 0 \\ \sqrt{2}e^{-i\omega t} & 0 & \sqrt{2}e^{+i\omega t} \\ 0 & \sqrt{2}e^{-i\omega t} & 0 \end{pmatrix} \end{array} = \frac{\hbar\omega_1}{\sqrt{2}}\begin{pmatrix} 0 & e^{+i\omega t} & 0 \\ e^{-i\omega t} & 0 & e^{+i\omega t} \\ 0 & e^{-i\omega t} & 0 \end{pmatrix}$$

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = i\hbar\begin{pmatrix} \dot{b}_{-1}(t)\, e^{-iE_{-1}t/\hbar} \\ \dot{b}_0(t)\, e^{-iE_0 t/\hbar} \\ \dot{b}_1(t)\, e^{-iE_1 t/\hbar} \end{pmatrix} + \begin{pmatrix} b_{-1}(t)\, E_{-1} \\ b_0(t)\, E_0 \\ b_1(t)\, E_1 \end{pmatrix}$$

$$H_0\,|\psi(t)\rangle = \begin{pmatrix} b_{-1}(t)\, E_{-1}\, e^{-iE_{-1}t/\hbar} \\ b_0(t)\, E_0\, e^{-iE_0 t/\hbar} \\ b_1(t)\, E_1\, e^{-iE_1 t/\hbar} \end{pmatrix}$$

$$\therefore \boxed{i\hbar\begin{pmatrix} \dot{b}_{-1}(t)\, e^{-iE_{-1}t/\hbar} \\ \dot{b}_0(t)\, e^{-iE_0 t/\hbar} \\ \dot{b}_1(t)\, e^{-iE_1 t/\hbar} \end{pmatrix} = W(t)\,|\psi(t)\rangle = \frac{\hbar\omega_1}{\sqrt{2}}\begin{pmatrix} b_0\, e^{-it(E_0/\hbar - \omega)} \\ b_{-1}\, e^{-it(E_{-1}/\hbar + \omega)} + b_1 e^{-it(E_1/\hbar - \omega)} \\ b_0\, e^{-it(E_0/\hbar + \omega)} \end{pmatrix}}$$

b. assume at $t=0$ $|\psi(0)\rangle = |\phi_{-1}\rangle$

Show that $b_1(t)$ must be calc. to 2nd order

From 1st order TDPT $b_1^{(1)}(t) = \frac{1}{i\hbar}\int_0^t W_{1,-1}\, e^{i\omega_{1,-1}t}\, dt$

$$W_{1,-1} = \langle 1|W|-1\rangle = \frac{\omega_1}{2}\langle 1|J_+e^{-i\omega t} + J_- e^{i\omega t}|-1\rangle$$

$$= \frac{\omega_1}{2}\langle 1|\sqrt{2}e^{-i\omega t}|0\rangle + 0 = 0$$

so $\boxed{b_1^{(1)}(t) = 0}$

calc. $b_1^{(2)}(t) = \frac{1}{\hbar^2}\sum_K \int_0^t dt' \int_0^{t'} dt''\, e^{i\omega_{1,K}t'}\, W_{1,K}(t')\, e^{i\omega_{K,-1}t''}\, W_{K,-1}(t'')$

$$W_{K,-1}(t) = \langle K|W(t)|-1\rangle = \frac{\omega_1}{2}\langle K|\sqrt{2}\,e^{-i\omega t}|0\rangle = \frac{\omega_1}{\sqrt{2}}e^{-i\omega t}\delta_{K0}$$

$$\langle 1|W(t)|K\rangle = \frac{\omega_1}{2}\langle 0|\sqrt{2}e^{-i\omega t}|K\rangle = \frac{\omega_1}{\sqrt{2}}e^{-i\omega t}\delta_{K0}$$

$$b_1^{(2)}(t) = \frac{1}{\hbar^2}\int_0^t dt' \int_0^{t'} dt''\, e^{i\omega_{1,0}t'}\frac{\omega_1}{\sqrt{2}}e^{-i\omega t'}\, e^{i\omega_{0,-1}t''}\frac{\omega_1}{\sqrt{2}}e^{-i\omega t''}$$

$\omega_{10} = \omega_0 \qquad \omega_{0,-1} = \omega_0'$

$$= \frac{\omega_1^2}{2\hbar^2}\int_0^t dt'\, e^{i\omega_0 t'}e^{-i\omega t'}\int_0^{t'} e^{i(\omega_0'-\omega)t''}\,dt''$$

$$= \frac{\omega_1^2}{2\hbar^2}\int_0^t dt'\, e^{i(\omega_0-\omega)t'}\left(\frac{e^{i(\omega_0'-\omega)t'}-1}{i(\omega_0'-\omega)}\right)$$

$$= \frac{\omega_1^2}{2i\hbar^2(\omega_0'-\omega)}\int_0^t e^{i(\omega_0+\omega_0'-2\omega)t'} - e^{i(\omega_0-\omega)t'}\,dt'$$

$$= \frac{\omega_1^2}{2i\hbar^2(\omega_0'-\omega)}\left(\frac{e^{i(\omega_0+\omega_0'-2\omega)t}-1}{i(\omega_0+\omega_0'-2\omega)} - \frac{e^{i(\omega_0-\omega)t}-1}{i(\omega_0-\omega)}\right)$$

$$b_1^{(2)}(t) = \frac{-\omega_1^2}{2\hbar^2(\omega_0'-\omega)}\left(\frac{e^{i(\omega_0+\omega_0'-2\omega)t}-1}{(\omega_0+\omega_0'-2\omega)} - \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right)$$

$$= \frac{-\omega_1^2}{2\hbar^2(\omega_0'-\omega)}\left(e^{+i(\omega_0+\omega_0'-2\omega)t/2}\frac{\left(e^{i(\omega_0+\omega_0'-2\omega)t/2} - e^{-i(\omega_0+\omega_0'-2\omega)t/2}\right)}{\omega_0+\omega_0'-2\omega} - e^{i(\omega_0-\omega)t/2}\frac{\left(e^{i(\omega_0-\omega)t/2} - e^{-i(\omega_0-\omega)t/2}\right)}{\omega_0-\omega}\right)$$

$$= \frac{-\omega_1^2}{2\hbar^2(\omega_0'-\omega)}\left(e^{i(\omega_0+\omega_0'-2\omega)t/2}\frac{2i\sin\left((\omega_0+\omega_0'-2\omega)t/2\right)}{\omega_0+\omega_0'-2\omega} - e^{i(\omega_0-\omega)t/2}\frac{2i\sin\left(\frac{\omega_0-}{2}\right.}{\omega_0-\omega}\right.$$

Let $\Delta = \omega_0+\omega_0'-2\omega \qquad \Delta' = \omega_0-\omega$

$$\boxed{b_1^{(2)}(t) = \frac{-\omega_1^2 i}{\hbar^2(\omega_0'-\omega)}\left(e^{i\Delta t/2}\frac{\sin\left(\frac{\Delta t}{2}\right)}{\Delta} - e^{i\Delta' t/2}\frac{\sin\left(\Delta' t/2\right)}{\Delta'}\right)}$$

$$P_{1\to 1}(t) = \left|b_1^{(2)}(t)\right|^2 = \frac{\omega_1^4}{\hbar^4(\omega_0'-\omega)^2}\left(\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2} + \frac{\sin^2\Delta' t/2}{\Delta'^2} - \frac{2\sin\frac{\Delta t}{2}\sin\frac{\Delta' t}{2}}{\Delta\Delta'}\cos\left(\frac{\omega_0'}{2}\right.\right.$$

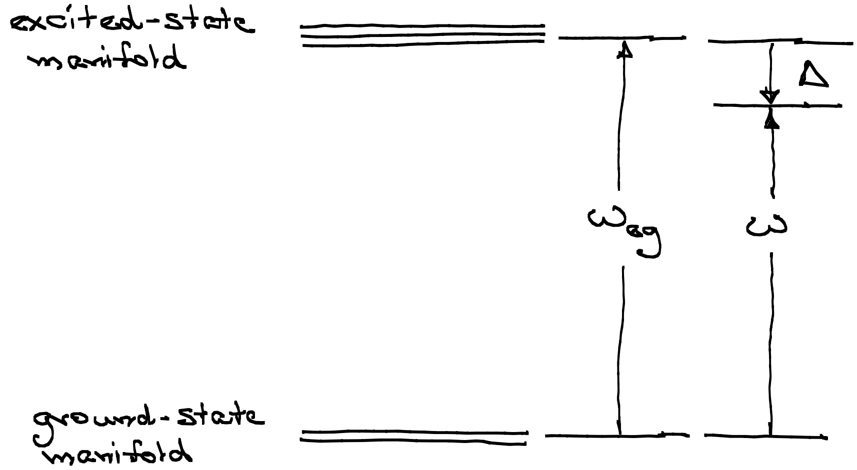
$$\boxed{= \frac{\omega_1^4}{\hbar^4(\omega_0'-\omega)^2}\left(\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2} + \frac{\sin^2\frac{\Delta' t}{2}}{\Delta'^2} - \frac{2\sin\frac{\Delta t}{2}\sin\frac{\Delta' t}{2}\cos\left(\frac{\omega_0'-\omega}{2}t\right)}{\Delta\Delta'}\right)}$$

So we see resonances for $\omega' = \omega$,

$\Delta' = 0 \Rightarrow \omega = \omega_0$

$\Delta = 0 \Rightarrow \omega = \frac{\omega_0+\omega_0'}{2}$

5.4.



Evolution is governed by

$$|\psi_S(t)\rangle = \mathcal{U}(t,t_0)|\psi(t_0)\rangle = e^{-iH_0 t/\hbar}\,\mathcal{U}_I(t,t_0)|\psi(t_0)\rangle,$$

where the IP evolution operator satisfies

$$i\hbar\frac{d\,\mathcal{U}_I(t,t_0)}{dt} = W_I(t)\,\mathcal{U}_I(t,t_0), \quad W_I(t) = e^{iH_0(t-t_0)/\hbar}\,W(t)\,e^{-iH_0(t-t_0)/\hbar}$$

$$= \underbrace{e^{iH_0(t-t_0)/\hbar}\,W\,e^{-iH_0(t-t_0)/\hbar}}_{\equiv \tilde{W}_I(t)}\cos\omega t$$

$$= \tilde{W}_I(t)\cos\omega t$$

$$\mathcal{U}_I(t,t_0) = I - \frac{i}{\hbar}\int_{t_0}^{t} dt'\,W_I(t') - \frac{1}{\hbar^2}\int_{t_0}^{t} dt'\,W_I(t')\int_{t_0}^{t'} dt''\,W_I(t'') + \ldots$$

$$= I - \frac{i}{\hbar}\int_{t_0}^{t} dt'\,\tilde{W}_I(t')\cos\omega t' - \frac{1}{\hbar^2}\int_{t_0}^{t} dt'\,\tilde{W}_I(t')\cos\omega t'\int_{t_0}^{t'} dt''\,\tilde{W}_I(t'')\cos\omega t'' + \ldots$$

(a) We can consider arbitrary evolution <u>within</u> the ground-state manifold by projecting onto the ground-state manifold both before and after the evolution, i.e., calculating

$$P_g\,\mathcal{U}(t+\tau,t)\,P_g = P_g\,e^{-iH_0\tau/\hbar}\,\mathcal{U}_I(t,t+\tau)\,P_g$$

$$= P_g\,e^{-iH_0\tau/\hbar}\,P_g\,\underbrace{P_g\,\mathcal{U}_I(t+\tau,t)\,P_g}_{\text{This is what we want to calculate}} \leftarrow P_g^2 = P_g \text{ and } P_g \text{ commutes with } H_0$$

2

$P_g \tilde{W}_I(t') P_g$ varies on the time scale $1/\Delta_e$ and thus is essentially constant over a time $\tau \gg 1/\Delta_e$. Meanwhile, $\cos\omega t'$ integrates essentially to zero over a time $\omega\tau \gg 1$. Thus we neglect the 1st-order term. What we are saying is that transitions within the ground-state manifold are not energy-conserving, so can be neglected.

$$P_g U_I(t, t+\tau) P_g = P_g - \frac{i}{\hbar}\int_t^{t+\tau} dt' P_g \tilde{W}_I(t') P_g \cos\omega t'$$

$$- \frac{1}{\hbar^2}\int_t^{t+\tau} dt' \, P_g \tilde{W}_I(t') \cos\omega t' \int_t^{t'} dt'' \tilde{W}_I(t'') P_g \cos\omega t'' + \ldots$$

The second-order term is nonvanishing, so we neglect higher-order terms.

We are only going to keep secular terms, i.e., those that integrate to be proportional to τ.

$$P_g U_I(t+\tau, t) P_g - P_g = -\frac{1}{\hbar^2}\int_t^{t+\tau} dt' \, P_g \tilde{W}_I(t') \cos\omega t' \int_t^{t'} dt'' \tilde{W}_I(t'') P_g \cos\omega t''$$

We can put $P_e + P_g$ here, but from the argument for first-order transitions, the contribution from the P_g term is negligible.

$$= -\frac{1}{\hbar^2}\int_t^{t+\tau} dt' \, \underbrace{P_g \tilde{W}_I(t') P_e}\cos\omega t' \int_t^{t'} dt'' \underbrace{P_e \tilde{W}_I(t'') P_g}\cos\omega t''$$

$$= \sum_{j,k} |e_k\rangle\langle e_k| \tilde{W}_I(t') |g_j\rangle\langle g_j|$$

$$= \sum_{j,k} e^{i(e_k - g_j)(t'' - t)} |e_k\rangle\langle e_k| W |g_j\rangle\langle g_j|$$

Over time τ, we can approximate $e_k - g_j = \omega_{eg}$, since $\Delta_g\tau, \Delta_e\tau \ll 1$

$$= e^{-i\omega_{eg}(t'-t)} P_g W P_e \qquad\qquad = e^{i\omega_{eg}(t''-t)} P_e W P_g$$

$$P_g U_I(t,t+\tau) P_g - P_g = -\frac{1}{\hbar^2} P_g W P_e W P_g \int_t^{t+\tau} dt'\, e^{-i\omega_{eg}(t'-t)} \cos\omega t'$$

cancel

$$\times \int_t^{t'} dt''\, e^{i\omega_{eg}(t''-t)} \cos\omega t''$$

$$= -\frac{1}{\hbar^2} P_g W P_e W P_g \underbrace{\int_t^{t+\tau} dt'\, e^{-i\omega_{eg} t'} \cos\omega t'}\ \underbrace{\int_t^{t'} dt''\, e^{i\omega_{eg} t''} \cos\omega t''}$$

$$= \frac{1}{2}\int_t^{t'} dt'' \left(e^{i(\omega_{eg}+\omega)t''} + e^{i\overbrace{(\omega_{eg}-\omega)}^{-\Delta} t''} \right)$$

antiresonant term integrates to be nearly zero (energy nonconserving)

resonant term

keeping only the resonant term

$$= \frac{1}{2}\int_t^{t+\tau} dt'\, e^{i\Delta t'}$$

$$= \frac{1}{2}\int_t^{t'} dt''\, e^{-i\Delta t''}$$

$$P_g U_I(t,t+\tau) P_g - P_g = -\frac{1}{4\hbar^2} P_g W P_e W P_g \int_t^{t+\tau} dt'\, e^{i\Delta t'} \underbrace{\int_t^{t'} dt''\, e^{-i\Delta t''}}_{\frac{i}{\Delta}\left(e^{-i\Delta t'} - e^{-i\Delta t}\right)}$$

keeping only strictly resonant (secular terms) →

$$= \frac{i}{\Delta}\int_t^{t+\tau} dt' \left(1 - e^{i\Delta(t'-t)}\right)$$

$$= \frac{i}{\Delta}\int_0^{\tau} du \left(1 - e^{i\Delta u}\right)$$

$$= \frac{i}{\Delta}\left(\tau - \underbrace{\frac{e^{i\Delta\tau} - 1}{i\Delta}}\right)$$

This term we can neglect because $\Delta\tau \gg 1$. Physically we have chosen τ to be large enough that the detuning can be resolved, making the transitions into the excited-state manifold entirely virtual.

$$P_g U_I(t, t+\tau) P_g = P_g - \frac{i}{\hbar}\tau \frac{1}{4} \frac{P_g W P_e W P_g}{\hbar\Delta}$$

$$= P_g \left(I - \frac{i}{\hbar}\tau\frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta} \right) P_g$$

The total evolution operator in the ground-state manifold is

$$P_g U(t+\tau, t) P_g = P_g e^{-iH_0\tau/\hbar} P_g P_g U_I(t+\tau, t) P_g$$

$$= e^{-iE_e\tau/\hbar} \underbrace{P_g e^{-i(H_0 - E_e)\tau/\hbar} P_g}_{P_g\left(I - \frac{i}{\hbar}\underbrace{\tau(H_0 - E_g)}\right)P_g} \underbrace{P_g U_I(t+\tau, t) P_g}_{P_g\left(I - \frac{i}{\hbar}\tau\frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta}\right)P_g}$$

take out the typical g-s energy

$\Delta_e \tau << 1$ makes this small, so we can expand the exponential

$$P_g U(t+\tau, t) P_g = e^{-iE_e\tau/\hbar} P_g \left[I - \frac{i}{\hbar}\tau\left(P_g H_0 P_g - E_g + \frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta} \right)\right] P_g$$

$$= P_g \exp\left[-\frac{i}{\hbar}\tau \underbrace{\left(P_g H_0 P_g + \frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta}\right)}\right] P_g$$

We have successive short-time evolutions for time τ. The accumulated effect is an exponential.

This is an effective Hamiltonian within the g-s manifold

$$\boxed{H_{eff} = P_g H_0 P_g + \frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta}} \qquad \Delta = \omega - \omega_{eg}$$

(b) The new term in the Hamiltonian now takes the form

$$\frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta} = \frac{1}{2}\vec{E}_0 \cdot \frac{1}{2}\frac{P_g \vec{p} P_e \vec{p} P_g}{\hbar(\omega-\omega_{eg})} \cdot \vec{E}_0 \qquad W = -\vec{p}\cdot\vec{E}_0$$

$$= \frac{1}{2}\vec{E}_0 \cdot \underbrace{\left(\frac{1}{2}\sum_{\substack{j,j' \\ k}} |g_{j'}\rangle\langle g_{j'}| \frac{\langle g_{j'}|\vec{p}|e_k\rangle\langle e_k|\vec{p}|e_j\rangle}{\hbar(\omega-\omega_{eg})}\right)}_{\text{polarizability tensor}} \cdot \vec{E}_0$$

This certainly looks like the energy of an induced dipole interacting with the field that induced it. The field and dipole have a harmonic dependence; i.e., this is the AC Stark effect. The polarization is bigger than for the DC Stark effect because we are driving the dipole oscillator near its resonance frequency. If $\Delta<0$ ($\Delta>0$), we are driving the dipole oscillator below (above) its resonant frequency, the induced dipole will be (in) (out of) phase with the field, and the resulting energy will be negative (positive).

What we want to calculate to check this is the time-averaged (over time τ) energy of the induced dipole, i.e.,

time average over t' from t to $t+\tau$

$$-\frac{1}{2}\overline{\langle\psi(t')|\vec{p}|\psi(t')\rangle\cdot\vec{E}} = -\frac{1}{2}\overline{\langle\psi(t')|\vec{p}|\psi(t')\rangle\cos\omega t'}\cdot\vec{E}_0 .$$

So we need

$$\langle\psi(t')|\vec{p}|\psi(t')\rangle = \langle\psi(t)|\, U_I^\dagger(t',t)\, e^{iH_0(t'-t)/\hbar}\,\vec{p}\, e^{-iH_0(t'-t)/\hbar}\, U_I(t',t)\,|\psi(t)\rangle$$

put P_g here since $|\psi(t)\rangle$ is in the g-s manifold

put P_g here

We'll only need the first-order terms in $U_I(t',t')$

$$P_g U_I^\dagger(t',t)\, e^{iH_0(t'-t)/\hbar}\,\vec{p}\,e^{-iH_0(t'-t)/\hbar}\, U_I(t',t)\, P_g \longleftarrow$$

This will be averaged against $\cos\omega t'$, so we only need to keep terms that oscillate at $\pm\omega$

$$= P_g\left(I + \frac{i}{\hbar}\int_t^{t'} dt''\, W_I(t'')\right) e^{iH_0(t'-t)/\hbar}\,\vec{p}\,e^{-iH_0(t'-t)/\hbar}\left(I - \frac{i}{\hbar}\int_t^{t'} dt''\, W_I(t'')\right) P_g$$

all in the g-s manifold, so no frequencies near ω

too high order

So we only need the cross terms, which are conjugates of one another

$$= -\frac{i}{\hbar}\underbrace{P_g\, e^{iH_0(t'-t)/\hbar}\,\vec{p}\,e^{-iH_0(t'-t)/\hbar}}_{=\,P_g \vec{p} P_e\, e^{-i\omega_{eg}(t'-t)}} \int_t^{t'} dt''\, P_e \tilde{W}_I(t'') P_g \cos\omega t'' + (hc)$$

can put P_e here since the g-s part averages to near zero

$$\tilde{W}_I(t'') = e^{iH_0(t''-t)}\, W\, e^{-iH_0(t''-t)}$$

$$= -e^{+iH_0(t''-t)}\,\vec{p}\,e^{-iH_0(t''-t)}\cdot\vec{E}_0$$

$$P_e \tilde{W}_I(t'') P_g = -e^{i\omega_{eg}(t''-t)} P_e \vec{p} P_g\cdot\vec{E}_0$$

$$= \frac{i}{\hbar} P_g \vec{p} P_e\, e^{-i\omega_{eg}t'} P_e \vec{p} P_g\cdot\vec{E}_0 \underbrace{\int_t^{t'} dt''\, e^{i\omega_{eg}t''}\cos\omega t''}_{} + (hc)$$

Keeping only the resonant term. We're repeating what we did in (a) $\longrightarrow$

$$= \frac{1}{2}\int_t^{t'} dt''\, e^{i\underbrace{(\omega_{eg}-\omega)}_{-\Delta}t''}$$

$$= \frac{i}{2\Delta}\left(e^{-i\Delta t'} - e^{-i\Delta t}\right)$$

$$= -\frac{1}{2\hbar\Delta} P_g \vec{p} P_e\, P_e \vec{p} P_g\cdot\vec{E}_0\left(e^{-i\omega t'} - e^{-i\Delta t}e^{-i\omega_{eg}t'}\right) + (hc)$$

oscillation at the driving frequency

averages to near zero against $\cos\omega t'$, so we drop it

$$= -\frac{P_g\vec{p}P_e\vec{p}P_g\cdot\vec{E}_0}{\hbar\Delta}\cos\omega t'$$

adding the (hc)

From above the time-averaged dipole energy for any state $|\psi\rangle$ in the ground-state manifold is

average over t'

$$-\frac{1}{2}\vec{E}_0\cdot\overline{\langle\psi|\left(\qquad\right)|\psi\rangle\cos\omega t'}$$

$$= +\frac{1}{2}\vec{E}_0\cdot\frac{\langle\psi|P_g\vec{p}P_e\vec{p}P_g|\psi\rangle}{\hbar\Delta}\cdot\vec{E}_0\,\underbrace{\overline{\cos^2\omega t'}}_{1/2}$$

$$= \vec{E}_0\cdot\langle\psi|\underbrace{\frac{1}{4}\frac{P_g\vec{p}P_e\vec{p}P_g}{\hbar\Delta}}_{\text{operator that gives g-s dipole energies, as in our effective Hamiltonian}}|\psi\rangle\cdot\vec{E}_0$$