Phys 522

Lectures 1-2

Electron spin

An electron is a particle with an intrinsic angular momentum ($J = 1/2$) or spin.

rest frame

Translational degrees of freedom

Hilbert space $\mathcal{E}_{\vec{r}}$

Position operator $\vec{R}$ $\quad |\vec{r}\rangle \quad [R_j, P_k] = i\hbar\,\delta_{jk}$

Momentum operator $\vec{P}$ $\quad |\vec{p}\rangle \quad \langle\vec{r}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{r}\right)$

Angular-momentum operator $\vec{L} = \vec{R}\times\vec{P}$

$[L_j, L_k] = i\hbar\,\epsilon_{jkl} L_l$ $\quad$ Irreps $|l,m\rangle$, $m = -l,\ldots,l$

$$\vec{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$$

$$L_z|l,m\rangle = m\hbar|l,m\rangle$$

Magnetic moment $\vec{M}_{orb} = \frac{\mu_B}{\hbar}\vec{L}$, $\quad \mu_B = q\hbar/2m_e \quad (q = -e)$

Bohr magneton

Energy in a magnetic field: $-\vec{M}_{orb}\cdot\vec{B}$

Where does this come from?
Zeeman splitting
Stern-Gerlach effect

Spin degree of freedom

2-dimensional Hilbert space $\mathcal{E}_S$ $\quad [S_j, S_k] = i\hbar\,\epsilon_{jkl} S_l$

Intrinsic angular momentum (spin) $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ $\quad [\sigma_j, \sigma_k] = 2i\,\epsilon_{jkl}\sigma_l$

Pauli operators

$|s = \tfrac{1}{2}, m = \tfrac{1}{2}\epsilon\rangle$, $\epsilon = \pm 1$

$$\vec{S}^2|\tfrac{1}{2},m\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},m\rangle$$

$$S_z|\tfrac{1}{2},m\rangle = m\hbar|\tfrac{1}{2},m\rangle$$

Magnetic moment $\vec{M}_{spin} = 2\frac{\mu_B}{\hbar}\vec{S}$

Zeeman splitting
Stern-Gerlach effect (entanglement)

$\rightarrow s = 1/2 \Rightarrow 2$ levels

nonclassical

Spin-$\frac{1}{2}$ Hilbert space

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |+\rangle_z \qquad \sigma_z|\epsilon\rangle_z = \epsilon|\epsilon\rangle_z$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \equiv |-\rangle_z$$

often omit this z

In this basis

$$\sigma_z = |+\rangle\langle+| - |-\rangle\langle-| \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = |+\rangle\langle-| + |-\rangle\langle+| \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = -i|+\rangle\langle-| + i|-\rangle\langle+| \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Raising and lowering operators:

$$S_\pm = S_x \pm i S_y = \hbar\sigma_\pm$$

$$S_+|+\rangle = 0 \qquad S_-|+\rangle = \hbar|-\rangle$$

$$S_+|-\rangle = \hbar|+\rangle \qquad S_-|-\rangle = 0$$

One phase must be chosen here, which amounts to choosing the directions of the x and y axes.

$$\sigma_\pm = \tfrac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$\sigma_x|+\rangle = |-\rangle \qquad \sigma_y|+\rangle = i|-\rangle$$

$$\sigma_x|-\rangle = |+\rangle \qquad \sigma_y|-\rangle = -i|+\rangle$$

Special properties for spin-$\frac{1}{2}$:

1. $\sigma_j\sigma_k = \delta_{jk}1_s + i\epsilon_{jkl}\sigma_l$
2. $1_s, \sigma_x, \sigma_y, \sigma_z$ are orthogonal operators, i.e., $\mathrm{tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$; that span the 4d vector space of operators: $A = \sum_\alpha A_\alpha\sigma_\alpha \Leftrightarrow A_\alpha = \frac{1}{2}\mathrm{tr}(A\sigma_\alpha)$

Spin states and the Bloch sphere

An arbitrary spin state can be written as

$$\cos(\theta/2)|+\rangle + e^{i\varphi}\sin(\theta/2)|-\rangle, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi$$

$$\equiv |\vec{u}\rangle \equiv |+\rangle_{\vec{u}} \longleftarrow$$ single-valued representation

CT distributes phases evenly

double-valued representation

$$|-\vec{u}\rangle = |+\rangle_{-\vec{u}} = \sin(\theta/2)|+\rangle - e^{i\varphi}\cos(\theta/2)|-\rangle \equiv |-\rangle_{\vec{u}}$$

$\theta \to \pi - \theta$
$\varphi \to \varphi + \pi$

$$1_s = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle+| + |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle-|$$

$$\vec{\sigma}\cdot\vec{u}\,|\pm\rangle_{\vec{u}} = \pm|\pm\rangle_{\vec{u}} \qquad \vec{\sigma}\cdot\vec{u} = |+\rangle_{\vec{u}}\,{}_{\vec{u}}\langle+| - |-\rangle_{\vec{u}}\,{}_{\vec{u}}\langle-|$$

$$\vec{u} = \vec{e}_x: \quad |+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z) \qquad \vec{u} = \vec{e}_y: \quad |+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle_z + i|-\rangle_z)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z - |-\rangle_z) \qquad |-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle_z - i|-\rangle_z)$$

Bloch sphere

General electron states

Tensor-product Hilbert space $\mathcal{E}_{\vec{r}} \otimes \mathcal{E}_S = \mathcal{E}$

Product basis $|\vec{r}\rangle \otimes |\epsilon\rangle_z \equiv |\vec{r},\epsilon\rangle$

Orthogonality: $\langle \vec{r},\epsilon|\vec{r}\,',\epsilon'\rangle = \delta_{\epsilon\epsilon'}\delta(\vec{r}-\vec{r}\,')$

Completeness: $\sum_\epsilon \int d^3r\, |\vec{r},\epsilon\rangle\langle\vec{r},\epsilon| = \mathbb{1}$

Arbitrary state: $$|\psi\rangle = \sum_\epsilon \int d^3r\, |\vec{r},\epsilon\rangle \underbrace{\langle \vec{r},\epsilon|\psi\rangle}_{\psi_\epsilon(\vec{r})}$$

$\psi_\epsilon(\vec{r}) \longleftarrow$ 2 wave functions $\psi_\pm(\vec{r})$

Operators: spin ops commute with translational operators

$$A = \sum_{\epsilon,\epsilon'} \int d^3r\, d^3r'\, |\vec{r},\epsilon\rangle \underbrace{\langle \vec{r},\epsilon|A|\vec{r}\,',\epsilon'\rangle}_{A_{\epsilon\epsilon'}(\vec{r},\vec{r}\,')} \langle \vec{r}\,',\epsilon'|$$

Spinor representation: $|\psi\rangle \longleftrightarrow \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$ position-valued 2-d column vector

$$A \longleftrightarrow \begin{pmatrix} A_{++}(\vec{r},\vec{r}\,') & A_{+-}(\vec{r},\vec{r}\,') \\ A_{-+}(\vec{r},\vec{r}\,') & A_{--}(\vec{r},\vec{r}\,') \end{pmatrix}$$

Inner product: $$\langle\phi|\psi\rangle = \sum_\epsilon \int d^3r\, \langle\phi|\vec{r},\epsilon\rangle\langle\vec{r},\epsilon|\psi\rangle$$
$$= \sum_\epsilon \int d^3r\, \phi_\epsilon^*(\vec{r})\,\psi_\epsilon(\vec{r})$$
$$= \int d^3r \begin{pmatrix} \phi_+^*(\vec{r}) & \phi_-^*(\vec{r}) \end{pmatrix} \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

Interpretation: $\mathcal{P}(\vec{r},\epsilon) = |\langle\vec{r},\epsilon|\psi\rangle|^2 = |\psi_\epsilon(\vec{r})|^2$ S-G experiment

Two-slit experiment

Standard: $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)\otimes|+\rangle_z$

$\langle\psi_1|\psi_2\rangle = 0$

$$\mathcal{P}(\vec{r},+) = \frac{1}{2}\left(\psi_1^*(\vec{r}) + \psi_2^*(\vec{r})\right)\left(\psi_1(\vec{r}) + \psi_2(\vec{r})\right)$$
$$= \frac{1}{2}\left(|\psi_1|^2 + |\psi_2|^2\right) + \mathrm{Re}\left(\psi_1^*(\vec{r})\psi_2(\vec{r})\right)$$

Spin flip: $\frac{1}{\sqrt{2}}\left(|\psi_1\rangle\otimes|+\rangle_z + |\psi_2\rangle\otimes|-\rangle_z\right)$

$\mathcal{P}(\vec{r},+) = \frac{1}{2}|\psi_1|^2$

$\mathcal{P}(\vec{r},-) = \frac{1}{2}|\psi_2|^2$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)\otimes|+\rangle_x + \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)\otimes|-\rangle_x\right)$$

$$P(\vec{r},+_x) = \frac{1}{4}\left(|\psi_1|^2+|\psi_2|^2\right)+\frac{1}{2}\mathrm{Re}\left(\psi_1^*(\vec{r})\,\psi_2(\vec{r})\right)$$
$$P(\vec{r},-_x) = \frac{1}{4}\left(|\psi_1|^2+|\psi_2|^2\right)-\frac{1}{2}\mathrm{Re}\left(\psi_1^*(\vec{r})\,\psi_2(\vec{r})\right)$$

Rotation operators

$$\vec{J} = \vec{L}+\vec{S} = \vec{L}\otimes 1_S + 1_{\vec{r}}\otimes\vec{S}$$

$$\begin{pmatrix}\text{unitary rotation operator}\\ \text{for rotation by } \alpha \text{ about } \vec{u}\end{pmatrix} = R_{\vec{u}}(\alpha) = \exp\left(-\frac{i}{\hbar}\alpha\vec{J}\cdot\vec{u}\right) = e^{-i\alpha\vec{L}\cdot\vec{u}/\hbar}\,\underbrace{e^{-i\alpha\vec{S}\cdot\vec{u}/\hbar}}$$

$$= e^{-i\vec{\sigma}\cdot\vec{u}\,\alpha/2} = \cos(\alpha/2)\,1_S - i\sin(\alpha/2)\,\vec{\sigma}\cdot\vec{u}$$

all 2d unitaries up to a phase

$R_{\vec{u}}(2\pi) = -1$

double-valued representation

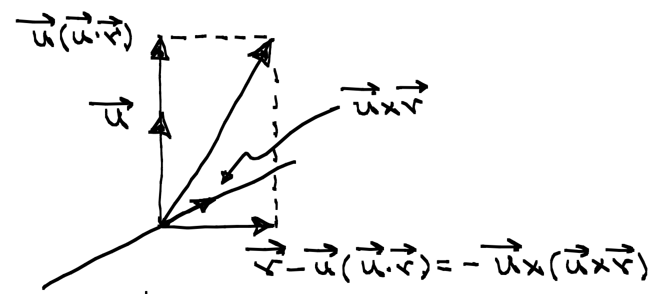
$$\langle\vec{r},\epsilon|R_{\vec{u}}(\alpha)|\psi\rangle = \left(\langle\vec{r}|\otimes{}_z\langle\epsilon|\right)\left(e^{-i\alpha\vec{L}\cdot\vec{u}/\hbar}\,e^{-i\alpha\vec{S}\cdot\vec{u}/\hbar}\right)|\psi\rangle$$
$$=\left(\langle\vec{r}|\,e^{-i\alpha\vec{L}\cdot\vec{u}/\hbar}\right)\otimes\left({}_z\langle\epsilon|\,e^{-i\alpha\vec{S}\cdot\vec{u}/\hbar}\right)|\psi\rangle$$

$$e^{-i\alpha\vec{L}\cdot\vec{u}/\hbar}|\vec{r}\rangle = |\mathcal{R}\vec{r}\rangle$$

no phase

$$\langle\vec{r}|\,e^{-i\alpha\vec{L}\cdot\vec{u}/\hbar} = \langle\mathcal{R}^{-1}\vec{r}|$$

$$\mathcal{R}\vec{r} = \sum_{j,k}\vec{e}_j\,O_{jk}\,x_k$$
$$= \vec{u}(\vec{u}\cdot\vec{r}) - \cos\alpha\,\vec{u}\times(\vec{u}\times\vec{r}) + \sin\alpha\,\vec{u}\times\vec{r}$$
$$= \vec{r}\cos\alpha + (1-\cos\alpha)\,\vec{u}(\vec{u}\cdot\vec{r}) + \sin\alpha\,\vec{u}\times\vec{r}$$



$$e^{-i\alpha\vec{S}\cdot\vec{u}/\hbar}|\epsilon\rangle_z = e^{-i\vec{\sigma}\cdot\vec{u}\,\alpha/2}|\epsilon\rangle_z = e^{i\delta}|\epsilon\rangle_{\mathcal{R}\vec{e}_z}$$

$$\Longrightarrow \langle \vec{r}, \epsilon | R_{\vec{u}}(\alpha) | \psi \rangle = \left(\langle \mathcal{R}^{-1}\vec{r} | \otimes {}_{\mathcal{R}^{-1}\vec{e}_z}\langle \epsilon | \right) | \psi \rangle e^{-i\mu}$$

All this follows, up to a phase, from the fact that position and angular momentum are vector operators:

$$R^{\dagger}_{\vec{u}}(\alpha) \vec{V} R_{\vec{u}}(\alpha) = \mathcal{R}\vec{V} = \sum_k \vec{e}_j O_{jk} V_k \longleftarrow$$

How do we show this? From the commutators that imply the infinitesimal version.

$$R_{\vec{u}}(\alpha) \vec{V} R^{\dagger}_{\vec{u}}(\alpha) = \mathcal{R}^{-1}\vec{V}$$

$$R_{\vec{u}}(\alpha) V_j R^{\dagger}_{\vec{u}}(\alpha) = R_{\vec{u}}(\alpha) \vec{V} R^{\dagger}_{\vec{u}}(\alpha) \cdot \vec{e}_j = \vec{e}_j \cdot \mathcal{R}^{-1}\vec{V} = \mathcal{R}\vec{e}_j \cdot \vec{V}$$

① $$\vec{R}\, e^{-i\alpha \vec{L}\cdot\vec{u}/\hbar} |\vec{r}\rangle = e^{-i\alpha \vec{L}\cdot\vec{u}/\hbar} \mathcal{R}\vec{R} |\vec{r}\rangle = \mathcal{R}\vec{r}\, e^{-i\alpha \vec{L}\cdot\vec{u}/\hbar} |\vec{r}\rangle$$

$$\Longrightarrow e^{-i\alpha \vec{L}\cdot\vec{u}/\hbar} |\vec{r}\rangle = e^{i\nu} |\mathcal{R}\vec{r}\rangle$$

② $$(\vec{\sigma}\cdot\mathcal{R}\vec{e}_z) e^{-i\alpha \vec{S}\cdot\vec{u}/\hbar} |\epsilon\rangle_z = e^{-i\alpha \vec{S}\cdot\vec{u}/\hbar} \underbrace{(\mathcal{R}\vec{\sigma}\cdot\mathcal{R}\vec{e}_z)}_{\vec{\sigma}\cdot\vec{e}_z = \sigma_z} |\epsilon\rangle_z = \epsilon\, e^{-i\alpha \vec{S}\cdot\vec{u}/\hbar} |\epsilon\rangle_z$$

$$\Longrightarrow e^{-i\alpha \vec{S}\cdot\vec{u}/\hbar} |\epsilon\rangle_z = e^{i\delta} |\epsilon\rangle_{\mathcal{R}\vec{e}_z}$$