Phys 522

Time-dependent perturbation theory

Lectures 17-21

The problem

System: unperturbed Hamiltonian H_0

$$H_0|\varphi_n\rangle = E_n|\varphi_n\rangle$$

why t-d

Perturbed Hamiltonian $H(t) = H_0 + \underbrace{W(t)}_{\lambda \hat{W}(t)}$

Initial state: $|\psi(0)\rangle = |\varphi_i\rangle$

IC:

$$\xrightarrow[SE]{} |\psi(t)\rangle = \sum_n c_n(t)|\varphi_n\rangle \quad , \quad c_n(0) = \delta_{ni}$$

Transition probability: $P_{i\to f}(t) = |c_f(t)|^2 = |\langle\varphi_f|\psi(t)\rangle|^2$

Schrödinger equation: $c_n(t) = b_n(t)\, e^{-i\omega_n t}$, $\omega_n = E_n/\hbar$

why? IC: $b_n(0) = \delta_{ni}$

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H(t)|\psi(t)\rangle$$

$$\frac{d|\psi(t)\rangle}{dt} = \sum_n \dot{c}_n|\varphi_n\rangle = \sum_n (\dot{b}_n - i\omega_n b_n)\, e^{-i\omega_n t}|\varphi_n\rangle$$

$$-\frac{i}{\hbar}H(t)|\psi(t)\rangle = \sum_n \left(-i\omega_n - \frac{i}{\hbar}W\right) b_n e^{-i\omega_n t}|\varphi_n\rangle$$

$$\Longrightarrow \sum_n \dot{b}_n e^{-i\omega_n t}|\varphi_n\rangle = \sum_k -\frac{i}{\hbar} b_k e^{-i\omega_k t}\, W|\varphi_k\rangle$$

$$\Longrightarrow \dot{b}_n e^{-i\omega_n t} = \sum_k -\frac{i}{\hbar} b_k e^{-i\omega_k t} \underbrace{\langle \varphi_n | W | \varphi_k \rangle}_{W_{nk}}$$

$$\boxed{i\hbar \dot{b}_n = \sum_k W_{nk}(t)\, e^{i\omega_{nk} t}\, b_k \quad , \quad \omega_{nk} = \omega_n - \omega_k}$$

Think about structure

driving transitions, etc.

Perturbation expansion:

Write $b_n(t) = b_n^{(0)}(t) + \lambda b_n^{(1)}(t) + \lambda^2 b_n^{(2)}(t) + \dots$

Easy: $$\boxed{i\hbar \dot{b}_n^{(r)} = \sum_k \hat{W}_{nk}(t)\, e^{i\omega_{nk} t}\, b_k^{(r-1)}}$$

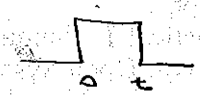
$r=0$: $i\hbar \dot{b}_n^{(0)} = 0$

Solution:

① $b_n^{(0)}(t) = \delta_{ni}$

② $i\hbar \dot{b}_n^{(1)} = \hat{W}_{ni}(t)\, e^{i\omega_{ni} t}$

$$\Longrightarrow \boxed{b_n^{(1)}(t) = -\frac{i}{\hbar} \underbrace{\int_0^t dt'\, \hat{W}_{ni}(t')\, e^{i\omega_{ni} t'}}_{\text{FT of } B(t')\hat{W}_{ni}(t')}}$$



"spatial" and temporal overlap

1st-order normalization:

$$|\psi(t)\rangle = \sum_n b_n(t)\, e^{-i\omega_n t} |\varphi_n\rangle$$

$$= e^{-i\omega_i t} |\varphi_i\rangle + \lambda \sum_n b_n^{(1)}(t)\, e^{-i\omega_n t} |\varphi_n\rangle$$

$$= e^{-i\omega_i t} \left(1 + \lambda b_i^{(1)}(t)\right) |\varphi_i\rangle + \lambda \sum_{n \neq i} b_n^{(1)}(t)\, e^{-i\omega_n t}$$

$\uparrow -\frac{i}{\hbar}\int_0^t dt'\, \hat{W}_{ii}(t')$ pure imaginary

First-order corrections contribute at 2nd order to $\langle\psi|\psi\rangle$.

First-order transition probability:

$$P_{i\to f}(t) = |\langle \varphi_f | \psi(t)\rangle|^2 = |b_f(t)|^2 = |\lambda b_f^{(1)}(t)|^2$$

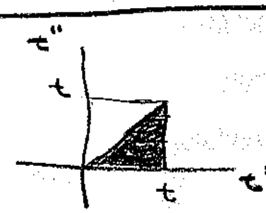
$f \neq i$:

$$\boxed{P_{i\to f}(t) = \frac{1}{\hbar^2}\left|\int_0^t dt'\, W_{ni}(t')\, e^{i\omega_{ni}t}\right|^2}$$

3 $i\hbar\, \dot{b}_n^{(2)} = \sum_k \hat{W}_{nk}(t)\, e^{i\omega_{nk}t}\, b_k^{(1)}(t)$

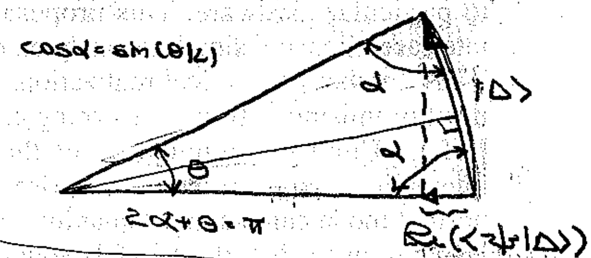
$$\boxed{\begin{aligned} b_n^{(2)}(t) &= -\frac{i}{\hbar}\sum_k \int_0^t dt'\, \hat{W}_{nk}(t')\, e^{i\omega_{nk}t'}\, b_k^{(1)}(t') \\ &= \left(-\frac{i}{\hbar}\right)^2 \sum_k \int_0^t dt'\, \hat{W}_{nk}(t')\, e^{i\omega_{nk}t'} \\ &\qquad \times \int_0^{t'} dt''\, \hat{W}_{ki}(t'')\, e^{i\omega_{ki}t''} \end{aligned}}$$

2nd-order transitions
(coherent; how different from
2 1st-order transitions)
virtual intermediate state



(A)

$$(\langle\psi| + \langle\Delta|)(|\psi\rangle + |\Delta\rangle)$$
$$= 1 + 2\mathrm{Re}(\langle\psi|\Delta\rangle) + \langle\Delta|\Delta\rangle$$



$$|\psi(t)\rangle = \sum_n b_n(t) e^{-i\omega_n t} |\varphi_n\rangle$$

$$1 = \langle\psi(t)|\psi(t)\rangle = \sum_n |b_n(t)|^2$$

$$= |1 + \delta b_i(t)|^2 + \sum_{n\neq i} |\delta b_n(t)|^2$$

$$= 1 + 2\mathrm{Re}(\delta b_i(t)) + \sum_n |\delta b_n(t)|^2$$

$$1 = \langle\psi|\psi\rangle \iff \boxed{\mathrm{Re}(\delta b_i(t)) = -\frac{1}{2}\sum_n |\delta b_n(t)|^2}$$ Optical theorem

$$\mathrm{Re}\langle\psi|\Delta\rangle = -\langle\Delta|\Delta\rangle^{1/2}\cos\alpha$$
$$\frac{1}{2}\langle\Delta|\Delta\rangle^{1/2} = \sin(\theta/2)$$
$$\mathrm{Re}\langle\psi|\Delta\rangle = -\frac{1}{2}\langle\Delta|\Delta\rangle$$

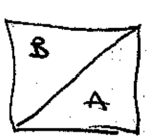
To 2nd order:

$$\sum_n |\delta b_n(t)|^2 = \sum_n \lambda^2 |b_n^{(1)}(t)|^2$$

$$= \frac{1}{\hbar^2}\left|\int_0^t dt'\, W_{ni}(t') e^{-i\omega_{ni}t'}\right|^2$$

$$\mathrm{Re}(\delta b_i(t)) = \lambda^2 \mathrm{Re}(b_i^{(2)}(t))$$

$$= -\frac{1}{\hbar^2}\mathrm{Re}\left(\sum_k \int_0^t dt'\, W_{ik}(t') e^{i\omega_{ik}t'} \times \int_0^{t'} dt''\, W_{ki}(t'') e^{i\omega_{ki}t''}\right)$$



Re is $\frac{1}{2}$ sum of A and B

$$= -\frac{1}{2\hbar^2}\sum_k \left|\int_0^t dt'\, W_{ki}(t') e^{i\omega_{ki}t'}\right|^2$$

special observable - yes-no

Pictures:

$$P_{a_i} = \sum_k |a_i,k\rangle\langle a_i,k|$$

State: $|\psi\rangle$

Observable: $A = \sum_{i,k} a_i |a_i,k\rangle\langle a_i,k| = \sum_i a_i P_{a_i}$

Representation of state: $|\psi\rangle = \sum_{a_i,k} |a_i,k\rangle\langle a_i,k|\psi\rangle = \sum_i P_{a_i}|\psi\rangle$

Interpretation:

$$\mathcal{P}(a_i) = \sum_k \mathcal{P}(a_i,k) = \sum_k |\langle a_i,k|\psi\rangle|^2$$

$$= \sum_k \langle\psi|a_i,k\rangle\langle a_i,k|\psi\rangle$$

$$= \langle\psi|P_{a_i}|\psi\rangle$$

$$\boxed{\mathcal{P}(a_i) = \langle\psi|P_{a_i}|\psi\rangle}$$ ← special yes-no observable

$$\boxed{\langle A\rangle = \sum_i a_i \mathcal{P}(a_i) = \langle\psi|A|\psi\rangle}$$

Equivalent description: unitary operator V

$$|\overline{\psi}\rangle = V|\psi\rangle \quad \text{and} \quad \overline{A} = VAV^\dagger$$ ← preserves commutators

$\overline{|a_i,k\rangle}$ (eigenvectors of $\overline{A}$)

$$\Longrightarrow \langle\overline{\psi}|\overline{A}|\overline{\psi}\rangle = \langle\psi|A|\psi\rangle$$

Representation: $\overline{A} = \sum_{i,k} a_i V|a_i,k\rangle\langle a_i,k|V^\dagger = \sum_i a_i \overline{P}_{a_i}$

$$|\overline{\psi}\rangle = \sum_{i,k} \overline{|a_i,k\rangle}\,\overline{\langle a_i,k|}\overline{\psi}\rangle = \sum_i \overline{P}_{a_i}|\overline{\psi}\rangle$$

eigenvalues of $\overline{A}$

$$\mathcal{P}(a_i) = \sum_k |\overline{\langle a_i,k|}\overline{\psi}\rangle|^2 = \langle\overline{\psi}|\overline{P}_{a_i}|\overline{\psi}\rangle$$

Example: rotation R $\qquad |\psi'\rangle = R|\psi\rangle$

$\uparrow$ rotated state

$$|\bar{\psi}\rangle = \underbrace{R^\dagger}|\psi'\rangle = |\psi\rangle$$

$$\bar{A} = R^\dagger A R$$

Time development and pictures

Schrödinger picture: SP

Hamiltonian $H_S(t)$

Examples: $H_0 \leftarrow$ t.i. special representation

$$H_0|E_n\rangle = E_n|E_n\rangle$$

$$H(t) = H_0 + W(t)$$

States: $|\psi_S(t)\rangle$

$$\text{SE:} \quad i\hbar \frac{d|\psi_S(t)\rangle}{dt} = H(t)|\psi_S(t)\rangle \qquad i\hbar\frac{d\langle\psi_S(t)|}{dt} = -\langle\psi_S(t)|H_S(t)$$

Observables: $A_S(t) \longleftarrow$ generally no t-d but notice $H_S(t)$

$$\text{Interp:} \quad \langle A\rangle_t = \langle\psi_S(t)|A_S(t)|\psi_S(t)\rangle \Rightarrow i\hbar\frac{d\langle A\rangle_t}{dt} = \langle[A_S, H_S]\rangle_t + i\hbar\frac{\partial\langle A\rangle_t}{\partial t}$$

Energy rep: $(H_S(t) = H_0)$

$$|\psi_S(t)\rangle = \sum_n |E_n\rangle\underbrace{\langle E_n|\psi_S(t)\rangle}_{e^{-\frac{i}{\hbar}E_n t}\langle E_n|\psi_S(0)\rangle} = \sum_n |E_n\rangle e^{-\frac{i}{\hbar}E_n t}\langle E_n|\psi_S(0)\rangle$$

$$A_S = \sum_{nm} |E_n\rangle\langle E_n|A_S|E_m\rangle\langle E_m|$$

$$\langle A\rangle_t = \sum_{n,m} e^{\frac{i}{\hbar}(E_n - E_m)t} \times \langle\psi_S(0)|E_n\rangle\langle E_n|A_S|E_m\rangle\langle E_m|\psi_S(0)\rangle$$

Time-evolution operator: $|\psi_S(t)\rangle = \mathcal{U}(t,t_0)|\psi_S(t_0)\rangle$

$$i\hbar\frac{d\,\mathcal{U}(t,t_0)}{dt} = H_S(t)\,\mathcal{U}(t,t_0), \qquad \mathcal{U}(t_0,t_0) = 1$$

Other pictures: $|\bar{\psi}(t)\rangle = V(t)|\psi_S(t)\rangle$

$$\bar{A}(t) = V(t) A_S V^\dagger(t)$$

Any $V(t)$ will do. The t-dep makes it interesting

Heisenberg picture: HP $\qquad V(t) = U^\dagger(t,t_0)$

States: $|\psi_H(t)\rangle = U^\dagger(t,t_0)|\psi_S(t)\rangle = |\psi_S(t_0)\rangle$ no t-d

Observables: $A_H(t) = U^\dagger(t,t_0) A_S(t) U(t,t_0)$

HE: $i\hbar \dfrac{dA_H}{dt} = [A_H, H_H] + i\hbar \dfrac{\partial A_H}{\partial t}$

What if $A_H = H_H$? Hamiltonian is special.

A_S t-i: $A_S = \sum_{n,k} a_i |a_i,k\rangle\langle a_i,k|$

$$A_H(t) = \sum_{n,k} a_i \underbrace{U^\dagger |a_i,k\rangle}_{\text{eigenstates of } A_H(t)} \langle a_i,k| U$$

Ex: HO

Interp: $\langle A\rangle_t = \langle \psi_H(t)| A_H(t) |\psi_H(t)\rangle$ $\Rightarrow i\hbar \dfrac{d\langle A\rangle_t}{dt}$

Energy rep: $H_S = H_0 \Rightarrow U(t,t_0) = e^{-\frac{i}{\hbar}H_0(t-t_0)}$

$\Rightarrow H_H(t) = H_S = H_0$ and $H_H|E_n\rangle = E_n|E_n\rangle$

$$|\psi_H(t)\rangle = \sum_n |E_n\rangle\langle E_n|\psi_S(t_0)\rangle$$

$$A_H(t) = \sum_{n,m} e^{\frac{i}{\hbar}(E_n-E_m)t} |E_n\rangle\langle E_n| A_S |E_m\rangle\langle E_m|$$

$$\langle A\rangle_t = \sum_{n,m} e^{\frac{i}{\hbar}(E_n-E_m)t} \langle \psi_S(0)|E_n\rangle\langle E_n| A_S |E_m\rangle\langle E_m|\psi_S(0)\rangle$$

Interaction picture: IP

$H(t) = H_0 + W(t)$

$V(t) = U_0^\dagger(t,t_0) = e^{\frac{i}{\hbar}H_0(t-t_0)}$

States: $|\psi_I(t)\rangle = U_0^\dagger(t,t_0)|\psi_S(t)\rangle$

Observables: $A_I(t) = U_0^\dagger(t,t_0)\, A_S(t)\, U_0(t,t_0)$

$$\boxed{\begin{aligned} i\hbar \frac{d|\psi_I(t)\rangle}{dt} &= W_I(t)|\psi_I(t)\rangle \\ i\hbar \frac{dA_I}{dt} &= [A_I, H_0] + i\hbar\frac{\partial A_I}{\partial t} \end{aligned}}$$

Perturbation evolves state vector

Unperturbed H evolves operators

$$A_I(t) = \sum_{n,m} e^{\frac{i}{\hbar}(E_n-E_m)t}\,|E_n\rangle\langle E_n|A_S|E_m\rangle\langle E_m| \quad \leftarrow$$ unperturbed t-d

Interp: $\langle A\rangle_t = \langle\psi_I(t)|A_I(t)|\psi_I(t)\rangle$

$$i\hbar\frac{dU_I(t,t_0)}{dt} = W_I(t)\,U_I(t,t_0), \qquad U_I(t_0,t_0) = 1$$

$$\Longrightarrow |\psi_I(t)\rangle = U_I(t,t_0)|\psi_I(t_0)\rangle = U_I(t,t_0)|\psi_S(t_0)\rangle$$

$$|\psi_S(t)\rangle = \boxed{\underbrace{U_0(t,t_0)\,U_I(t,t_0)}_{U(t,t_0)}}\,|\psi_S(t_0)\rangle$$

$$\boxed{|\psi_S(t)\rangle = \sum_n |E_n\rangle\langle E_n|\psi_S(t)\rangle = \sum_n |E_n\rangle\, e^{-\frac{i}{\hbar}E_n t}\,\underbrace{\langle E_n|U_I(t,t_0)|\psi_S(t_0)\rangle}_{b_n(t)}}$$

$$\boxed{\langle A\rangle_t = \sum_{nm} e^{\frac{i}{\hbar}(E_n-E_m)t}\, b_n^*(t)\, b_m(t)\,\langle E_n|A_S|E_m\rangle}$$

Dyson expansion:

$$H = H_0 + W(t)$$

$$U_0(t, t_0=0) = U_0(t) = e^{-\frac{i}{\hbar} H_0 t}$$

$$W_I(t) = U_0^\dagger(t)\, W(t)\, U_0(t) = e^{\frac{i}{\hbar} H_0 t}\, W(t)\, e^{-\frac{i}{\hbar} H_0 t}$$

↑ this is t-d even if W is constant

$$i\hbar \frac{d U_I(t)}{dt} = W_I(t)\, U_I(t)$$

$$\boxed{U_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt'\, W_I(t')\, U_I(t')}$$

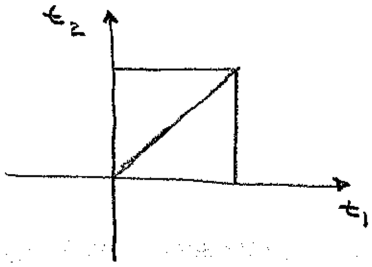
Integral equation for $U_I(t)$

Solve iteratively:

$$U_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt_1\, W_I(t_1)$$
$$\times \left(1 - \int_0^{t_1} dt_2\, W_I(t_2)\right.$$
$$\times \left(1 - \int_0^{t_2} dt_3\, W_I(t_3)\right.$$
$$\times \cdots$$

3rd-order:

$$U_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt_1\, W_I(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1\, W_I(t_1) \int_0^{t_1} dt_2\, W_I(t_2)$$
$$+ \left(-\frac{i}{\hbar}\right)^3 \int_0^t dt_1\, W_I(t_1) \int_0^{t_2} dt_2\, W_I(t_2) \int_0^{t_3} dt_3\, W_I(t_3)$$



$$\int_0^t dt_1\, W_I(t_1) \int_0^{t_1} dt_2\, W_I(t_2)$$

$$= \frac{1}{2} \mathbf{T} \int_0^t dt_1 \int_0^t dt_2\, W_I(t_1)\, W_I(t_2)$$

$$= \frac{1}{2} \mathbf{T} \left(\int_0^t dt'\, W_I(t') \right)^2$$

rth order

$$\int_0^t dt_1\, W_I(t_1) \int_0^{t_1} dt_2\, W_I(t_2) \dots \int_0^{t_{r-1}} dt_r\, W_I(t_r)$$

$$= \frac{1}{r!} \mathbf{T} \int_0^t dt_1 \int_0^t dt_2 \dots \int_0^t dt_r\, W_I(t_1) \dots W_I(t_r)$$

$$= \frac{1}{r!} \mathbf{T} \left(\int_0^t dt'\, W_I(t') \right)^r$$

$$\boxed{U_I(t) = \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{T} \left(-\frac{i}{\hbar} \int_0^t dt'\, W_I(t') \right)^r = \mathbf{T} \exp\left(-\frac{i}{\hbar} \int_0^t dt'\, W_I(t') \right)}$$

Solved problem independent of IC

$$C_n(t) = \langle \varphi_n | \overbrace{U(t)}^{U_0(t)\, U_I(t)} | \psi_S(0) \rangle = e^{-\frac{i}{\hbar} E_n t} \underbrace{\langle \varphi_n | U_I(t) | \varphi_i \rangle}_{b_n(t)}$$

$$b_n(t) = \delta_{ni} - \frac{i}{\hbar} \int_0^t dt_1\, \langle \varphi_n | W_I(t_1) | \varphi_i \rangle$$
$$+ \left(-\frac{i}{\hbar}\right)^2 \sum_k \int_0^t dt_1\, \langle \varphi_n | W_I(t_1) | \varphi_k \rangle \int_0^{t_1} dt_2\, \langle \varphi_k | W_I(t_2) | \varphi_i \rangle$$
$$+ \left(-\frac{i}{\hbar}\right)^3 \sum_{k,l} \int_0^t dt_1\, \langle \varphi_n | W_I(t_1) | \varphi_k \rangle \int_0^{t_1} dt_2\, \langle \varphi_k | W_I(t_2) | \varphi_l \rangle$$
$$\times \int_0^{t_2} dt_3\, \langle \varphi_l | W_I(t_3) | \varphi_i \rangle$$

$$\langle \varphi_k | W_I(t) | \varphi_l \rangle = e^{i\omega_{kl} t}\, W_{kl}(t)$$

Two-level system with sinusoidal perturbation

$$H = \underbrace{E_g|g\rangle\langle g| + E_e|e\rangle\langle e|}_{} + C\big(\underbrace{\overbrace{|e\rangle\langle g|}^{\sigma_+} + \overbrace{|g\rangle\langle e|}^{\sigma_-}}_{}\big)\cos\omega t$$

$$\frac{1}{2}(E_g+E_e)(|e\rangle\langle e| + |g\rangle\langle g|)$$

$$+\frac{1}{2}(E_e-E_g)(|e\rangle\langle e| - |g\rangle\langle g|) \rightarrow \frac{1}{2}\hbar\omega_0\sigma_3$$

irrelevant: choose $E_e = -E_g = \frac{1}{2}\hbar\omega_0$

$$\omega_{eg} = \frac{E_e - E_g}{\hbar} = \omega_0$$

$\sigma_1 = \sigma_+ + \sigma_-$

$\sigma_2 = -i|e\rangle\langle g| + i|g\rangle\langle e|$

$= -i(\sigma_+ - \sigma_-)$

$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$[\sigma_\pm, \sigma_3] = \frac{1}{2}\big(\underbrace{[\sigma_1,\sigma_3]}_{-2i\sigma_2} \pm i\underbrace{[\sigma_2,\sigma_3]}_{2i\sigma_1}\big)$

$= \mp(\sigma_1 \pm i\sigma_2)$

$= \mp 2\sigma_\pm$

$$H = \underbrace{\frac{1}{2}\hbar\omega_0\sigma_3}_{H_0} + \underbrace{C\sigma_1\cos\omega t}_{W(t)}$$

1st order perturbation theory: $|\psi(0)\rangle = |g\rangle$

$$b_e(t) = -\frac{i}{\hbar}\int_0^t dt'\, W_{eg}(t')\, e^{i\omega_{eg}t'}$$

$$= -\frac{i}{\hbar}\int_0^t dt'\, C\,\underbrace{\cos\omega t'\, e^{i\omega_0 t'}}_{\frac{1}{2}\big(\underbrace{e^{i(\omega+\omega_0)t'}}_{\text{antiresonant}} + \underbrace{e^{-i(\omega-\omega_0)t'}}_{\text{resonant}}\big)}$$

Assume $|\omega-\omega_0| \ll \omega_0$: drop antiresonant term

$$= -\frac{i}{2\hbar}C\int_0^t dt'\, e^{-i(\omega-\omega_0)t'}$$

$$= \frac{C}{2\hbar}\,\frac{e^{-i(\omega-\omega_0)t} - 1}{(\omega-\omega_0)}$$

$$= \frac{C}{2\hbar}\, e^{-i(\omega-\omega_0)t/2}\left(\frac{-2i\sin(\omega-\omega_0)t/2}{\omega-\omega_0}\right)$$

$$= -i\Omega\, e^{-i(\omega-\omega_0)t/2}\,\frac{\sin(\omega-\omega_0)t/2}{(\omega-\omega_0)/2} \qquad \frac{C}{2\hbar} = \Omega$$

$$c_e(t) = e^{-i\omega_0 t/2}\, b_e(t) = -i\Omega\, e^{-i\omega t/2}\,\frac{\sin(\omega-\omega_0)t/2}{(\omega-\omega_0)/2}$$

$$|b_e(t)|^2 = \Omega^2 \left(\frac{\sin (\omega-\omega_0)t/2}{(\omega-\omega_0)/2} \right)^2 \xrightarrow[(\omega-\omega_0)t/2 \ll 1]{} (\Omega t)^2$$

Right for long times only if $\left(\frac{\Omega}{\omega-\omega_0}\right)^2 \ll 1$

Exact treatment in RWA:

$$H = \tfrac{1}{2}\hbar\omega_0\sigma_3 + 2\hbar\Omega\sigma_1 \cos\omega t \qquad (C = 2\hbar\Omega)$$

$$= \hbar\Omega(\sigma_+ + \sigma_-)(e^{i\omega t} + e^{-i\omega t})$$

$$= \hbar\Omega\Big(\underbrace{\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}}_{\text{resonant}} + \underbrace{\sigma_+ e^{+i\omega t} + \sigma_- e^{-i\omega t}}_{\text{antiresonant}}\Big)$$

$\omega - \omega_0 \equiv \Delta =$ (detuning) drop: RWA

$$H = \tfrac{1}{2}\hbar\omega_0\sigma_3 + \hbar\Omega(\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t})$$

IP 1: $H_0 = \tfrac{1}{2}\hbar\omega_0\sigma_3$, $W(t) = \hbar\Omega(\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t})$

$$W_I(t) = e^{\frac{i}{\hbar}H_0 t}\, W(t)\, e^{-\frac{i}{\hbar}H_0 t}$$

$$(\sigma_\pm)_I = e^{\frac{i}{\hbar}H_0 t}\sigma_\pm e^{-\frac{i}{\hbar}H_0 t} = e^{\pm i\omega_0 t}\sigma_\pm$$

$$i\hbar \frac{d(\sigma_\pm)_I}{dt} = [(\sigma_\pm)_I, H_0] = \tfrac{1}{2}\hbar\omega_0 \underbrace{[(\sigma_\pm)_I, (\sigma_3)_I]}_{\mp 2\sigma_\pm} = \mp\hbar\omega_0(\sigma_\pm)_I$$

$$\frac{(d\sigma_\pm)_I}{dt} = \pm i\omega_0(\sigma_\pm)_I$$

$$W_I(t) = \hbar\Omega(\sigma_+ e^{-i\Delta t} + \sigma_-^{\;i\Delta t}) \longleftarrow \text{Still a function of } t$$

IP 2: $H_0 = \frac{1}{2}\hbar\omega\sigma_3$, $W(t) = -\frac{1}{2}\hbar\Delta\sigma_3 + \hbar\Omega(\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t})$

$\uparrow$ pretend ω is Bohr frequency

$$W_I(t) = e^{\frac{i}{\hbar}H_0 t} W(t)\, e^{-\frac{i}{\hbar}H_0 t} = -\frac{1}{2}\hbar\Delta\sigma_3 + \hbar\Omega\underbrace{(\sigma_+ + \sigma_-)}_{\sigma_1}$$

$\Omega' = (\Omega^2 + \Delta^2/4)^{1/2}$

$u_1 = \frac{\Omega}{\Omega'}$, $u_3 = -\frac{\Delta/2}{\Omega'}$

$$= \hbar\Omega'\,\vec{u}\cdot\vec{\sigma}$$

$$U_I(t) = e^{-\frac{i}{\hbar}W_I t} = \underbrace{e^{-i\Omega' t\,\vec{u}\cdot\vec{\sigma}}}_{\substack{\text{rotation by } 2\Omega' t \\ \text{about } \vec{u} = \frac{1}{\Omega'}(\Omega\vec{e}_1 - (\Delta/2)\vec{e}_3)}} = \cos\Omega' t\, 1 - i\vec{u}\cdot\vec{\sigma}\sin\Omega' t$$

$$U_0(t) = e^{-\frac{i}{\hbar}H_0 t} = e^{-i(\omega_0 t/2)\sigma_3} \cdot \begin{pmatrix}\text{rotation by } \omega_0 t \\ \text{about } \vec{e}_3\end{pmatrix}$$

$|\psi(0)\rangle = |g\rangle$

$|\psi(t)\rangle = U_0(t)\, U_I(t)\, |g\rangle$

$$\langle e|\psi(t)\rangle = \underbrace{\langle e|U_0(t)\, U_I(t)|g\rangle}_{c_e(t)} = e^{-i\omega t/2}\underbrace{\langle e|U_I(t)|g\rangle}_{b_e(t)}$$

$$b_e(t) = \langle e|U_I(t)|g\rangle = -i u_1 \sin\Omega' t\, \underbrace{\langle e|\sigma_1|g\rangle}_{1}$$

$$b_e(t) = -i\Omega\,\frac{\sin\Omega' t}{\Omega'}\,, \qquad c_e(t) = -i\Omega' e^{-i\omega t/2}\,\frac{\sin\Omega' t}{\Omega'}$$

agrees with perturbation for small times $\Omega t' \ll 1$
and at all times for large detuning $|\Delta| \gg \Omega$ ($\Omega' \approx \Delta/2$)

$$\boxed{|b_e(t)|^2 = \left(\frac{\Omega}{\Omega'}\right)^2 \sin^2\Omega' t = \frac{1}{2}\left(\frac{\Omega}{\Omega'}\right)^2\left(1 - \cos 2\Omega' t\right)}$$

Resonant behavior: $\Delta = 0 \Rightarrow \Omega' = \Omega$

Full solution:

① $|\psi(0)\rangle = |g\rangle$

$$|\psi_I(t)\rangle = U_I(t)|g\rangle = \cos\Omega' t\,|g\rangle - i\sin\Omega' t\,\Big(\underbrace{u_3\sigma_3|g\rangle}_{= -u_3|g\rangle = \frac{\Delta}{2\Omega'}|g\rangle} + \underbrace{u_1\sigma_1|g\rangle}_{= u_1|e\rangle = \frac{\Omega}{\Omega'}|e\rangle}\Big)$$

$$\boxed{\begin{aligned}|\psi_I(t)\rangle &= \left(\cos\Omega' t - i\frac{\Delta}{2\Omega'}\sin\Omega' t\right)|g\rangle - i\frac{\Omega}{\Omega'}\sin\Omega' t\,|e\rangle \\ &\xrightarrow[\Delta = 0]{} \cos\Omega t\,|g\rangle - i\sin\Omega t\,|e\rangle\end{aligned}}$$

② $|\psi(0)\rangle = |e\rangle$

$$|\psi_I(t)\rangle = \cos\Omega' t\,|e\rangle - i\sin\Omega' t\,\Big(\underbrace{u_3\sigma_3|e\rangle}_{= u_3|e\rangle = -\frac{\Delta}{2\Omega'}|e\rangle} + \overbrace{u_1\sigma_1|e\rangle}^{u_1|g\rangle = \frac{\Omega}{\Omega'}|g\rangle}\Big)$$

$$\boxed{\begin{aligned} |\Psi_I(t)\rangle &= \left(\cos\Omega' t + i\frac{\Delta}{2\Omega'}\sin\Omega' t\right)|e\rangle - i\frac{\Omega}{\Omega'}\sin\Omega' t\,|g\rangle \\ &\xrightarrow[\Delta=0]{} \cos\Omega' t\,|e\rangle - i\sin\Omega' t\,|g\rangle \end{aligned}}$$

$$U_I(t) = \begin{array}{c c} & \begin{array}{cc} |e\rangle & |g\rangle \end{array} \\ \begin{array}{c} |e\rangle \\ \\ |g\rangle \end{array} & \begin{pmatrix} \cos\Omega' t + i\frac{\Delta}{2\Omega'}\sin\Omega' t & -i\frac{\Omega}{\Omega'}\sin\Omega' t \\ \\ -i\frac{\Omega}{\Omega'}\sin\Omega' t & \cos\Omega t' - i\frac{\Delta}{2\Omega'}\sin\Omega t' \end{pmatrix} \end{array}$$

$$\xrightarrow[\Delta=0]{} \begin{pmatrix} \cos\Omega t & -i\sin\Omega t \\ -i\sin\Omega t & \cos\Omega t \end{pmatrix}$$

Fermi's golden rule of constant transition rates

① Density of states

② $\Delta E \, \Delta t \sim h$

$$W(t) = \hat{W} \quad \text{or} \quad W(t) = \hat{W} \cos \omega t$$

$$\int_0^t dt' e^{i\omega_{fi} t'} = \frac{e^{i\omega_{fi} t} - 1}{i\omega_{fi}} = e^{i\omega_{fi} t/2} \frac{2i \sin \omega_{fi} t/2}{i\omega_{fi}} = e^{i\omega_{fi} t/2} \frac{\sin(\omega_{fi} t/2)}{\omega_{fi}/2}$$

sinc function

$$b_f(t) = -\frac{i}{\hbar} \int_0^t dt' \, W_{fi}(t') \, e^{i\omega_{fi} t'}$$

$$= \begin{cases} -\frac{i}{\hbar} \hat{W}_{fi} \int_0^t dt' \, e^{i\omega_{fi} t'} \\ -\frac{i}{2\hbar} \hat{W}_{fi} \underbrace{\int_0^t dt' \left(e^{i\omega t'} + e^{-i\omega t'} \right) e^{i\omega_{fi} t'}} \end{cases}$$

$$= \int_0^t dt' \left(e^{i(\omega_{fi} + \omega) t'} + e^{i(\omega_{fi} - \omega) t'} \right)$$

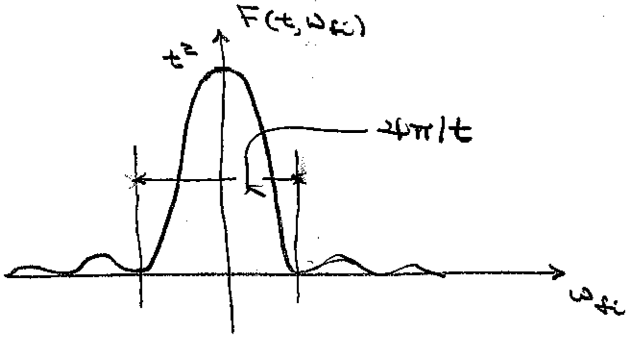
↑ drop antiresonant

$$|\omega_{fi} - \omega| \ll \omega_{fi} \simeq \omega$$

$$= \int_0^{t'} dt' \, e^{i(\omega_{fi} - \omega) t'}$$

$$= \begin{cases} -\frac{i}{\hbar} \hat{W}_{fi} \, e^{i\omega_{fi} t/2} \, \frac{\sin(\omega_{fi} t/2)}{\omega_{fi}/2} \\ -\frac{i}{2\hbar} \hat{W}_{fi} \, e^{i(\omega_{fi} - \omega) t/2} \, \frac{\sin[(\omega_{fi} - \omega) t/2]}{(\omega_{fi} - \omega)/2} \end{cases}$$

$$|b_f(t)|^2 = \begin{cases} \frac{1}{\hbar^2} |\hat{W}_{fi}|^2 \, F(t, \omega_{fi}) \\ \frac{1}{4\hbar^2} |\hat{W}_{fi}|^2 \, F(t, \omega_{fi} - \omega) \end{cases}$$



$$F(t, \omega_{fi}) = \frac{\sin^2(\omega_{fi} t/2)}{(\omega_{fi}/2)^2}$$

$$(\Delta \omega_{fi}) t \sim 2\pi$$

$$(\Delta E_{fi}) t \sim 2\pi\hbar = h$$

Final states: $|\alpha\rangle$

$$H_0|\alpha\rangle = E(\alpha)|\alpha\rangle$$

$$\langle\alpha'|\alpha\rangle = \delta(\alpha-\alpha')$$

$$\int d\alpha\, |\alpha\rangle\langle\alpha| = 1$$

Example:

$$|\alpha\rangle = |\vec{p}\rangle$$

$$\langle\vec{r}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

$$E(\vec{p}) = \frac{\vec{p}^2}{2m}$$

$$\langle\vec{p}'|\vec{p}\rangle = \delta(\vec{p}-\vec{p}')$$

$$\int d^3p\, |\vec{p}\rangle\langle\vec{p}| = 1$$

$$\delta P_{i\to\alpha_f\in D_f}(t) = \int_{\alpha_f\in D_f} d\alpha\, |b_\alpha(t)|^2$$

$$d\alpha = \rho(E,\beta)\, dE\, d\beta$$

$$= \int_{\substack{E\in\delta E_f \\ \beta\in\delta\beta_f}} dE\, d\beta\, |b_\alpha(t)|^2$$

$$d^3p = p^2\, dp\, d\Omega_{\vec{p}}$$

$$= 2mE\sqrt{2m}\, d\sqrt{E}\, d\Omega_{\vec{p}}$$

$$= (2m)^{3/2} E \frac{1}{2\sqrt{E}}\, dE\, d\Omega_{\vec{p}}$$

$$= \underbrace{\left(2m^3E\right)^{1/2}}_{\rho(E)} dE\, d\Omega_{\vec{p}}$$

w.o. case:

$$\delta P_{i\to\delta E_f,\delta\beta_f} = \frac{1}{\hbar^2}\int_{\substack{E\in\delta E_f \\ \beta\in\delta\beta_f}} dE\, d\beta\, \underbrace{\rho(E,\beta)\,|\langle \underbrace{E,\beta}_{\alpha}|\hat{W}|i\rangle|^2}_{\text{essentially a constant across } \delta E_f,\ \delta\beta_f}\, \underbrace{F\left(t, \frac{E-E_i}{\hbar}\right)}_{\text{rapidly varying}}$$

$$\delta E_f \gg \frac{4\pi\hbar}{t}$$

For given δE_f, must be able to make t big enough without getting outside validity of 1st-order perturbation theory

$$= \frac{1}{\hbar^2}\,\delta\beta_f\,\rho(E_f,\beta_f)\,|\langle E_f,\beta_f|\hat{W}|i\rangle|^2$$

$$\times \underbrace{\int_{\delta E_f} dE\, F\left(t, \frac{E-E_i}{\hbar}\right)}_{0 \text{ unless } E_f = E_i}$$

if $E_f = E_i$, extend limits to $\pm\infty$

$$\int_{-\infty}^{\infty} dE\, F\left(t, \frac{E-E_i}{\hbar}\right) = \int_{-\infty}^{\infty} dE\, \frac{\sin^2\left((E-E_i)t/2\hbar\right)}{\left((E-E_i)/2\hbar\right)^2}$$

$$u = \frac{(E-E_i)t}{2\hbar} \qquad = \frac{2\hbar}{t}\, t^2 \underbrace{\int_{-\infty}^{\infty} du\, \frac{\sin^2 u}{u^2}}_{\pi}$$

$$= 2\pi\hbar t$$

$$F\left(t, \frac{E-E_i}{\hbar}\right) \xrightarrow[t\to\infty]{} 2\pi\hbar t\, \delta(E-E_i)$$

$$\boxed{\begin{aligned} \delta P_{i\to\delta E_f,\delta\beta_f} &= \frac{2\pi}{\hbar}\, t\, \delta\beta_f\, \rho(E_f = E_i, \beta_f)\left|\langle E_f = E_i, \beta_f|\hat{W}|i\rangle\right|^2 \\ \delta W_{i\to E_f,\beta_f} &= \begin{pmatrix} \text{transition rate per} \\ \text{unit } \delta\beta_f \end{pmatrix} \\ &= \frac{2\pi}{\hbar}\, \rho(E_f = E_i, \beta_f)\left|\langle E_f = E_i, \beta_f|\hat{W}|i\rangle\right|^2 \end{aligned}}$$

(1) $\omega \neq 0$: divide by t, $E_f = E_i + \hbar\omega$

(2) Relation to exponential decays.

(3) Momentum case:

$$\Theta(\omega) = \begin{cases} \pi, & |\omega| \leq 1 \\ 0, & |\omega| \geq 1 \end{cases}$$

$$\tilde{\Theta}(u) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \Theta(\omega)\, e^{-i\omega u} = \frac{1}{2}\int_{-1}^{+1} d\omega\, e^{-i\omega u} = \frac{e^{-iu} - e^{+iu}}{-2iu} = \frac{-2i\sin u}{-2iu} = \frac{\sin u}{u}$$

Parseval: $\int_{-\infty}^{\infty} du\, \frac{\sin^2 u}{u^2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, |\Theta(\omega)|^2 = \frac{\pi}{2}\int_{-1}^{+1} d\omega = \pi$

Periodic boundary conditions: $k_j L = 2\pi n_j$, $p_j L = 2\pi\hbar n_j$

$$\langle \vec{r}|\vec{p}\rangle = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \qquad \rho_{\vec{p}}\,\Delta^3 p = \frac{V}{(2\pi\hbar)^3}\,\Delta^3 p$$

$$\delta P_{i\to\delta^3 p} = \sum_{\vec{p}\in\delta^3 p} |b_{\vec{p}}(t)|^2$$

$$= \frac{1}{\hbar^2}\sum_{\vec{p}\in\delta^3 p} |\langle\vec{p}|\hat{W}|i\rangle|^2 F(t,\omega_{fi})$$

$$= \left|\int_V d^3r\, \langle\vec{p}|\vec{r}\rangle\langle\vec{r}|\hat{W}|i\rangle\right|^2$$

$$= \frac{1}{V}\left|\int_V d^3r\, e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\langle\vec{r}|\hat{W}|i\rangle\right|^2$$

$$= \frac{1}{\hbar^2}\int_{\delta^3 p} d^3p\, \underbrace{\rho_{\vec{p}}\frac{1}{V}}_{\frac{1}{(2\pi\hbar)^3}} \left| \qquad \right|^2 F(t,\omega_{fi})$$

$$= \frac{1}{\hbar^2}\int_{\delta^3 p} d^3p \left|\underbrace{\int d^3r\, \frac{e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}}}{(2\pi\hbar)^{3/2}}\langle\vec{r}|\hat{W}|i\rangle}_{\langle\vec{p}|\hat{W}|i\rangle}\right|^2 F(t,\omega_{fi})$$

$$= \frac{1}{\hbar^2}\int_{\delta^3 p} d^3p\, |\langle\vec{p}|\hat{W}|i\rangle|^2 F(t,\omega_{fi})$$

So you will often see a density of states

$$\rho(E) = \frac{V}{(2\pi\hbar)^3}(2m^3E)^{1/2}$$