Phys 522

Lectures 14-16

Hydrogen spectroscopy

Hydrogen zero-order

$$H_0 = \vec{P}^2/2\mu + V(R)$$

$$\mu = \frac{m_e M_p}{m_e + M_p} = m_e \frac{1}{1 + m_e/M_p}$$

$$V(R) = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{R} = -\frac{e^2}{R} \qquad e = \frac{q}{\sqrt{4\pi\epsilon_0}}$$

$$H_0|n\ell m\rangle = E_n|n\ell m\rangle$$

$n = 1, 2, 3, \ldots$

$\ell = 0, 1, \ldots, n-1$
s p d f

$m = -\ell, \ldots, \ell$

$$\langle \vec{r}|n\ell m\rangle = R_{n\ell}(r)\, Y_\ell^m(\theta, \varphi)$$

$$E_n = -\frac{e^2}{2an^2}, \qquad a = \frac{\hbar^2}{\mu e^2}$$

$$= -\frac{13.6\ \text{eV}}{n^2} \qquad a_0 = \frac{\hbar^2}{m_e e^2} = 0.529 \times 10^{-10}\ \text{m}$$

$$= -\frac{1}{2}\mu c^2 \alpha^2 \frac{1}{n^2} \qquad \alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}$$

$$V(R) = -\frac{e^2}{a_0}\frac{a_0}{R} = -m_e c^2 \alpha^2 \frac{a_0}{R}$$

Additional effects

① Relativistic effects — relativistic mass increase, spin-orbit, Darwin term

$\to v^4/c^4$

$$\frac{\Delta E_{FS}}{|E|} \sim \frac{|E_n|}{\mu c^2} \sim \alpha^2 \sim 10^{-4} \longleftarrow \text{fine structure}$$

Treat in perturbation theory — gives physical interp that can be generalized to more complicated description

(Dirac gives fundamental description)

$$\Delta E_{FS} \sim m_e c^2 \alpha^4$$

(2) Nuclear magnetic effects — spin-spin (mag dipole)
orbit-nuclear spin

$$\Delta E_{HF} \sim \frac{m_e}{M_p} \Delta E_{FS} \longleftarrow \text{hyperfine structure}$$

S-O is electron magnetic moment interacting with magnetic field of nucleus

O-NS is nuclear magnetic moment interacting with magnetic field of electron

$$\frac{\Delta E_{HF}}{\Delta E_{FS}} \sim \frac{m_e}{M_p} \sim \frac{1}{2000}$$

(3) Lamb shift — interaction with vacuum em field (roughly Stark effect of vacuum field)

Splits $2S_{1/2}$ and $2p_{1/2}$ by $\Delta E_{LS} \sim \frac{1}{10} \Delta E_{FS}$

Fine structure:

States: $|n l m_L m_S m_I\rangle$ ← omit the $s = \frac{1}{2}$ throughout

m_I: leave out for FS; everything 2-fold degenerate

$|n l m_L m_S\rangle$ or $|n l J m_J\rangle$

$\vec{J} = \vec{L} + \vec{S}$

spectroscopic notation $n l_J$: $2S_{1/2}$, $2p_{1/2}$, $2p_{3/2}$

Hamiltonian:

$$H = m_e c^2 + H_0 + W_{FS}$$

H_0: put in reduced mass

$$W_{FS} = \underbrace{-\frac{\vec{p}^4}{8m_e^3c^2}}_{W_{mv}} + \underbrace{\frac{1}{2m_e^2c^2}\frac{1}{R}\frac{dV}{dR}\vec{L}\cdot\vec{S}}_{W_{SO}} + \underbrace{\frac{\hbar^2}{8m_e^2c^2}\nabla^2 V}_{W_D}$$

classical relativistic effects — quantum effect

W_{mv}:
$$E = (c^2p^2 + m_e^2c^4)^{1/2}$$
$$= m_ec^2\left(1+\frac{p^2}{m_e^2c^2}\right)^{1/2}$$
$$\simeq m_ec^2\left(1+\frac{p^2}{2m_e^2c^2}-\frac{1}{8}\frac{p^4}{m_e^4c^4}+\dots\right)$$
$$= m_ec^2 + \frac{p^2}{2m_e} - \frac{p^4}{8m_e^3c^2}$$

$$\frac{W_{mv}}{H_0} \sim \left(\frac{v}{c}\right)^2 \sim \alpha^2$$

relativistic increase in mass

W_{SO}: Rest frame of electron $\vec{B}' = -\frac{1}{c^2}\vec{v}\times\vec{E}$

$$\vec{M}_S = g_e \mu_B \vec{S}/\hbar = q\vec{S}/m_e$$
$$g_e \simeq 2, \quad \mu_B = \frac{q\hbar}{2m_e}$$

$$W' = -\vec{M}_S\cdot\vec{B}'$$

$$q\vec{E} = -\frac{dV}{dR}\vec{e}_r = -\frac{dV}{dR}\frac{\vec{r}}{r}$$
$$\vec{B}' = +\frac{1}{c^2}\frac{\vec{p}}{m_e}\times\frac{1}{q}\frac{dV}{dR}\frac{\vec{r}}{r} = -\frac{1}{qm_ec^2}\frac{1}{r}\frac{dV}{dr}\vec{L}$$
$$W' = -\vec{M}_S\cdot\vec{B}' = +\frac{1}{m_e^2c^2}\frac{1}{R}\frac{dV}{dR}\vec{L}\cdot\vec{S}$$

Thomas precession: rest frame rotates backwards at half the rate predicted by this torque (purely kinematic effect), so in lab frame

$$W_{SO} = \frac{1}{2m_e^2c^2}\underbrace{\frac{1}{R}\frac{dV}{dR}}_{e^2/R^3}\vec{L}\cdot\vec{S} = \frac{e^2}{2m_e^2c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S}$$

$$\frac{W_{SO}}{H_0} \sim \frac{\frac{\cancel{e^2}\hbar^2}{m_e^2c^2}\frac{1}{R^3}}{\cancel{e^2}/R} \sim \frac{\hbar^2}{m_e^2c^2}\frac{1}{a_0^2} \sim \frac{e^4}{\hbar^2c^2} = \alpha^2$$

W_D: Darwin term — quantum effect — zitterbewegung interaction of electron with field over a Compton wavelength $= \hbar/m_ec$

Slowly varying field - interacts only at quadrupole order w/ 2nd derivatives of V — diagonalizes 2nd derivative matrix, only $\nabla^2 V$

$$W_D \sim \nabla^2 V\left(\frac{\hbar}{m_ec}\right)^2$$

$$W_D = \frac{\hbar^2}{8m_e^2c^2}\underbrace{\nabla^2 V}_{} = \frac{\pi e^2\hbar^2}{2m_e^2c^2}\,\delta(\vec{R}) \longleftarrow \text{only affects S-states}$$

$$-e^2\nabla^2\left(\frac{1}{R}\right) = +e^2\,4\pi\,\delta(\vec{R})$$

$$W_D \sim \frac{e^2\hbar^2}{m_e^2c^2}\underbrace{|\psi(0)|^2}_{1/a_0^3} \sim W_{SO}$$

$$\frac{W_D}{H_0} \sim \alpha^2$$

General conclusions:

① W_{FS} is a scalar $\Leftrightarrow$ rotationally invariant $\Leftrightarrow$ conserves $\vec{J} \Leftrightarrow [W_{FS}, \vec{J}] = 0$

W_{FS}, $\vec{J}^2$, and J_z have simultaneous eigenstates. Does this mean $|n\ell j m_j\rangle$ are eigenstates of W_{FS}? No. Explain

$\langle n'l'J'm_J'|W_{FS}|nlJm_J\rangle = 0$ unless $J = J'$ and $m_J = m_J'$.

But there is more: W-E $\Rightarrow$

$$\langle n'l'J'm_J'|W_{FS}|nlJm_J\rangle = \underbrace{\langle J0m_J0|J'm_J'\rangle}_{\delta_{JJ'}\delta_{m_Jm_{J'}}} \langle n'l'J'\|W_{FS}\|nlJ\rangle$$

$$= \delta_{J'J}\delta_{m_J'm_J}\langle n'l'J\|W_{FS}\|nlJ\rangle$$

Can see this directly from commutators:

$$0 = \langle n'l'J'm_J'|[W_{FS},\vec{J}^2]|nlJm_J\rangle = \hbar^2[J(J+1)-J'(J'+1)]\langle n'l'J'm_J'|W_{FS}|nlJm_J\rangle$$

$$0 = \qquad [W_{FS},J_z] \qquad = \hbar(m_J-m_J')$$

$\Rightarrow$ matrix element vanishes unless $J = J'$ and $m_J = m_J'$.

$$0 = \langle n'l'J,m_J+1|[W_{FS},J_+]|nlJm_J\rangle = \hbar C_+(J,m_J)\langle n'l'J,m_J+1|W_{FS}|nlJ,m_J+1\rangle$$
$$- \hbar\underbrace{C_-(J,m_J+1)}_{=C_+(J,m_J)}\langle n'l'J,m_J|W_{FS}|nlJm_J\rangle$$

Same conclusion for 3 pieces of W.

We are interested in doing degenerate perturbation theory in each degenerate subspace (each $\underset{n'=n}{n}$). Still need to investigate l degeneracy.

② W_{mv} and W_D are scalars wrt $\vec{L}$ and $\vec{S}$ $\Leftrightarrow$ invariant under spatial and spin rotations separately $\Leftrightarrow$ conserve $\vec{L}$ and $\vec{S}$ $\Leftrightarrow$

$$[W_{mv},\vec{L}] = 0 = [W_D,\vec{L}]$$

$$[W_{mv},\vec{S}] = 0 = [W_D,\vec{S}]$$

$$\langle n\,l' m_L' m_S' | W_{mv}^{D} | n\,l\,m_L m_S\rangle = \delta_{m_S' m_S} \underbrace{\langle n\,l' m_L' | W_{mv}^{D} | n\,l\,m_L\rangle}_{\delta_{m_L' m_L}\,\delta_{l' l}\,\langle n l \| W_{mv}^{D} \| n l\rangle}$$

$$\boxed{\langle n\,l' m_L' m_S' | W_{mv}^{D} | n\,l\,m_L m_S\rangle = \delta_{l' l}\,\delta_{m_L' m_L}\,\delta_{m_S' m_S}\underbrace{\langle n l \| W_{mv}^{D} \| n l\rangle}_{\langle W\rangle_{nl}}}$$

$n=2$:

$$\begin{array}{c|cccccccc}
 & 00\tfrac{1}{2} & 00,-\tfrac{1}{2} & 11\tfrac{1}{2} & 11,-\tfrac{1}{2} & 10\tfrac{1}{2} & 10,-\tfrac{1}{2} & 1,-1\tfrac{1}{2} & 1,-1,-\tfrac{1}{2} \\
\hline
00\tfrac{1}{2} & \langle W\rangle_{2s} & & & & 0 & & & \\
00,-\tfrac{1}{2} & & \langle W\rangle_{2s} & & & & & & \\
11\tfrac{1}{2} & & & \langle W\rangle_{2p} & & & & & \\
11,-\tfrac{1}{2} & & & & \langle W\rangle_{2p} & & & 0 & \\
10\tfrac{1}{2} & 0 & & & & \langle W\rangle_{2p} & & & \\
10,-\tfrac{1}{2} & & & & 0 & & \langle W\rangle_{2p} & & \\
1,-1\tfrac{1}{2} & & & & & & & \langle W\rangle_{2p} & \\
1,-1,-\tfrac{1}{2} & & & & & & & & \langle W\rangle_{2p}
\end{array}$$

How does this look in the $|n l J M_J\rangle$ basis?

$$\langle n l' J' m_J' | W_{mv}^{D} | n l J m_J\rangle = \delta_{l' l}\,\delta_{J' J}\,\delta_{m_J' m_J}\,\langle W_{mv}^{D}\rangle_{nl}$$

How evaluate reduced matrix element?

(a) $$\langle W_{mv}\rangle_{nl} = \langle n l \| W_{mv} \| n l\rangle = \langle n l 0 | W_{mv} | n l 0\rangle$$

$$\frac{\vec{P}^2}{2m_e} = H_0 - V(R) \Longrightarrow \vec{P}^4 = 4m_e^2\left(H_0 - V(R)\right)^2$$

$$W_{mv} = -\frac{\vec{P}^4}{8m_e^3 c^2} = -\frac{1}{2m_e c^2}\left(H_0 - V(R)\right)^2$$

$$\langle W_{mv}\rangle_{nl} = -\frac{1}{2m_e c^2}\langle n l 0 | \left(H_0^2 - V(R) H_0 - H_0 V(R) + V^2(R)\right) | n l 0\rangle$$

$$= -\frac{1}{2m_e c^2}\left(E_n^2 - 2E_n\langle n l 0|V(R)|n l 0\rangle + \langle n l 0|V^2(R)|n l 0\rangle\right)$$

$$\langle W_{mv}\rangle_{nl} = -\frac{1}{2}m_e c^2\alpha^4\left(\frac{1}{4n^2} - \frac{1}{n^2}\left\langle\frac{a_0}{R}\right\rangle_{nl} + \left\langle\left(\frac{a_0}{R}\right)^2\right\rangle_{nl}\right)$$

$E_n = -\frac{1}{2}m_e c^2\alpha^2\frac{1}{n^2}$

$V(R) = -m_e c^2\alpha^2\frac{a_0}{R}$

$$\boxed{\langle W_{mv}\rangle_{nl} = -\frac{1}{2}m_e c^2\alpha^4\left(\frac{1}{4n^4} - \frac{1}{n^2}\left\langle\frac{a_0}{R}\right\rangle_{nl} + \left\langle\left(\frac{a_0}{R}\right)^2\right\rangle_{nl}\right)}$$

Smaller as $n\uparrow$

Why?

$$\left\langle\left(\frac{a_0}{R}\right)^k\right\rangle_{nl} = \int r^2\,dr\,d\Omega\left(\frac{a_0}{r}\right)^k \underbrace{|\psi_{nl0}(\vec{r})|^2}_{R_{nl}^2(r)\,Y_l^{0\,2}}$$

$\sim \frac{1}{n^{2k}}$

$$= \int_0^\infty dr\, r^2 R_{nl}^2(r)\left(\frac{a_0}{r}\right)^k$$

Tabulate (C-T B_{XII}):

$$\left\langle\frac{a_0}{R}\right\rangle_{1s} = 1,\quad \left\langle\frac{a_0}{R}\right\rangle_{2s} = \frac{1}{4},\quad \left\langle\frac{a_0}{R}\right\rangle_{2p} = \frac{1}{4}$$

$$\left\langle\left(\frac{a_0}{R}\right)^2\right\rangle_{1s} = 2,\quad \left\langle\left(\frac{a_0}{R}\right)^2\right\rangle_{2s} = \frac{1}{4},\quad \left\langle\left(\frac{a_0}{R}\right)^2\right\rangle_{2p} = \frac{1}{12}$$

$$\left\langle\left(\frac{a_0}{R}\right)^3\right\rangle_{2p} = \frac{1}{24}$$

$$\langle W_{mv}\rangle_{1s} = -\frac{5}{8}m_e c^2\alpha^4$$

$$\langle W_{mv}\rangle_{2s} = -\frac{13}{128}m_e c^2\alpha^4$$

$$\langle W_{mv}\rangle_{2p} = -\frac{7}{384}m_e c^2\alpha^4$$

We can get $\langle a_0/R\rangle_{nl}$ from the virial theorem

$-\langle e^2/R\rangle = \langle V(R)\rangle = 2E_n = -e^2/a_0 n^2$

$\Rightarrow \langle 1/R\rangle = 1/a_0 n^2$

$\langle a_0/R\rangle_{nl} = 1/n^2$

(b) $\langle W_D \rangle_{nl} = \langle nl \| W_D \| nl \rangle = \delta_{l0} \langle n00 | W_D | n00 \rangle$

$$\langle n00 | W_D | n00 \rangle = \frac{\pi e^2 \hbar^2}{2 m_e^2 c^2} \underbrace{|\psi_{n00}(0)|^2}_{R_{n0}^2 Y_0^{0\,2} = \frac{1}{4\pi} R_{n0}^2(0)} \sim 1/n^6$$

$$= \frac{1}{4\pi a_0^3} a_0^3 R_{n0}^2(0)$$

$$= \frac{1}{8} \underbrace{\frac{e^2 \hbar^2}{m_e^2 c^2 a_0^3}}_{m_e c^2 \alpha^4} a_0^3 R_{n0}^2(0)$$

$$\boxed{\langle W_D \rangle_{nl} = \delta_{l0} \frac{1}{8} m_e c^2 \alpha^4 \underbrace{a_0^3 R_{n0}^2(0)}}$$

Smaller as $n \uparrow$ Why?

$n=1$: 4
$n=2$: $1/2$

$$\langle W_D \rangle_{1s} = \frac{1}{2} m_e c^2 \alpha^4$$

$$\langle W_D \rangle_{2s} = \frac{1}{16} m_e c^2 \alpha^4$$

$$\langle W_D \rangle_{2p} = 0$$

(9)

(3) W_{SO} commutes with $\vec{L}^2$: $0 = [W_{SO}, \vec{L}^2]$

$$0 = \langle n l' J' m_J' | [W_{SO}, \vec{L}^2] | n l J m_J \rangle$$

$$= \hbar^2 \underbrace{[l(l+1) - l'(l'+1)]}_{} \underbrace{\langle n l' J' m_J' | W_{SO} | n l J m_J \rangle}_{=0 \text{ unless } l = l'}$$

$\langle W_{SO} \rangle_{nlJ}$ $= 0$ if $l = 0$

not diagonal in $m_L m_S$ basis

$$\boxed{\therefore \langle n l' J' m_J' | W_{SO} | n l J m_J \rangle = \delta_{ll'} \delta_{J'J} \delta_{m_J' m_J} \langle n l J \| W_{SO} \| n l J \rangle}$$

$n = 2$

$$\begin{array}{ll|cccccccc|}
 & & 0\tfrac{1}{2}\tfrac{1}{2} & 0\tfrac{1}{2},-\tfrac{1}{2} & 1\tfrac{1}{2}\tfrac{1}{2} & 1\tfrac{1}{2},-\tfrac{1}{2} & 1\tfrac{3}{2}\tfrac{3}{2} & 1\tfrac{3}{2}\tfrac{1}{2} & 1\tfrac{3}{2},-\tfrac{1}{2} & 1\tfrac{3}{2},-\tfrac{3}{2} \\
2S_{1/2} & 0\tfrac{1}{2}\tfrac{1}{2} & \langle W \rangle_{2S_{1/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0\tfrac{1}{2},-\tfrac{1}{2} & 0 & \langle W \rangle_{2S_{1/2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
2P_{1/2} & 1\tfrac{1}{2}\tfrac{1}{2} & 0 & 0 & \langle W \rangle_{2P_{1/2}} & 0 & 0 & 0 & 0 & 0 \\
 & 1\tfrac{1}{2},-\tfrac{1}{2} & 0 & 0 & 0 & \langle W \rangle_{2P_{1/2}} & 0 & 0 & 0 & 0 \\
2P_{3/2} & 1\tfrac{3}{2}\tfrac{3}{2} & 0 & 0 & 0 & 0 & \langle W \rangle_{2P_{3/2}} & 0 & 0 & 0 \\
 & 1\tfrac{3}{2}\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & \langle W \rangle_{2P_{3/2}} & 0 & 0 \\
 & 1\tfrac{3}{2},-\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \langle W \rangle_{2P_{3/2}} & 0 \\
 & 1\tfrac{3}{2},-\tfrac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \langle W \rangle_{2P_{3/2}}
\end{array}$$

$\delta_{ll'}$ essential to making this block zero.

Same is true for each piece of W

mv: $2S_{1/2} = 2S$ $\quad 2P_{1/2} = 2P_{3/2} = 2P$

D: " $\quad$ " $= 0$

SO: $2S_{1/2} = 0$

Same is true for W_{FS} : $\langle W_{FS} \rangle_{2S_{1/2}}, \langle W_{FS} \rangle_{2P_{1/2}}, \langle W_{FS} \rangle_{2P_{3/2}}$ are the energy level shifts for $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$ (remaining degeneracy in m_J); $|n l J m_J\rangle$ are appropriate zero-order eigenstates.

Need $\langle W_{so}\rangle_{n\ell J} = \langle n\ell J m_J | W_{so} | n\ell J m_J\rangle$

$$= \underbrace{\frac{1}{2}\frac{e^2\hbar^2}{m_e^2c^2}\frac{1}{a_0^3}}_{m_e c^2\alpha^4} \langle n\ell J m_J | \left(\frac{a_0}{R}\right)^3 \frac{\vec{L}\cdot\vec{S}}{\hbar^2} | n\ell J m_J\rangle$$

$$= \frac{1}{2} m_e c^2 \alpha^4 \langle n\ell J m_J | \left(\frac{a_0}{R}\right)^3 \frac{\vec{L}\cdot\vec{S}}{\hbar^2} | n\ell J m_J\rangle$$

$$\vec{J} = \vec{L} + \vec{S} \Rightarrow \vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$$

Same trick we used for the Landé g-factor

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$\frac{1}{2}\left[J(J+1) - \ell(\ell+1) - \frac{3}{4}\right] \underbrace{\langle n\ell J m_J | \left(\frac{a_0}{R}\right)^3 | n\ell J m_J\rangle}_{\left\langle \left(\frac{a_0}{R}\right)^3 \right\rangle_{n\ell}}$$

$$\boxed{\langle W_{so}\rangle_{n\ell J} = \frac{1}{4} m_e c^2 \alpha^4 \underbrace{\left[J(J+1) - \ell(\ell+1) - \frac{3}{4}\right]}_{= \begin{cases} \ell, & J = \ell + 1/2 \\ -(\ell+1), & J = \ell - 1/2 \end{cases}} \underbrace{\left\langle \left(\frac{a_0}{R}\right)^3 \right\rangle_{n\ell}}_{\sim \frac{1}{n^6}}}$$

$$\langle W_{so}\rangle_{1S_{1/2}} = 0$$

$$\langle W_{so}\rangle_{2S_{1/2}} = 0$$

$$\langle W_{so}\rangle_{2P_{1/2}} = -\frac{1}{48} m_e c^2 \alpha^4$$

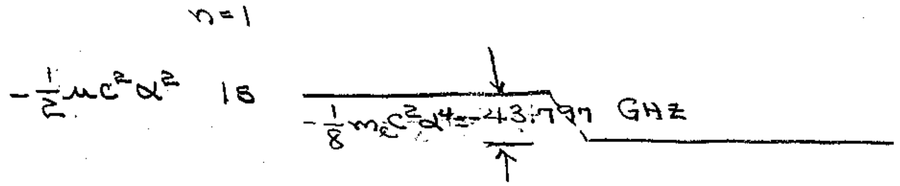
$$\langle W_{so}\rangle_{2P_{3/2}} = \frac{1}{96} m_e c^2 \alpha^4$$

$m_e c^2 \alpha^4 / h = 350.37$ GHz (11)

Gather results: $m_e = 9.1095 \times 10^{-31}$ kg $\quad \alpha = 1/137.036$

$c = 2.9979 \times 10^{8}$ m/s $\quad h = 6.6262 \times 10^{-34}$ J·s

$$1S_{1/2}: \quad \langle W_{FS} \rangle_{1S_{1/2}} = \underset{-\frac{5}{8}\alpha^2}{\langle W_{mv} \rangle_{1S}} + \underset{\frac{1}{2}\alpha^2}{\langle W_D \rangle_{1S}} + \underset{0}{\langle W_{SO} \rangle_{1S_{1/2}}} = -\frac{1}{8}$$



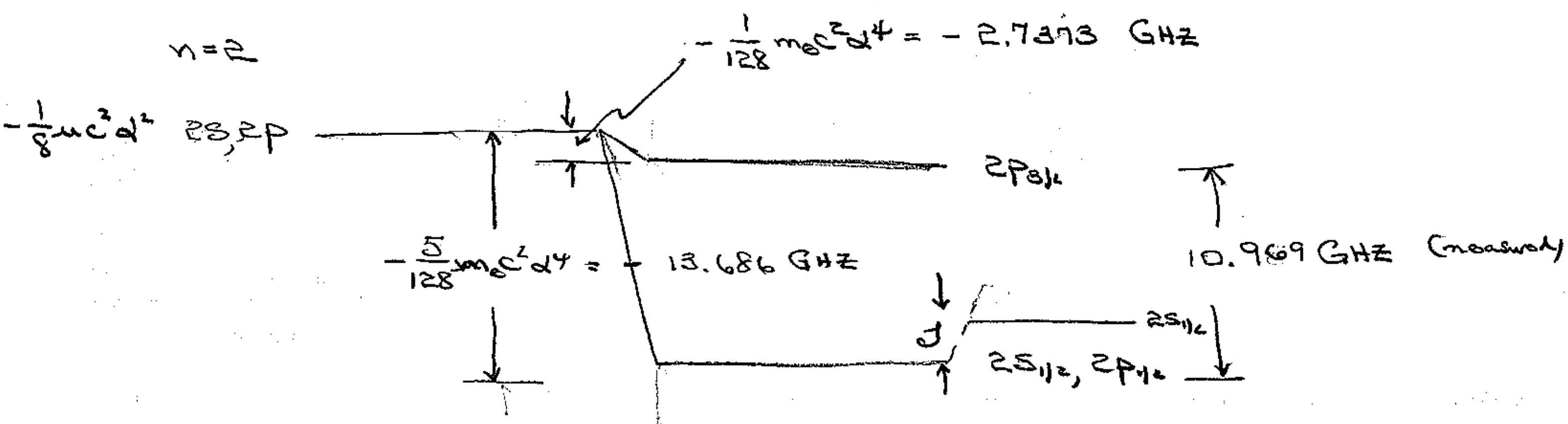
$$2S_{1/2}: \quad \langle W_{FS} \rangle_{2S_{1/2}} = \underset{-\frac{13}{128}}{\langle W_{mv} \rangle_{2S}} + \underset{\frac{1}{16} = \frac{8}{128}}{\langle W_D \rangle_{2S}} + \underset{0}{\langle W_{SO} \rangle_{2S_{1/2}}} = -\frac{5}{128}$$

accidental

$$2P_{1/2}: \quad \langle W_{FS} \rangle_{2P_{1/2}} = \underset{-\frac{7}{384}}{\langle W_{mv} \rangle_{2P}} + \underset{0}{\langle W_D \rangle_{2P}} + \underset{-\frac{1}{48} = -\frac{8}{384}}{\langle W_{SO} \rangle_{2P_{1/2}}} = -\frac{15}{384} = -\frac{5}{128}$$

Spin-orbit split

$$2P_{3/2}: \quad \langle W_{FS} \rangle_{2P_{3/2}} = \underset{-\frac{7}{384}}{\langle W_{mv} \rangle_{2P}} + \underset{0}{\langle W_D \rangle_{2P}} + \underset{\frac{1}{96}}{\langle W_{SO} \rangle_{2P_{3/2}}} = -\frac{3}{384} = -\frac{1}{128}$$



Lamb shift $\mathcal{J} = 1.058$ GHz