Physics 522

Homework # 6

Solution Set

6.1. Consider a localized central potential $V(r)$. The "in" scattering state satisfies

$$\varphi^{(+)}_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int d^3r' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} U(r')\varphi^{(+)}_{\vec{k}}(\vec{r}\,')$$

The Born approximation consists of replacing $\varphi^{(+)}_{\vec{k}}(\vec{r}\,')$ by $\frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}\,'}$ in the scattered wave. Thus we are neglecting the scattered wave relative to the incident plane wave inside the scattering region. We can estimate the validity of this approximation by taking $\vec{r}=0$ as a typical point inside the source and estimating the scattered wave there:

$$\begin{pmatrix}\text{Scattered wave} \\ \text{at } \vec{r}=0\end{pmatrix} = -\frac{1}{4\pi}\int d^3r' \frac{e^{ikr'}}{r'} U(r')\varphi^{(+)}_{\vec{k}}(\vec{r}\,')$$

$$\simeq -\frac{1}{4\pi}\int d^3r' \frac{e^{ikr'}}{r'} U(r') \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}\,'}$$

$$= \frac{1}{(2\pi)^{3/2}}\left(-\frac{1}{4\pi}\int r'\,dr'\,e^{ikr'}U(r') \times \underbrace{\int d\Omega_{\hat{r}'}\, e^{i\vec{k}\cdot\vec{r}\,'}}_{4\pi \sin kr'/kr'}\right.$$

$$= \frac{1}{(2\pi)^{3/2}}\left(-\frac{1}{k}\int_0^\infty dr'\, e^{ikr'}\sin kr'\, U(r')\right)$$

Require

$$\frac{1}{(2\pi)^{3/2}}\left| -\frac{1}{k}\int dr'\, e^{ikr'} \sin kr'\, U(r') \right| \ll \frac{1}{(2\pi)^{3/2}}$$

$$\Longleftrightarrow \left| \frac{2\mu}{\hbar^2 k}\int_0^\infty dr'\, e^{ikr'} \sin kr'\, V(r') \right| \ll 1$$

We can make this simpler in two regimes, defined relative to the range a of $V(r)$:

① Low energy: $ka \lesssim 1 \Rightarrow e^{ikr'} \sin kr' \simeq kr'$

$$\underbrace{\left| \int_0^\infty dr'\, r'\, V(r') \right|}_{\sim V_0 a^2} \ll \frac{\hbar^2}{2\mu} \Longleftrightarrow \frac{\hbar^2}{2\mu a^2} \gg V_0$$

② High energy: $ka \gtrsim 1$

$$e^{ikr'} \sin kr' = \frac{1}{2i}\left(e^{2ikr'} - 1 \right)$$

averaged away

$$\underbrace{\left| \int_0^\infty dr'\, V(r') \right|}_{\sim V_0 a} \ll \frac{\hbar^2 k}{\mu} \Longleftrightarrow \frac{\hbar^2 k}{\mu a} \gg V_0$$

Born approximation tends to work at high energy.

①

6.2. $V(r) = \begin{cases} -V_0 , & r < r_0 \\ 0 , & r > r_0 \end{cases}$

$$f^{(B)}_{\vec{k}}(\hat{r}) = -\frac{1}{4\pi}\int d^3r' \underbrace{U(\vec{r}\,')}_{\frac{2\mu}{\hbar^2}V(\vec{r}\,')} e^{-i\overbrace{(\vec{k}\,'-\vec{k})}^{\Delta\vec{k}}\cdot\vec{r}\,'} , \quad \vec{k}\,' = k\hat{r}$$

$$= \frac{2\mu V_0}{\hbar^2}\int_0^{r_0} r'^2 dr' \underbrace{\frac{1}{4\pi}\int d\Omega_{\hat{r}'} e^{-i\Delta\vec{k}\cdot\vec{r}\,'}}_{\frac{\sin|\Delta\vec{k}|r'}{|\Delta\vec{k}|r'}}$$

$$= \frac{2\mu V_0}{\hbar^2}\frac{1}{|\Delta\vec{k}|}\underbrace{\int_0^{r_0} dr'\, r' \sin|\Delta\vec{k}|r'}$$

$$= -\frac{d}{d|\Delta\vec{k}|}\int_0^{r_0} dr' \cos|\Delta\vec{k}|r'$$

$$= -\frac{d}{d|\Delta\vec{k}|}\left(+\frac{\sin|\Delta\vec{k}|r_0}{|\Delta\vec{k}|}\right)$$

$$= -\left(\frac{r_0\cos|\Delta\vec{k}|r_0}{|\Delta\vec{k}|} - \frac{\sin|\Delta\vec{k}|r_0}{|\Delta\vec{k}|^2}\right)$$

$$= +\frac{1}{|\Delta\vec{k}|^2}\left(\sin|\Delta\vec{k}|r_0 - |\Delta\vec{k}|r_0\cos|\Delta\vec{k}|r_0\right)$$

$$\boxed{f^{(B)}_{\vec{k}}(\hat{r}) = \frac{2\mu V_0}{\hbar^2}\frac{\sin|\Delta\vec{k}|r_0 - |\Delta\vec{k}|r_0\cos|\Delta\vec{k}|r_0}{|\Delta\vec{k}|^3} \qquad |\Delta\vec{k}| = 2k\sin(\theta/2)}$$

$$\sigma(\hat{r}) = |f_{\vec{k}}(\hat{r})|^2$$

$$u = |\Delta \vec{k}| r_0 = 2kr_0 \sin(\theta/2)$$

$$f_{\vec{k}}(\hat{r}) = -\frac{2\mu V_0 r_0^3}{\hbar^2}\frac{1}{u}\frac{d}{du}\left(\frac{\sin u}{u}\right)$$

$$\sigma = \int_0^\pi 2\pi \sin\theta \, d\theta \, |f_{\vec{k}}(\hat{r})|^2$$

$$= \left(\frac{2\mu V_0 r_0^2}{\hbar^2}\right)^2 2\pi r_0^2 \int_0^\pi \sin\theta \, d\theta \, \frac{1}{u^2}\left(\frac{d}{du}\left(\frac{\sin u}{u}\right)\right)^2$$

$$du = kr_0 \cos(\theta/2) \, d\theta$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

$$\sin\theta \, d\theta = \frac{2}{kr_0}\sin(\theta/2)\cancel{\cos(\theta/2)}\frac{1}{\cancel{\cos(\theta/2)}} du$$

$$= \frac{1}{k^2 r_0^2} u \, du$$

$$\sigma = \left(\frac{2\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{2\pi}{k^2}\int_0^{2kr_0} u \, du \, \frac{1}{u^2}\left(\frac{d}{du}\left(\frac{\sin u}{u}\right)\right)^2$$

$$\sigma = \left(\frac{2\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{2\pi}{k^2}\underbrace{\int_0^{2kr_0} \frac{du}{u}\left(\frac{d}{du}\left(\frac{\sin u}{u}\right)\right)^2}_{\text{dimensionless integral}}$$

$$= \frac{-1 - 8k^2r_0^2 + 32k^4r_0^4 + \cos 4kr_0 + 4kr_0 \sin 4kr_0}{128 k^4 r_0^4}$$

$$\sigma = \left(\frac{V_0}{\hbar^2/2\mu r_0^2}\right)^2 \overbrace{\frac{2\pi}{k^2}}^{\lambda^2/2\pi} \underbrace{\frac{-1-8k^2r_0^2+32k^4r_0^4-\cos 4kr_0+4kr_0\sin 4kr_0}{128\,k^4r_0^4}}$$

↑ potential in units of characteristic kinetic energy

Low-energy $kr_0 \ll 1$: $\frac{2}{9}k^2r_0^2$

high-energy $kr_0 \gg 1$: $\frac{1}{4}$

Low energy: $kr_0 \ll 1$, $\sigma = \underbrace{\left(\frac{V_0}{\hbar^2/2\mu r_0^2}\right)^2}\frac{4\pi}{9}r_0^2$

This needs to be small for Born approximation to be valid

At low energy the particle is so spread out that it sees the whole scattering center

High energy: $kr_0 \gg 1$, $\sigma = \left(\frac{V_0}{\hbar^2/2\mu r_0^2}\right)^2 \frac{\pi}{2k^2}$

$$= \underbrace{\left(\frac{V_0}{\hbar^2 k/2\mu r_0}\right)^2}\frac{\pi}{2}r_0^2$$

This needs to be small for Born approximation to be valid.

At high energy the particle is spread out over a wavelength

6.3. $V(r) = V_0 e^{-\alpha r}$

$$f^{(B)}_{\vec{k}}(\hat{r}) = -\frac{1}{4\pi}\int d^3r'\, U(\vec{r}')\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}'} \quad , \quad \vec{k}' = k\hat{r}$$

$$= -\frac{2\mu V_0}{\hbar^2}\int r'^2 dr'\, e^{-\alpha r'} \underbrace{\frac{1}{4\pi}\int d\Omega\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}'}}_{\frac{\sin|\Delta\vec{k}|r'}{|\Delta\vec{k}|r'}}$$

$$= -\frac{2\mu V_0}{\hbar^2}\frac{1}{|\Delta\vec{k}|}\underbrace{\int_0^\infty dr'\, r' \sin \underbrace{|\Delta\vec{k}|}_{q = 2k\sin(\theta/2)} r'\, e^{-\alpha r'}}$$

$$= -\frac{d}{d\alpha}\int_0^\infty dr' \sin qr'\, e^{-\alpha r'}$$

$$= -\frac{d}{d\alpha}\frac{1}{2i}\int_0^\infty dr'\, e^{-\alpha r'}\left(e^{iqr'} - e^{-iqr'}\right)$$

$$= -\frac{d}{d\alpha}\frac{1}{2i}\left(-\frac{1}{iq-\alpha} - \frac{1}{+iq+\alpha}\right)$$

$$= -\frac{d}{d\alpha}\left(-\frac{q}{q^2+\alpha^2}\right)$$

$$= \frac{d}{d\alpha}\left(\frac{q}{q^2+\alpha^2}\right)$$

$$= -\frac{2\alpha q}{(q^2+\alpha^2)^2}$$

$$\boxed{|\Delta\vec{k}| = 2k\sin(\theta/2)}$$

$$\boxed{\begin{aligned} f^{(B)}_{\vec{k}}(\hat{r}) &= \frac{2\mu V_0}{\hbar^2}\frac{2\alpha}{(|\Delta\vec{k}|^2+\alpha^2)^2} = \frac{4\mu V_0\alpha}{\hbar^2}\frac{1}{(|\Delta\vec{k}|^2+\alpha^2)^2} \\ \sigma(\hat{r}) &= |f^{(B)}_{\vec{k}}(\hat{r})|^2 = \left(\frac{4\mu V_0\alpha}{\hbar^2}\right)^2\frac{1}{(|\Delta\vec{k}|^2+\alpha^2)^4}\end{aligned}}$$

$$\sigma(\hat{r}) = \left(\frac{4\mu V_0}{\hbar^2 \alpha^3}\right)^2 \frac{1}{(1+\underbrace{|\Delta\vec{k}|^2/\alpha^2}_{\equiv u^2})^4}$$

$$u = \frac{|\Delta\vec{k}|}{\alpha} = \frac{2k\sin(\theta/2)}{\alpha}$$

$$du = \frac{k}{\alpha}\cos(\theta/2)\,d\theta$$

$$\sin\theta\,d\theta = \frac{\alpha^2}{k^2}\,u\,du$$

$$\sigma = \int d\Omega_{\hat{r}}\,\sigma(\hat{r}) = 2\pi\left(\frac{4\mu V_0}{\hbar^2\alpha^3}\right)^2 \int_0^\pi \sin\theta\,d\theta\,\frac{1}{(1+u^2)^4}$$

$$= \left(\frac{4\mu V_0}{\hbar^2\alpha^2}\right)^2 \frac{2\pi}{k^2}\int_0^{2k/\alpha} du\,\frac{u}{(1+u^2)^4}$$

$$v = 1+u^2$$

$$dv = 2u\,du$$

$$= \left(\frac{4\mu V_0}{\hbar^2\alpha^2}\right)^2 \frac{\pi}{k^2}\underbrace{\int_1^{1+4k^2/\alpha^2}\frac{dv}{v^4}}$$

$$= -\frac{1}{3}v^{-3}\Big|_1^{1+4k^2/\alpha^2}$$

$$= -\frac{1}{3}\left(\frac{1}{1+4k^2/\alpha^2} - 1\right)$$

$$= +\frac{1}{3}\,\frac{4k^2/\alpha^2}{1+4k^2/\alpha^2}$$

$$\sigma = \left(\frac{4mV_0}{\hbar^2 \alpha^2}\right)^2 \frac{4\pi}{3\alpha^2} \frac{1}{1+4k^2/\alpha^2}$$

$$\boxed{\sigma = \left(\frac{4mV_0}{\hbar^2 \alpha^2}\right)^2 \frac{4\pi}{3} \cdot \frac{1}{\alpha^2+4k^2}}$$

6.4. C-T C_{VIII}. 3.b

$$V(r) = \begin{cases} -V_0, & r < r_0 \\ 0, & r > r_0 \end{cases}$$

$$k_0 \equiv \sqrt{\frac{2\mu V_0}{\hbar^2}}$$

$\underline{l = 0 \text{ throughout}}$

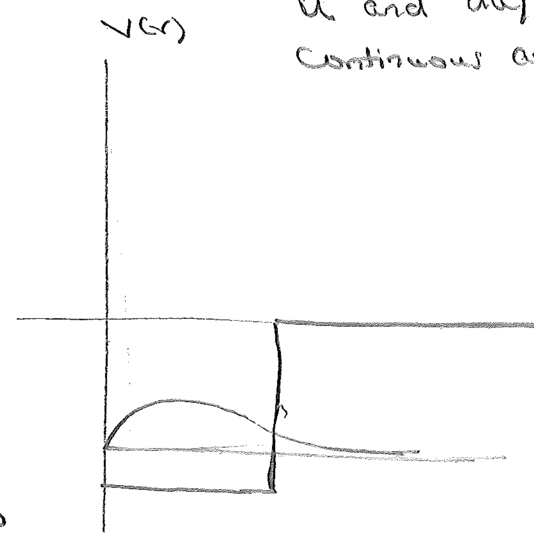
$$\varphi(\vec{r}) = R(r)\, Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \frac{u}{r}$$

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r)\right) u = E u$$

$$r < r_0: \quad \frac{d^2 u}{dr^2} + \frac{2\mu}{\hbar^2}(E + V_0) u = 0$$

$$r > r_0: \quad \frac{d^2 u}{dr^2} + \frac{2\mu E}{\hbar^2} u = 0$$

BC's:
$u(r=0) = 0$
u and du/dr continuous at $r = r_0$.



(a) Bound states: $E < 0$

$$\rho = \sqrt{-\frac{2\mu E}{\hbar^2}}, \quad 0 < \rho < k_0$$

$$K = \sqrt{k_0^2 - \rho^2} = \sqrt{\frac{2\mu}{\hbar^2}} \sqrt{E + V_0}$$

$$0 < K < k_0$$

(i) $r < r_0: \quad \dfrac{d^2 u}{dr^2} + K^2 u = 0 \qquad\Longrightarrow\qquad u(r) = B \sin Kr, \quad r < r_0$

$r > r_0: \quad \dfrac{d^2 u}{dr^2} - \rho^2 u = 0 \qquad\qquad u(r) = A e^{-\rho r}, \quad r > r_0$

u must vanish as $r \to \infty$

u continuous at $r = r_0$: $\quad B \sin K r_0 = A e^{-\rho r_0}$

u' continuous at $r = r_0$: $\quad B K \cos K r_0 = -A \rho e^{-\rho r_0}$

(ii)

$$\Rightarrow \frac{1}{K}\tan K r_0 = -\frac{1}{\rho}$$

$$\boxed{\tan K r_0 = -\frac{K}{\rho}} \Rightarrow$$

(iii)

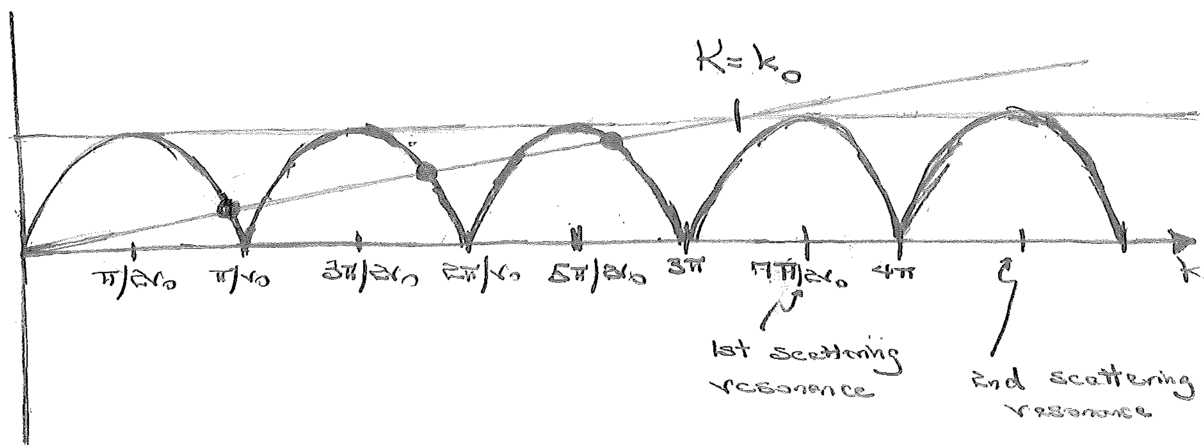
$$\cot^2 K r_0 = \frac{\rho^2}{K^2} = \frac{k_0^2 - K^2}{K^2} = \frac{k_0^2}{K^2} - 1$$

$$\frac{k_0^2}{K^2} = 1 + \cot^2 K r_0 = \frac{1}{\sin^2 K r_0}$$

$$\boxed{|\sin K r_0| = \frac{K}{k_0} \quad \text{and} \quad \tan K r_0 < 0}$$

These are the odd bound states for a well of width $2r_0$.

$$\boxed{\frac{(\text{phase-space volume})}{h} = \frac{2p_0 r_0}{h} = \frac{2\hbar k_0 r_0}{h} = \frac{k_0 r_0}{\pi}}$$



$$\boxed{\text{No bound states if } k_0 < \pi/2r_0 \Leftrightarrow k_0 r_0 < \pi/2.}$$

The $\sin Kr$ must accumulate at least $\pi/2$ of phase so that it can turn over before reaching r_0.

$$\left(\begin{array}{c}\#\text{ of bound}\\ \text{states}\end{array}\right) \equiv g_b = \underset{\substack{\uparrow \\ \text{integer part}}}{\mathrm{Int}}\left(\frac{k_0 + \pi/2r_0}{\pi/r_0}\right) = \mathrm{Int}\left(\frac{k_0 r_0}{\pi} + \frac{1}{2}\right)$$

B. Scattering resonances: $E > 0$

$$k = \sqrt{\frac{2\mu E}{\hbar^2}} > 0$$

$$K' = \sqrt{k_0^2 + k^2} = \sqrt{\frac{2\mu}{\hbar^2}}\sqrt{E + V_0} = K > k_0$$

(i) $r < r_0$: $\frac{d^2u}{dr^2} + K'^2 u = 0$

$r > r_0$: $\frac{d^2u}{dr^2} + k^2 u = 0$

$$\Longrightarrow \quad u = B \sin K'r, \quad r < r_0$$
$$u = A \sin(\underbrace{kr + \delta_0}_{\alpha(k)}), \quad r > r_0$$

u continuous at $r = r_0$: $B \sin K'r_0 = A \sin(kr_0 + \delta_0)$

u' continuous at $r = r_0$: $BK' \cos K'r_0 = Ak \cos(\underbrace{kr_0 + \delta_0}_{\alpha(k)})$

$$\Longrightarrow \frac{1}{K'} \tan K'r_0 = \frac{1}{k} \tan \alpha(k)$$

$$\boxed{\tan \alpha(k) = \frac{k}{K'} \tan K'r_0}$$

$$A^2k^2 = B^2(k^2 \sin^2 K'r_0 + K'^2 \cos^2 K'r_0)$$
$$= B^2(k^2 + \underbrace{(K'^2 - k^2)}_{k_0^2} \cos^2 K'r_0)$$

$$\boxed{\left(\frac{B}{A}\right)^2 = \frac{k^2}{k^2 + k_0^2 \cos^2 K'r_0} = \frac{1}{1 + (k_0^2/k^2)\cos^2 K'r_0}}$$

$(B/A)^2 \leq 1$, maximum achieved when $\cos K'r_0 = 0$

$$\Longrightarrow \boxed{K'r_0 = (2n+1)\frac{\pi}{2}, \quad K' > k_0} \quad \text{S-wave resonance condition}$$

$$\boxed{\begin{pmatrix}\text{phase accumulated} \\ \text{within well}\end{pmatrix} = 2K'r_0 = (2n+1)\pi}$$

If the well is deep, the $\ell=0$ bound states near the top of the well occur at $K r_0 = (2n+1)\pi/2$, and the s-wave resonances are an extension of this sequence to positive energies. In any case the first resonance occurs for $n = g_b + 1$.

Resonances at $K' r_0 = (2n+1)\frac{\pi}{2} \Longleftrightarrow k^2 = K'^2 - k_0^2 = \frac{(2n+1)^2\pi^2}{4 r_0^2} - k_0^2$

$$\left(\frac{k}{k_0}\right)^2 = -1 + \frac{(2n+1)^2\pi^2}{4 k_0^2 r_0^2}$$

$n = g_b + 1, g_b + 2, \ldots$

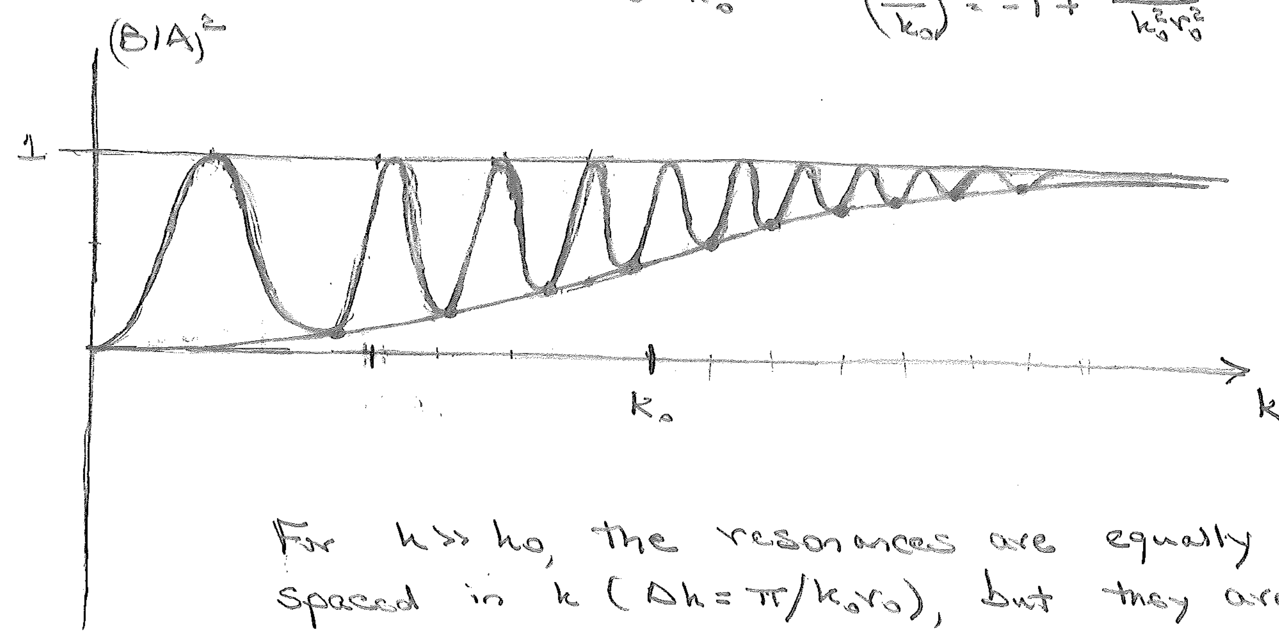
$\uparrow$

smallest value of n st $K' \geq k_0$, $k \geq 0$, is # of bound states plus 1

$$(B/A)^2 = 1$$

When $\cos K' r_0 = \pm 1$, i.e., $K' r_0 = 2n\pi \Longleftrightarrow k^2 = \frac{n^2\pi^2}{r_0^2} - k_0^2$,

$$(B/A)^2 = \frac{k^2}{k^2 + k_0^2} \qquad \left(\frac{k}{k_0}\right)^2 = -1 + \frac{n^2\pi^2}{k_0^2 r_0^2}$$



For $k \gg k_0$, the resonances are equally spaced in k ($\Delta k = \pi/k_0 r_0$), but they are very shallow. For $k \lesssim k_0$, the resonances are deeper and more widely spaced.

Scattering phase shift

$$\tan\alpha(k) = \frac{k}{K'}\tan K'r_0$$

$\alpha(k)$ advances along with $K'r_0$ as k increases.
For $k \gg k_0$, $k/K' \simeq 1$, and we have
$\alpha(k) \simeq K'r_0 \simeq kr_0$, which gives $\delta_0 \sim 0$, as required because the high-energy particle scarcely notices the potential well. For low energies, i.e., $k \lesssim k_0$, α advances by π as $K'r_0$ advances by π, with $\alpha = K'r_0 = (2n+1)\pi/2$ at resonance, but the advance of α is unsteady, mostly occuring near the resonance.
We can see this by looking at the behavior of $\alpha(k)$ near a resonance:

$$\cot\alpha(k) = \frac{K'}{k}\cot K'r_0$$

Take the derivative wrt k

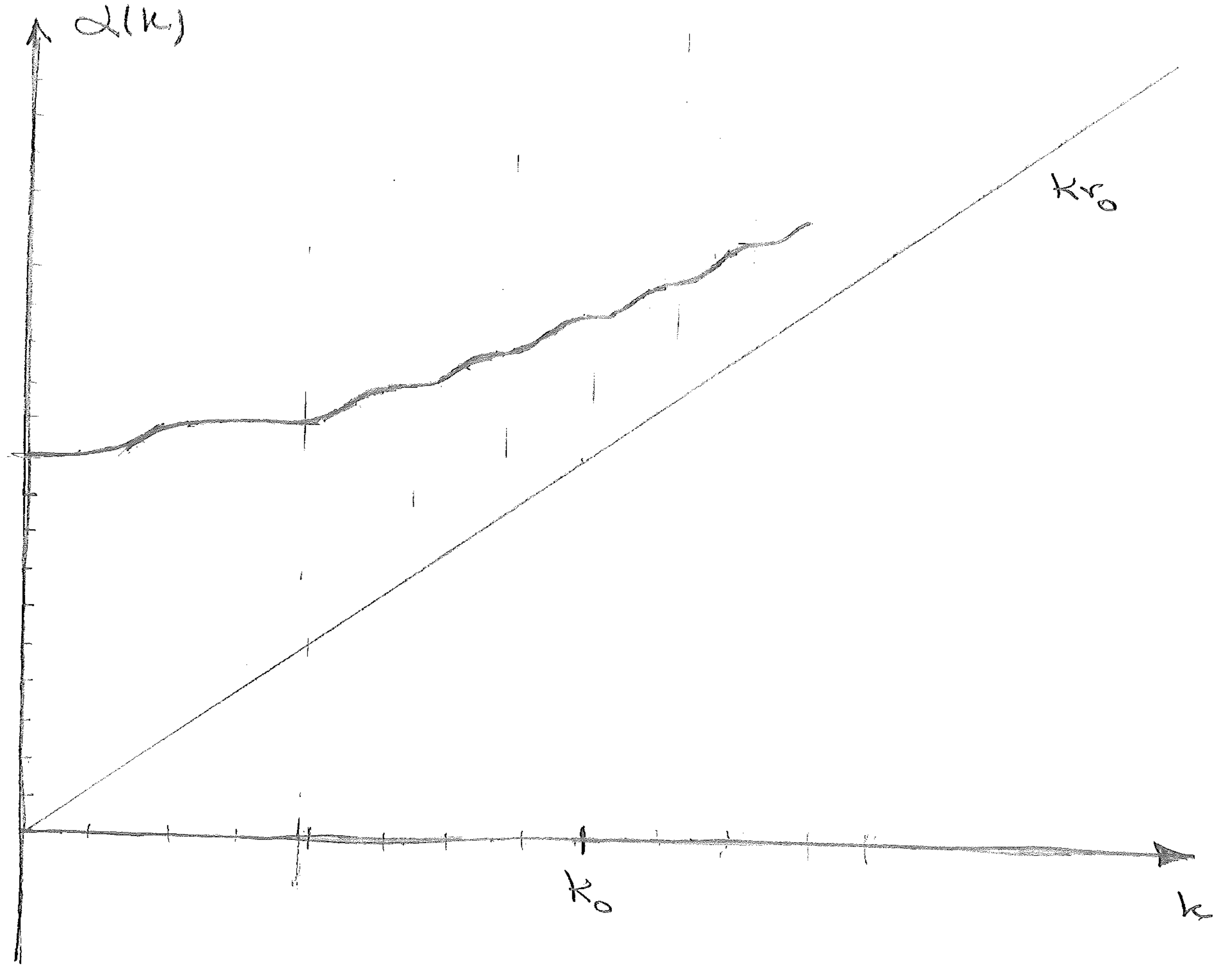
$$-\frac{1}{\sin^2\alpha(k)}\frac{d\alpha}{dk} = \frac{1}{k}\frac{dK'}{dk}\cot K'r_0 - \frac{K'}{k^2}\cot K'r_0$$
$$-r_0\frac{K'}{k}\frac{dK'}{dk}\frac{1}{\sin^2 K'r_0}$$

At resonance, $K'r_0 = \alpha(k) = (2n+1)\pi/2$, $\cot K'r_0 = 0$, and $\sin^2\alpha(k) = \sin^2 K'r_0 = 1$. Notice also that $dK'/dk = k/K'$.

∴ at resonance we have the simple result

$$\boxed{\left.\frac{d\alpha}{dk}\right|_{resonance} = r_0}$$

For $k \gg k_0$, this is true for all k. For $k \lesssim k_0$, since the resonances are farther apart in k, this derivative means the advance of phase is more concentrated near the low energy resonances



If there is an s-wave resonance for $kr_0 \ll 1$, then

$$\delta_0 = -kr_0 + \alpha = -kr_0 + \frac{(2n+1)\pi}{2}$$

$$\sin\delta_0 = \underbrace{\sin\left(\frac{(2n+1)\pi}{2}\right)}_{(-1)^n}\cos kr_0 - \underbrace{\cos\left(\frac{(2n+1)\pi}{2}\right)}_{0}\sin kr_0$$

$$= (-1)^n \cos kr_0$$

$$\simeq (-1)^n\left(1 - \frac{1}{2}(kr_0)^2\right)$$

The s-wave contribution to the total cross-section is

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \simeq \frac{4\pi}{k^2}\left(1 - (kr_0)^2\right)$$

$$= 4\left(\frac{\pi}{k^2} - \pi r_0^2\right)$$

geometrical cross section

Nearly has max value of $4\pi/k^2$.

(d) $k_0 r_0 = (2n+1)\pi/2 + \epsilon, \quad |\epsilon| \ll 1$

The number of bound states is

$$g_b = \mathrm{Int}\left(\frac{1}{2} + \frac{2n+1}{2} + \frac{\epsilon}{\pi}\right)$$

$$= \mathrm{Int}(n + 1 + \epsilon/\pi) = \begin{cases} n, & \epsilon < 0 \\ n+1, & \epsilon > 0 \end{cases}$$

See drawing on page 2, where $n=3$, $\epsilon<0$

(ii) $\epsilon > 0$:

The highest bound state has

$$Kr_0 = (2n+1)\pi/2 + \eta, \qquad 0 < \eta < \epsilon \ll 1$$

$$\sin Kr_0 = (-1)^n \cos\eta$$

Quantization condition: $|\sin Kr_0| = \frac{K}{k_0}$

$$\underset{\displaystyle 1-\frac{1}{2}\eta^2}{\underset{\shortparallel}{\cos\eta}} = \frac{K}{k_0} = \frac{(2n+1)\pi/2 + \eta}{(2n+1)\pi/2 + \epsilon}$$

$$\Longrightarrow \quad (2n+1)\pi/2 + \epsilon - \frac{1}{2}\frac{(2n+1)\pi}{2}\eta^2 = \frac{(2n+1)\pi}{2} + \eta$$

$$\eta = \epsilon - \frac{1}{2}\frac{(2n+1)\pi}{2}\eta^2 \simeq \epsilon - \frac{1}{2}\frac{(2n+1)\pi}{2}\epsilon^2$$

$$\Longrightarrow \quad \epsilon - \eta \simeq \frac{1}{2}\frac{(2n+1)\pi}{2}\epsilon^2 \simeq \frac{1}{2}k_0 r_0 \epsilon^2$$

$$E = -\frac{\hbar^2}{2\mu}\rho^2$$

$$\rho^2 = k_0^2 - K^2 = (k_0 + K)(k_0 - K)$$

$$(\rho r_0)^2 = \underbrace{(k_0 r_0 + K r_0)}_{2k_0 r_0}\underbrace{(k_0 r_0 - K r_0)}_{\epsilon - \eta = \frac{1}{2}k_0 r_0 \epsilon^2}$$

$$= (k_0 r_0)^2 \epsilon^2$$

$$\rho^2 = k_0^2 \epsilon^2 \quad \Longrightarrow \quad \boxed{\begin{aligned} &\rho = k_0 \epsilon \\ &E = -\frac{\hbar^2}{2\mu}\rho^2 = -\frac{\hbar^2 k_0^2}{2\mu}\epsilon \end{aligned}}$$

(iii) $\epsilon < 0$:

Resonance at $K' r_0 = (2n+1)\pi/2 = k_0 r_0 - \epsilon$

$$k^2 = K'^2 - k_0^2 = (K' + k_0)(K' - k_0)$$

$$(k r_0)^2 = \underbrace{(K' r_0 + k_0 r_0)}_{2 k_0 r_0} \underbrace{(K' r_0 - k_0 r_0)}_{-\epsilon}$$

$$= -2 k_0 r_0 \epsilon$$

$$\boxed{\begin{aligned} k^2 &= -\frac{2k_0}{r_0}\epsilon \\ E &= \frac{\hbar k^2}{2\mu} = -\frac{\hbar k_0}{\mu r_0}\epsilon \end{aligned}}$$

(iii) Obvious