Phys 522

Lectures 3-8

Angular momentum

Review:

$$\vec{J} = J_x \vec{e}_x + J_y \vec{e}_y + J_z \vec{e}_z$$

Commutation relations: $[J_j, J_k] = i\hbar \epsilon_{jkl} J_l$

Ladder operators: $J_\pm = J_x \pm i J_y \qquad (J_+^\dagger = J_-)$

$$[J_z, J_\pm] = \pm \hbar J_\pm$$

$$[J_+, J_-] = 2\hbar J_z$$

Square: $\vec{J}^2 = \vec{J}\cdot\vec{J} = \underbrace{J_x^2 + J_y^2}_{\frac{1}{2}(J_+J_- + J_-J_+)} + J_z^2 = J_z^2 + \begin{matrix} J_+J_- - \hbar J_z \\ J_-J_+ + \hbar J_z \end{matrix}$

$$[\vec{J}^2, J_j] = 0 = [\vec{J}^2, J_\pm]$$

CSCO: $A, \vec{J}^2, J_z$

Eigenstates: $|kjm\rangle$ possible $j = 0, 1/2, 1, 3/2, \ldots$
mandatory $m = -j, \ldots, j$ $(2j+1$ values$)$

$$\vec{J}^2 |kjm\rangle = j(j+1)\hbar^2 |kjm\rangle$$

$$J_z |kjm\rangle = m\hbar |kjm\rangle$$

$$J_\pm |kjm\rangle = \underbrace{C_\pm(j,m)}\hbar |kj, m\pm 1\rangle$$

$$\sqrt{j(j+1) - m(m\pm 1)} = \sqrt{(j\mp m)(j\pm m+1)}$$

$C_-(j,j) = \sqrt{2j} = C_+(j,-j)$

$C_-(j,j-1) = \sqrt{2(2j-1)} = C_+(j,1-j)$

$$C_\pm(j, m\mp 1) = C_\mp(j,m) = C_\pm(j,-m)$$

Rotations:

Classical $\mathcal{R}_{\vec{u}}(\alpha)$

$$\mathcal{R}_{\vec{u}}(\alpha)\vec{e}_j = \vec{e}_k O_{kj} \Rightarrow \mathcal{R}_{\vec{u}}(\alpha)\vec{r} = \vec{e}_k O_{kj} x_j$$

Infinitesimal $\mathcal{R}_{\vec{u}}(d\alpha)\vec{r} = \vec{r} + d\alpha\, \vec{u}\times\vec{r}$

$$O_{kj} = \delta_{kj} - d\alpha\, \epsilon_{kjl} u_l$$

Note that
$\vec{e}_k \cdot \mathcal{R}\vec{e}_j = O_{kj}$

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Finite: $\mathcal{R}_{\vec{u}}(\alpha)\vec{r} = (\vec{r}\cdot\vec{u})\vec{u} - \vec{u}\times(\vec{u}\times\vec{r})\cos\alpha + \vec{u}\times\vec{r}\sin\alpha$

$$= \vec{r}\cos\alpha + (\vec{r}\cdot\vec{u})\vec{u}(1-\cos\alpha) + \vec{u}\times\vec{r}\sin\alpha$$

$$O_{kj} = \delta_{kj}\cos\alpha + u_k u_j(1-\cos\alpha) - \epsilon_{kj\ell}u_\ell \sin\alpha$$

Quantum: $R_{\vec{u}}(\alpha) = e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}}$

Orbital

$R|\vec{r}\rangle = |\mathcal{R}\vec{r}\rangle$

$|\psi'\rangle = R_{\vec{u}}(\alpha)|\psi\rangle$

Vector operator: $\vec{V} = V_x\vec{e}_x + V_y\vec{e}_y + V_z\vec{e}_z$

$$\langle\psi'|\vec{V}|\psi'\rangle = \mathcal{R}\langle\psi|\vec{V}|\psi\rangle \qquad \forall\ |\psi\rangle$$

$$\parallel$$

$$\langle\psi|R^\dagger\vec{V}R|\psi\rangle$$

$$\Rightarrow \quad R^\dagger\vec{V}R = \mathcal{R}\vec{V} = \vec{e}_k O_{kj}V_j$$

$$R^\dagger V_k R\,\vec{e}_k \Leftrightarrow R^\dagger V_k R = O_{kj}V_j \Leftrightarrow R V_k R^\dagger = V_j O_{jk}$$

$$\Leftrightarrow \quad [V_j, J_k] = i\hbar\,\epsilon_{jk\ell}V_\ell$$

(how do we show both directions?)

Other way of deriving angular momentum

Conservation of angular momentum

Addition of angular momentum

$$\vec{J} = \vec{J}_1 \otimes 1_2 + 1_1 \otimes \vec{J}_2 = \vec{J}_1 + \vec{J}_2 \longleftarrow \text{actually 3 equations}$$

$\mathcal{E}$ $\quad \mathcal{E}_1 \quad \mathcal{E}_2$

Why bother? Conservation of total AM
Ex: $\vec{L}\cdot\vec{S}$ coupling

CSCO in $\mathcal{E}_1$: $A_1, \vec{J}_1^2, J_{1z}$ $\quad |k_1 j_1 m_1\rangle$

CSCO in $\mathcal{E}_2$: $A_2, \vec{J}_2^2, J_{2z}$ $\quad |k_2 j_2 m_2\rangle$

CSCO in $\mathcal{E}$: $A_1, A_2, \vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$

product basis $\longrightarrow |k_1 j_1 m_1\rangle \otimes |k_2 j_2 m_2\rangle = |k_1 k_2 j_1 j_2 m_1 m_2\rangle$

What's the problem? Find eigenstates of $\vec{J}^2, J_z$. $\quad$ JM jm

A_1, A_2 commute with all components of $\vec{J}_1$ and $\vec{J}_2$, hence with $\vec{J}^2, J_z$

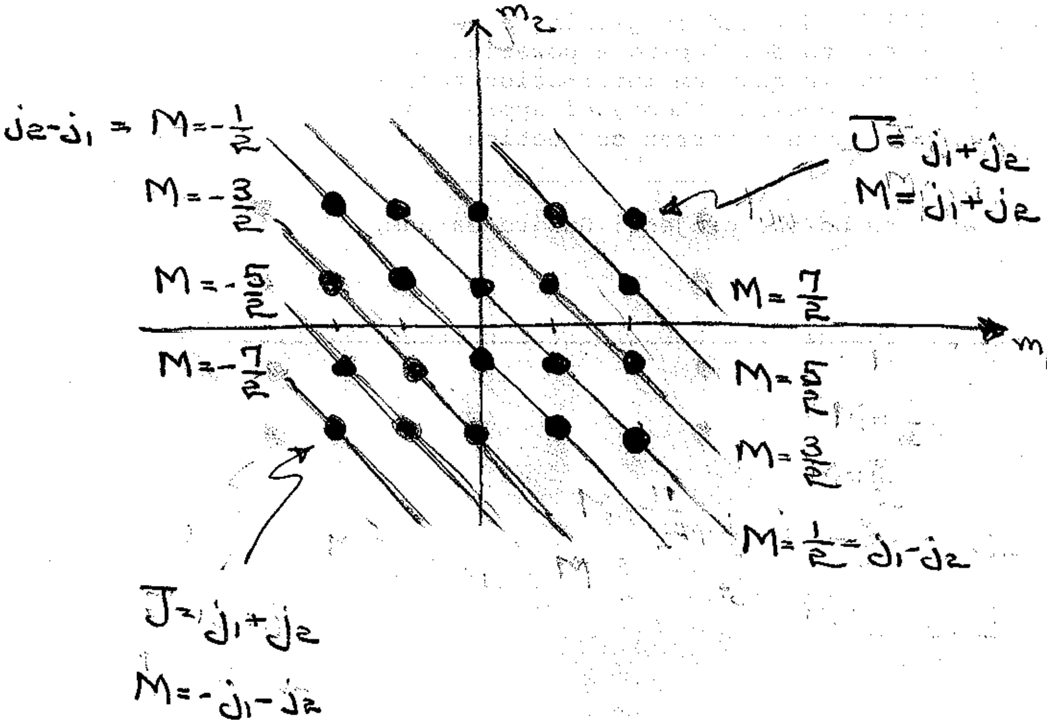
$\vec{J}_1^2$ commutes w/ J_{1j} and J_{2j}, hence with $\vec{J}^2, J_z$
$\vec{J}_2^2$ $\quad$ "

New CSCO in $\mathcal{E}$: $A_1, A_2, \vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z$
↑
have to show

Note J_z commutes with all of 1st set;
only $\vec{J}^2$ doesn't commute with J_{1z} and J_{2z}

In subspace w/ particular values of $\underbrace{k_1, k_2}_{\text{drop}}, \underbrace{j_1, j_2}_{\text{sometimes drop}}$,
diagonalize $\vec{J}^2$ in subspaces of constant $M = m_1 + m_2$

Dimension is $(2j_1+1)(2j_2+1)$
Find allowed values of J
Find $|j_1 j_2 JM\rangle$ in terms of $|j_1 j_2 m_1 m_2\rangle$ $\quad$ (Drop j_1, j_2?)



$j_1 = 2$

$j_2 = \frac{3}{2}$

Describe procedure

$J = j_1 + j_2,\ M = J$ orthogonalize $J = j_1 + j_2 - 1,\ M = J$ ... $J = j_1 - j_2,\ M = J$

lower J_- ⋮ J_- ⋮ ... J_- ⋮

$M = -J$ $M = -J$ $M = -J$

Allowed values of J: $|j_1 - j_2| \le J \le j_1 + j_2$ (triangle condition)

Gets dimension right

Procedure for calculating

$J \le j_1 + j_2$

$j_1 \le J + j_2$

$j_2 \le J + j_1$

$J \ge |j_1 - j_2|$

Clebsch-Gordan coefficients

$$|j_1 j_2 J M\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle \overbrace{\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle}^{\text{C-G}}$$

$$\langle j_1 j_2 m_1 m_2 | J M\rangle = \langle m_1 m_2 | J M\rangle$$

C-T

$$\boxed{m_1 + m_2 = M}$$

Fix j_1, j_2, J: C-G coefficients are a matrix on the above lattice. For different values of J, different parts non zero.

Recursion relations:

$$J_\pm |JM\rangle = C_\pm(J,M)\hbar\, |J, M\pm 1\rangle$$

$$J_{1\pm} |j_1 j_2 m_1 m_2\rangle = C_\pm(j_1, m_1)\hbar\, |j_1 j_2, m_1 \pm 1, m_2\rangle$$

$$\Rightarrow \langle j_1 j_2 m_1 m_2 | J_{1\mp} = \underset{\uparrow \atop \text{real}}{C_\pm(j_1, m_1)}\hbar\, \langle j_1 j_2, m_1 \pm 1, m_2|$$

$$\hbar C_\mp(J,M) \langle j_1 j_2 m_1 m_2 | J, M\mp 1\rangle$$

$$= \langle j_1 j_2 m_1 m_2 | J_\mp | J, M\rangle$$

$$= \langle j_1 j_2 m_1 m_2 | (J_{1\mp} + J_{2\mp}) | J, M\rangle$$

$$= \hbar C_\pm(j_1, m_1) \langle j_1 j_2 m_1 \pm 1, m_2 | JM\rangle$$

$$+ \hbar C_\pm(j_2, m_2) \langle j_1 j_2 m_1, m_2 \pm 1 | JM\rangle$$

$$C_\mp(J,M) \langle j_1 j_2 m_1 m_2 | J, M\mp 1\rangle$$

$$\boxed{M = m_1 + m_2 \pm 1}$$

$$= C_\pm(j_1, m_1) \langle j_1 j_2, m_1 \pm 1, m_2 | JM\rangle$$

$$+ C_\pm(j_2, m_2) \langle j_1 j_2 m_1, m_2 \pm 1 | JM\rangle$$

Upper sign: $\therefore$

Lower sign: $\because$

Outline how this works on lattice

Get all $\langle j_1 j_2 m_1 m_2 | J M \rangle$ in terms of
$M = m_1+m_2$

$\langle j_1 j_2 j_1, J-j_1 | JJ \rangle \longrightarrow$ amplitude – normalization of $|JJ\rangle$ (farthest right line)

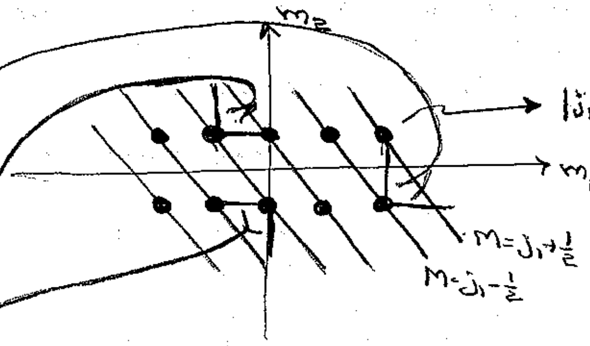
phase arbitrary (phase of $|JJ\rangle$) — choose <u>real</u> and <u>positive</u>

$\Longrightarrow$ all C-G coefficients real

<u>Example</u>: $j_2 = \frac{1}{2}$

$C_-(j,j) = \sqrt{2j} = C_+(j,-j)$

$C_+(\frac{1}{2},-\frac{1}{2}) = 1 = C_-(\frac{1}{2},\frac{1}{2})$



① $J = j_1 + \frac{1}{2}$ $\qquad |j_1+\frac{1}{2}, j_1+\frac{1}{2}\rangle = |j_1 \frac{1}{2} j_1 \frac{1}{2}\rangle$

$$\Longrightarrow \langle j_1 \tfrac{1}{2} j_1 \tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1+\tfrac{1}{2}\rangle = 1$$

$J = j_1+\frac{1}{2}$, $M = j_1+\frac{1}{2}$
$m_1 = j_1$, $m_2 = -\frac{1}{2}$

$$\underbrace{C_-(j_1+\tfrac{1}{2}, j_1+\tfrac{1}{2})}_{\sqrt{2j_1+1}} \langle j_1 \tfrac{1}{2} j_1, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1-\tfrac{1}{2}\rangle$$
$$= \underbrace{C_+(\tfrac{1}{2},-\tfrac{1}{2})}_{1} \underbrace{\langle j_1 \tfrac{1}{2} j_1 \tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1+\tfrac{1}{2}\rangle}_{1}$$

$$\langle j_1 \tfrac{1}{2} j_1, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2j_1+1}}$$

$J = j_1+\frac{1}{2}$, $M = m+1$
$m_1 = m-\frac{1}{2}$, $m_2 = \frac{1}{2}$

$$\underbrace{C_-(j_1+\tfrac{1}{2}, m+1)}_{\sqrt{(j_1+m+3/2)(j_1-m+1/2)}} \langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle$$
$$= \underbrace{C_+(j_1, m-\tfrac{1}{2})}_{\sqrt{(j_1-m+1/2)(j_1+m+1/2)}} \langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, \tfrac{1}{2} | j_1+\tfrac{1}{2}, m+1\rangle$$

Recursion along top row of lattice

$$\frac{\langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle}{\sqrt{j_1+m+1/2}} = \frac{\overbrace{\langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, \tfrac{1}{2} | j_1+\tfrac{1}{2}, m+1\rangle}^{1}}{\sqrt{j_1+m+3/2}}$$

$$\xrightarrow[m=j_1+\frac{1}{2}]{} \; = \frac{\langle j_1 \tfrac{1}{2} j_1 \tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1+\tfrac{1}{2}\rangle}{\sqrt{2j_1+1}}$$

$$\boxed{\langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle = \sqrt{\frac{j_1+m+1/2}{2j_1+1}}}$$

$J = j_1+\frac{1}{2}$, $M = m-1$

$m_1 = m+\frac{1}{2}$, $m_2 = -\frac{1}{2}$

$$\overbrace{C_+(j_1+\tfrac{1}{2}, m-1)}^{\sqrt{(j_1-m+3/2)(j_1+m+1/2)}} \langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle$$

$$= \underbrace{C_-(j_1, m+\tfrac{1}{2})}_{\sqrt{(j_1+m+1/2)(j_1-m+1/2)}} \langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, m-1\rangle$$

Recursion along bottom row of lattice

$$\frac{\langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle}{\sqrt{j_1-m+1/2}} = \frac{\langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, m-1\rangle}{\sqrt{j_1-m+3/2}}$$

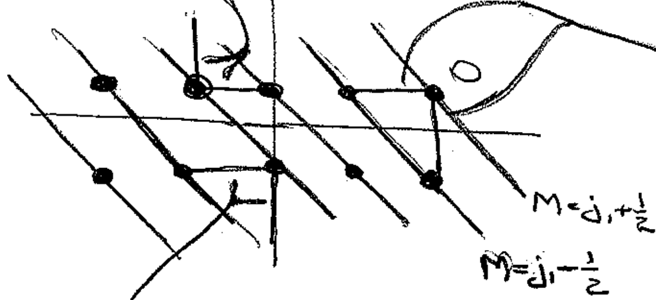
$$\xrightarrow[m=j_1-\frac{1}{2}]{} \; = \langle j_1 \tfrac{1}{2} j_1, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, j_1-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2j_1+1}}$$

$$\boxed{\langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1+\tfrac{1}{2}, m\rangle = \sqrt{\frac{j_1-m+1/2}{2j_1+1}}}$$

Check normalization and $m = \pm(j_1+\frac{1}{2})$

$$\boxed{|j_1+\tfrac{1}{2}, m\rangle = \sqrt{\frac{j_1+m+1/2}{2j_1+1}}\, |j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{\frac{j_1+m+1/2}{2j_1+1}}\, |j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2}\rangle}$$

(2) $J = j_1 - \frac{1}{2}$



$J = j_1 - \frac{1}{2}$, $M = j_1 - \frac{1}{2}$

$m_1 = j_1$, $m_2 = \frac{1}{2}$

$$0 = \underbrace{C_-(j_1, j_1)}_{\sqrt{2j_1}} \underbrace{\langle j_1 \tfrac{1}{2}, j_1 - 1, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, j_1 - \tfrac{1}{2} \rangle}_{-\frac{1}{\sqrt{2j_1+1}}}$$

$$+ \underbrace{C_-(\tfrac{1}{2}, \tfrac{1}{2})}_{1} \underbrace{\langle j_1 \tfrac{1}{2} j_1, -\tfrac{1}{2} | j_1 - \tfrac{1}{2}, j_1 - \tfrac{1}{2} \rangle}_{\sqrt{\frac{2j_1}{2j_1+1}}}$$

Normalization and phase convention

$J = j_1 - \frac{1}{2}$, $M = m+1$

$m_1 = m - \frac{1}{2}$, $m_2 = \frac{1}{2}$

$$\overbrace{C_-(j_1 - \tfrac{1}{2}, m+1)}^{\sqrt{(j_1+m+1/2)(j_1-m-1/2)}} \langle j_1 \tfrac{1}{2}, m - \tfrac{1}{2}, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m \rangle$$

$$= \underbrace{C_+(j_1, m - \tfrac{1}{2})}_{\sqrt{(j_1-m+1/2)(j_1+m+1/2)}} \langle j_1 \tfrac{1}{2}, m + \tfrac{1}{2}, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m+1 \rangle$$

$$\frac{\langle j_1 \tfrac{1}{2}, m - \tfrac{1}{2}, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m \rangle}{\sqrt{j_1 - m + 1/2}} = \frac{\langle j_1 \tfrac{1}{2}, m + \tfrac{1}{2}, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m+1 \rangle}{\sqrt{j_1 - m - 1/2}}$$

Recursion along top row of lattice

$$\xrightarrow{m = j_1 - \frac{1}{2}} \langle j_1 \tfrac{1}{2}, j_1 - 1, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, j_1 - \tfrac{1}{2} \rangle = -\frac{1}{\sqrt{2j_1+1}}$$

$$\boxed{\langle j_1 \tfrac{1}{2}, m - \tfrac{1}{2}, \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m \rangle = -\sqrt{\frac{j_1 - m + 1/2}{2j_1+1}}}$$

$J = j_1 - \frac{1}{2},\ M = m-1$

$m_1 = m + \frac{1}{2},\ m_2 = -\frac{1}{2}$

$$\overbrace{C_+(j_1-\tfrac{1}{2}, m-1)}^{\sqrt{(j_1-m+1/2)(j_1+m-1/2)}}\langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1 - \tfrac{1}{2}, m\rangle$$

$$= \underbrace{C_-(j_1, m+\tfrac{1}{2})}_{\sqrt{(j_1+m+1/2)(j_1-m+1/2)}}\langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}; \tfrac{1}{2} | j_1 - \tfrac{1}{2}, m-1\rangle$$

$$\frac{\langle j_1 \frac{1}{2}, m+\frac{1}{2}, -\frac{1}{2} | j_1 - \frac{1}{2}, m\rangle}{\sqrt{j_1+m+1/2}} = \frac{\langle j_1 \frac{1}{2}, m-\frac{1}{2}, -\frac{1}{2} | j_1 - \frac{1}{2}, m-1\rangle}{\sqrt{j_1+m-1/2}}$$

Recursion along bottom row of lattice

$$\xrightarrow{m = j_1 - \frac{1}{2}} \ = \frac{\langle j_1 \frac{1}{2} j_1, -\frac{1}{2} | j_1 - \frac{1}{2}, j_1 - \frac{1}{2}\rangle}{\sqrt{2j_1}} = \frac{1}{\sqrt{2j_1+1}}$$

$$\boxed{\langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1 - \tfrac{1}{2}, m\rangle = \sqrt{\frac{j_1+m+1/2}{2j_1+1}}}$$

Check normalization and orthogonality to $|j_1 + \frac{1}{2}, m\rangle$

$$\boxed{|j_1 - \tfrac{1}{2}, m\rangle = -\sqrt{\frac{j_1-m+1/2}{2j_1+1}}\,|j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{\frac{j_1+m+1/2}{2j_1+1}}\,|j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2}\rangle}$$

Gather results:

$$\boxed{\begin{aligned} \langle j_1 \tfrac{1}{2}, m-\tfrac{1}{2}, \tfrac{1}{2} | j_1 \pm \tfrac{1}{2}, m\rangle &= \pm\sqrt{\frac{j_1 \pm m + 1/2}{2j_1+1}} \\ \langle j_1 \tfrac{1}{2}, m+\tfrac{1}{2}, -\tfrac{1}{2} | j_1 \pm \tfrac{1}{2}, m\rangle &= \sqrt{\frac{j_1 \mp m + 1/2}{2j_1+1}} \end{aligned}}$$

Addition of two spins: $j_1 = j_2 = \frac{1}{2}$; $\vec{S} = \vec{S}_1 + \vec{S}_2$

$\langle \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} | 1 1\rangle = 1$

$\langle \frac{1}{2} \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} | 1 0\rangle = \langle \frac{1}{2} \frac{1}{2} \frac{1}{2}, -\frac{1}{2} | 1 0\rangle = \frac{1}{\sqrt{2}}$

$\langle \frac{1}{2} \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} | 1, -1\rangle = 1$

$$\left[\begin{aligned} |1 1\rangle &= |\tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2}\rangle = |+,+\rangle \\ |1 0\rangle &= \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2} \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\rangle\right) \\ &= \tfrac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right) \\ |1, -1\rangle &= |\tfrac{1}{2} \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |-,-\rangle \end{aligned}\right.$$

$\rightarrow$ Spin triplet (symmetric $1 \leftrightarrow 2$)

$$-\langle \tfrac12 \tfrac12, -\tfrac12 \tfrac12 | 00\rangle = \langle \tfrac12 \tfrac12 \tfrac12, -\tfrac12 | 00\rangle = \frac{1}{\sqrt{2}}$$

Commutation w/ the swap operator

It is easier to get this from $|11\rangle = |+,+\rangle$, $|1,-1\rangle = |-,-\rangle$, $S_-|11\rangle = \sqrt{2}|10\rangle$
" $S_{1-}|++\rangle + S_{2-}|++\rangle = |-+\rangle + |+-\rangle$, and orthogonality of $|00\rangle$ to $|10\rangle$.

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac12\tfrac12\tfrac12,-\tfrac12\rangle - |\tfrac12\tfrac12,-\tfrac12,\tfrac12\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$$

Spin singlet (antisymmetric $1 \leftrightarrow 2$)

Spinors: $|k\ell\tfrac12 m\epsilon\rangle \doteq |k\ell m\rangle \otimes |\epsilon\rangle$

$$[\psi_{k\ell\frac12 m\epsilon}](\vec r) = \begin{pmatrix} \langle \vec r, +|k\ell\tfrac12 m\epsilon\rangle \\ \langle \vec r, -|k\ell\tfrac12 m\epsilon\rangle \end{pmatrix}$$

$$[\psi_{k\ell\frac12 m+}](\vec r) = R_{k\ell}(r)\, Y_\ell^m(\theta,\varphi)\begin{pmatrix}1\\0\end{pmatrix}$$

$$[\psi_{k\ell\frac12 m-}](\vec r) = R_{k\ell}(r)\, Y_\ell^m(\theta,\varphi)\begin{pmatrix}0\\1\end{pmatrix}$$

$$|k\ell\tfrac12, \ell\pm\tfrac12, M\rangle = \pm\sqrt{\frac{\ell\pm M+1/2}{2\ell+1}}\,|k\ell\tfrac12, M-\tfrac12, \tfrac12\rangle + \sqrt{\frac{\ell\mp M+1/2}{2\ell+1}}\,|k\ell\tfrac12, M+\tfrac12, -\tfrac12\rangle$$

$$[\psi_{k\ell\frac12,\ell\pm\frac12,M}](\vec r) = \pm\sqrt{\frac{\ell\pm M+1/2}{2\ell+1}}\,R_{k\ell}Y_\ell^{M-1/2}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{\ell\mp M+1/2}{2\ell+1}}\,R_{k\ell}Y_\ell^{M+1/2}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boxed{[\psi_{k\ell\frac12,\ell\pm\frac12,M}](\vec r) = R_{k\ell}(r)\begin{pmatrix} \pm\sqrt{\frac{\ell\pm M+1/2}{2\ell+1}}\,Y_\ell^{M-1/2}(\theta,\varphi) \\ \sqrt{\frac{\ell\mp M+1/2}{2\ell+1}}\,Y_\ell^{M+1/2}(\theta,\varphi)\end{pmatrix}}$$

Completeness and orthogonality

$$1_{j_1j_2} = \sum_{m_1,m_2} |j_1j_2m_1m_2\rangle\langle j_1j_2m_1m_2| = \sum_{J,M} |j_1j_2JM\rangle\langle j_1j_2JM|$$

$$\delta_{m_1m_1'}\delta_{m_2m_2'} = \langle j_1j_2m_1m_2|j_1j_2m_1'm_2'\rangle$$

$$= \sum_{J,M} \langle j_1j_2m_1m_2|JM\rangle \underbrace{\langle JM|j_1j_2m_1'm_2'\rangle}_{\langle j_1j_2m_1'm_2'|JM\rangle}$$

because C-G coefficients are real

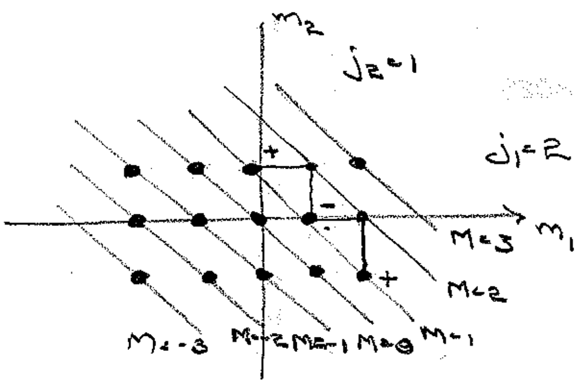
$$\delta_{JJ'}\delta_{MM'} = \langle JM|J'M'\rangle$$

$$= \sum_{m_1,m_2} \underbrace{\langle JM|j_1j_2m_1m_2\rangle}_{\langle j_1j_2m_1m_2|JM\rangle} \langle j_1j_2m_1m_2|J'M'\rangle$$

because C-G coefficients are real

Sum restricted by $M = m_1+m_2 = M'$
(visualize sum on lattice)

Signs on the border of the lattice:



Use $J=1$ to illustrate sign alternation of
$\langle j_1j_2m_1, J-m_1|JJ\rangle$

Choose $\langle j_1j_2j_1, J-j_1|JJ\rangle$ real and positive. Sign of $\langle j_1j_2m_1, J-m_1|JJ\rangle$ is $(-1)^{j_1-m_1}$
$(J-j_2 \le m_1 \le j_1)$

$\Longrightarrow$ sign of $\langle j_1j_2, J-j_2, j_2|JJ\rangle$ is $(-1)^{j_1-(J-j_2)} = (-1)^{j_1+j_2-J}$

(2) Use $J = 3$ to illustrate $\langle j_1 j_2 m_1 m_2 | J = j_1 + j_2, M \rangle > 0$

$\langle j_1 j_2 j_1 j_2 | j_1 + j_2, j_1 + j_2 \rangle = 1$

Recurrence relations along top and right side show these C-G's > 0. Recurrence relations into interior show all are > 0.

(3) Use $J = 2$ to illustrate that coefficients along right and bottom are positive, while those on top and left are nonzero and have the sign of $\langle j_1 j_2, J - j_2, j_2 | J J \rangle$, which is $(-1)^{j_1 + j_2 - J}$

$\langle j_1 j_2 j_1, J - j_1 | J J \rangle > 0 \Rightarrow$ right $\langle j_1 j_2 j_1, M - j_1 | J M \rangle > 0$ $(j_1 - j_2 \le M \le J)$

bottom $\langle j_1 j_2, M + j_2, -j_2 | J M \rangle > 0$ $(-J \le M \le j_1 - j_2)$

$\langle j_1 j_2, J - j_2, j_2 | J J \rangle \leftrightarrow (-1)^{j_1 + j_2 - J}$

top $\langle j_1 j_2, M - j_2, j_2 | J M \rangle$ $(j_2 - j_1 \le M \le J)$

left $\langle j_1 j_2, -j_1, M + j_1 | J M \rangle$ $(-J \le M \le j_2 - j_1)$

(4) Use $J = 1$ to illustrate the sign alternation of $\langle j_1 j_2 m_1, -J - m_1 | J, -J \rangle$.

The sign of $\langle j_1 j_2 m_1, -J - m_1 | J, -J \rangle$ is $(-1)^{m_2 + j_2} = (-1)^{j_2 - J - m_1}$

$(-j_1 \le m_1 \le j_2 - J)$

$\uparrow$

1 when $m_2 = -j_2$

In particular, the sign of $\langle j_1 j_2, -j_1, j_1 - J | J, -J \rangle$ is $(-1)^{j_2 + j_1 - J}$

Symmetries

Interchanging j_1 and j_2:

$$\boxed{\langle j_2 j_1 m_2 m_1 | JM\rangle = (-1)^{j_1+j_2-J}\langle j_1 j_2 m_1 m_2 | JM\rangle}$$

$$\big\langle j \big| j_1 j_2 m_1 m_2\rangle = |j_2 j_1 m_2 m_1\rangle$$

$$\langle j_2 j_1\, j_2, J-j_1 | JJ\rangle > 0$$

$|JJ\rangle$ and, hence, all $|JM\rangle$ change sign by $(-1)^{j_1+j_2-J}$

$\langle j_1 j_2, J-j_1, j_2 | JJ\rangle$ has sign $(-1)^{j_1+j_2-J}$

Changing sign of magnetic quantum numbers:

$$\boxed{\langle j_1 j_2, -m_1, -m_2 | J, -M\rangle = (-1)^{j_1+j_2-J}\langle j_1 j_2 m_1 m_2 | JM\rangle}$$

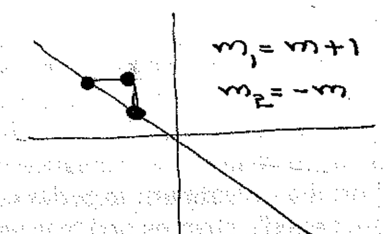
Use recursion relation starting with

$$\langle j_1 j_2, -j_1, -J+j_1 | J, -J\rangle = (-1)^{j_1+j_2-J}\langle j_1 j_2 j_1, J-j_1 | JJ\rangle$$

Coefficients of $\langle jj\, m, -m | 00\rangle$ $\quad (j_1 = j_2 = j)$

$$0 = \overbrace{C_+(j, m+1)}^{\sqrt{(j+m+1)(j-m)}}\langle jj\, m, -m | 00\rangle$$

$$+\ \underbrace{C_-(j, -m)}_{\sqrt{(j-m)(j+m+1)}}\langle jj, m+1, -m-1 | 00\rangle$$



$$\Longrightarrow \langle jj\, m, -m | 00\rangle = -\langle jj, m+1, -(m+1) | 00\rangle$$

$$\langle jj\, j, -j | 00\rangle = \frac{1}{\sqrt{2j+1}}$$

$$\boxed{\Longrightarrow \langle jj\, m, -m | 00\rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}} \qquad |00\rangle = \sum_{m=-j}^{+j}\frac{(-1)^{j-m}}{\sqrt{2j+1}}\,|jj\, m, -m\rangle}$$

$|00\rangle$ is the unique state that is invariant under all rotations.

Representations of rotation group:

$$R_{\vec{u}}(\alpha) \qquad R_2 R_1 = R_3$$

$$R_{\vec{u}}(\alpha) = e^{-\frac{i}{\hbar}\alpha \vec{u}\cdot\vec{J}} \qquad R_2 R_1 = \pm R_3$$

integral (single-valued) vs. half-integral (double-valued) representations

$$\langle jm' | R | jm \rangle = D^{(j)}_{m'm}(R)$$

$\longrightarrow$ matrix rep of SO(3)

irreducible reps

unitary matrix

Clebsh-Gordan series (apply matrix rep to addition of AM):

$$e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}} = e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}_1}\, e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}_2}$$

$$\langle j_1 j_2 m_1' m_2' | e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}} | j_1 j_2 m_1 m_2\rangle = D^{(j_1)}_{m_1'm_1}(R)\, \underset{\uparrow \text{ reducible}}{D^{(j_2)}_{m_2'm_2}(R)}$$

$$= \sum_{\substack{J'M' \\ JM}} \langle j_1 j_2 m_1' m_2' | J'M' \rangle \underbrace{\langle J'M' | e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}} | JM \rangle}_{\delta_{JJ'} D^{(J)}_{M'M}(R)} \langle JM | j_1 j_2 m_1 m_2 \rangle$$

C-G series

$$\boxed{D^{(j_1)}_{m_1'm_1}(R)\, D^{(j_2)}_{m_2'm_2}(R) = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1 j_2 m_1' m_2' | J, m_1'+m_2' \rangle \langle j_1 j_2 m_1 m_2 | J, m_1+m_2 \rangle \times D^{(J)}_{m_1'+m_2', m_1+m_2}(R)}$$

Invert:

$$\langle JM' | e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}} | JM \rangle = D^{(J)}_{M'M}(R)$$

$$= \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle JM' | j_1 j_2 m_1' m_2' \rangle \underbrace{\langle j_1 j_2 m_1' m_2' | e^{-\frac{i}{\hbar}\alpha\vec{u}\cdot\vec{J}} | j_1 j_2 m_1 m_2 \rangle}_{D^{(j_1)}_{m_1'm_1}(R)\, D^{(j_2)}_{m_2'm_2}(R)} \langle j_1 j_2 m_1 m_2 | JM \rangle$$

$$\boxed{D^{(J)}_{M'M}(R) = \sum_{\substack{m_1, m_1' \\ m_2, m_2'}} \langle j_1 j_2 m_1' m_2' | J M' \rangle \langle j_1 j_2 m_1 m_2 | J M \rangle D^{(j_1)}_{m_1' m_1}(R) \, D^{(j_2)}_{m_2' m_2}(R) \qquad m_1 + m_2 = M, \; m_1' + m_2' = M'}$$

Partial inverses:

$$\langle j_1 j_2 m_1' m_2' | e^{-\frac{i}{\hbar} \alpha \vec{u} \cdot \vec{J}} | J M \rangle = \sum_{M'} \langle j_1 j_2 m_1' m_2' | J M' \rangle D^{(J)}_{M'M}$$

$M' = m_1' + m_2'$

$$= \langle j_1 j_2 m_1' m_2' | J, m_1' + m_2' \rangle D^{(J)}_{m_1'+m_2', M}$$

$$= \sum_{m_1, m_2} D^{(j_1)}_{m_1' m_1} D^{(j_2)}_{m_2' m_2} \langle j_1 j_2 m_1 m_2 | J M \rangle$$

$$* \quad \boxed{\langle j_1 j_2 m_1' m_2' | J, m_1' + m_2' \rangle D^{(J)}_{m_1'+m_2', M} = \sum_{M'} \langle j_1 j_2 m_1' m_2' | J M' \rangle D^{(J)}_{M'M} = \sum_{m_1, m_2} \langle j_1 j_2 m_1 m_2 | J M \rangle D^{(j_1)}_{m_1' m_1} D^{(j_2)}_{m_2' m_2}}$$

Equivalent to * w/ $R \to R^{-1}$

Use C-G real

$$\langle J M' | e^{-\frac{i}{\hbar} \alpha \vec{u} \cdot \vec{J}} | j_1 j_2 m_1 m_2 \rangle = \sum_{M} D^{(J)}_{M'M} \langle j_1 j_2 m_1 m_2 | J M \rangle$$

$M = m_1 + m_2$

$$= \sum_{m_1', m_2'} \langle j_1 j_2 m_1' m_2' | J M' \rangle D^{(j_1)}_{m_1' m_1} D^{(j_2)}_{m_2' m_2}$$

$$* \quad \boxed{\langle j_1 j_2 m_1 m_2 | J, m_1 + m_2 \rangle D^{(J)}_{M', m_1 + m_2} = \sum_{M} \langle j_1 j_2 m_1 m_2 | J M \rangle D^{(J)}_{M'M} = \sum_{m_1', m_2'} \langle j_1 j_2 m_1' m_2' | J M' \rangle D^{(j_1)}_{m_1' m_1} D^{(j_2)}_{m_2' m_2}}$$

Now bring this to other side by multiplying by the inverse

$$\sum_{m_1, M} D^{(J)}_{M'M} \langle j_1 j_2 m_1 m_2 | J M \rangle D^{(j_1)*}_{m_1'' m_1}$$

$$= \sum_{m_2'} \langle j_1 j_2 m_1'' m_2' | J M' \rangle D^{(j_2)}_{m_2' m_2}$$

($J = j_3$, $M = m$, $M' = m_3$, $m_1'' = m_1'$)

$$\boxed{\sum_{m_2'} \langle j_1 j_2 m_1' m_2' | j_3 m_3 \rangle D^{(j_2)}_{m_2' m_2} = \sum_{m_1, m} D^{(j_3)}_{m_3 m} \langle j_1 j_2 m_1 m_2 | j_3 m \rangle D^{(j_1)*}_{m_1' m_1}}$$

Linear homogeneous relation for C-G coefficients

For infinitesimal rotations these are equivalent to the recursion relations (obvious from derivation)

Later on, we use $j_3 = j'$, $m_3 = m'$, $m = \mu'$

$j_1 = j$, $m_1' = m$, $m_1 = \mu$

$j_2 = k$, $m_2' = q'$, $m_2 = q$

$$\sum_{\mu, \mu'} D^{(j')}_{m'\mu'} \langle j k \mu q | j' \mu' \rangle D^{(j)*}_{m\mu} = \sum_{q'} \langle j k m q' | j' m' \rangle D^{(k)}_{q' q}$$

Addition relation for spherical harmonics

$$\langle \vec{r} | k l m \rangle = R_{kl}(r) Y_l^m(\theta,\varphi)$$

$$\langle \vec{r} | R | k l m \rangle = \langle R^{-1} \vec{r} | k l m \rangle = R_{kl}(r) Y_l^m(\theta',\varphi')$$

θ, φ: coördinates of $\vec{r}$

θ', φ': coördinates of $R^{-1}\vec{r}$ (active view) <u>OR</u> coördinates of $\vec{r}$ in coördinate system rotated by R (passive view)

$$= \sum_{m'} \underbrace{\langle \vec{r} | k l m' \rangle}_{R_{kl}(r) Y_l^{m'}(\theta,\varphi)} \underbrace{\langle l m' | R | l m \rangle}_{D^{(l)}_{m'm}(R)}$$

$$\boxed{\begin{aligned} &Y_l^m(\theta',\varphi') = \sum_{m'} D^{(l)}_{m'm}(R)\, Y_l^{m'}(\theta,\varphi) \\ &\text{Inverse: } Y_l^{m'}(\theta,\varphi) = \sum_m D^{(l)*}_{m'm} Y_l^m(\theta',\varphi') \end{aligned}}$$

Let $\theta = 0$: ($\vec{r} = \hat{e}_z$) Use $Y_l^{m'}(0,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m'0}$

$$\Rightarrow Y_l^m(\theta',\varphi') = \sqrt{\frac{2l+1}{4\pi}}\, D^{(l)}_{0m}(R)$$

$$\Rightarrow D^{(l)}_{0m}(R) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^m(\underbrace{\theta',\varphi'}_{})$$

coördinates of $R^{-1}\hat{e}_z$ or of $\hat{e}_z$ in rotated coordinate system

Let $\theta' = 0$: $Y_\ell^{m'}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{(\ell)*}_{m'0}(R)$

$(\vec{r} = R\hat{e}_z)$

$$\Longrightarrow \quad D^{(\ell)}_{m0}(R) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^{m*}(\theta,\varphi)$$

$(\theta,\varphi) \rightarrow$ coördinates of $R\hat{e}_z$

Now write

$$\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta') = Y_\ell^0(\theta',\varphi') = \sum_{m'} D^{(\ell)}_{m'0}(R)\, Y_\ell^{m'}(\theta,\varphi) = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m'} Y_\ell^{m'}(\beta,\alpha)\, Y_\ell^{m'}(\theta,\varphi)$$

$$P_\ell(\cos\theta') = \frac{4\pi}{2\ell+1} \sum_m Y_\ell^{m*}(\beta,\alpha)\, Y_\ell^m(\theta,\varphi)$$

θ, φ: coördinates of $\vec{r}$

θ', φ': coördinates of $\vec{r}$ in rotated coördinate system

β, α: coördinates of rotated $\hat{e}_z$ in original coördinate system

Evaluate C-G series at $j_1 = l_1$, $j_2 = l_2$, $m_1' = m_2' = 0$

$$\underbrace{D^{(l_1)}_{0m_1}(R)}_{\sqrt{\frac{4\pi}{2l_1+1}}\, Y_{l_1}^{m_1}(\theta',\varphi')} \underbrace{D^{(l_2)}_{0m_2}(R)}_{\sqrt{\frac{4\pi}{2l_2+1}}\, Y_{l_2}^{m_2}(\theta',\varphi')} = \sum_{l=|l_1-l_2|}^{l_1+l_2} \langle l_1 l_2 0 0 | l 0\rangle \langle l_1 l_2 m_1 m_2 | l, m_1+m_2\rangle \times \underbrace{D^{(l)}_{0,m_1+m_2}(R)}_{\sqrt{\frac{4\pi}{2l+1}}\, Y_l^{m_1+m_2}(\theta',\varphi')}$$

Addition relation

$$\boxed{Y_{l_1}^{m_1}(\theta,\varphi)\, Y_{l_2}^{m_2}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \sum_{l=|l_1-l_2|}^{l_1+l_2} \sqrt{\frac{(2l_1+1)(2l_2+1)}{2l+1}} \langle l_1 l_2 0 0 | l 0\rangle \times \langle l_1 l_2 m_1 m_2 | l, m_1+m_2\rangle\, Y_l^{m_1+m_2}(\theta,\varphi)}$$

Actually, $l \geq |m_1+m_2|$

Invert: evaluate starred form at $j_1 = l_1$, $j_2 = l_2$, $M' = m_1' = m_2' = 0$

$$\langle l_1 l_2 0 0 | l 0\rangle \underbrace{D^{(l)}_{0M}}_{\sqrt{\frac{4\pi}{2l+1}}\, Y_l^M} = \sum_{m_1, m_2} \langle l_1 l_2 m_1 m_2 | l M\rangle \underbrace{D^{(l_1)}_{0m_1}}_{\sqrt{\frac{4\pi}{2l_1+1}}\, Y_{l_1}^{m_1}} \underbrace{D^{(l_2)}_{0m_2}}_{\sqrt{\frac{4\pi}{2l_2+1}}\, Y_{l_2}^{m_2}}$$

$$\boxed{\frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2l_1+1)(2l_2+1)}{2l+1}} \langle l_1 l_2 0 0 | l 0\rangle\, Y_l^m(\theta,\varphi) = \sum_{m_1, m_2} \langle l_1 l_2 m_1 m_2 | l m\rangle\, Y_{l_1}^{m_1}(\theta,\varphi)\, Y_{l_2}^{m_2}(\theta,\varphi)}$$

$m_1 + m_2 = m$

Note $\langle l_1 l_2 0 0 | l 0\rangle$ is nonzero only if $l_1 + l_2 - l$ is even.

Triple integrals of spherical harmonics:

$$\int d\Omega\, Y_{\ell_1}^{m_1} Y_{\ell_2}^{m_2} Y_{\ell_3}^{m_3} = \frac{1}{\sqrt{4\pi}} \sum_{\ell} \sqrt{\quad} \langle \ell_1 \ell_2 0 0 | \ell 0 \rangle$$

$$\times \langle \ell_1 \ell_2 m_1 m_2 | \ell, m_1+m_2 \rangle \underbrace{\int d\Omega\, Y_{\ell}^{m_1+m_2} Y_{\ell_3}^{m_3}}$$

$$= (-1)^{m_3} \int d\Omega\, Y_{\ell}^{m_1+m_2} Y_{\ell_3}^{-m_3 *}$$

$$= (-1)^{m_3} \delta_{\ell \ell_3} \delta_{m_1+m_2, -m_3}$$

$$= \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{2\ell_3+1}} \langle \ell_1 \ell_2 0 0 | \ell_3 0 \rangle$$

$$\times \langle \ell_1 \ell_2 m_1 m_2 | \ell_3, -m_3 \rangle (-1)^{m_3}$$

$$\boxed{\int d\Omega\, Y_{\ell_1}^{m_1} Y_{\ell_2}^{m_2} Y_{\ell_3}^{m_3} = \frac{(-1)^{m_3}}{\sqrt{4\pi}} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{2\ell_3+1}} \langle \ell_1 \ell_2 0 0 | \ell_3 0 \rangle \langle \ell_1 \ell_2 m_1 m_2 | \ell_3 -m_3 \rangle}$$

Spherical tensors: Review scalars and vectors

2-tensor operator: $\overleftrightarrow{a} = a_{jk}\,\vec{e}_j \otimes \vec{e}_k$ ⟵ 9 components

rotate state

$$\langle \psi' | \overleftrightarrow{a} | \psi' \rangle = \langle \psi | a_{jk} | \psi \rangle \, R\vec{e}_j \otimes R\vec{e}_k$$

$$= \vec{e}_\ell \otimes \vec{e}_m \, O_{\ell j} O_{mk} \langle \psi | a_{jk} | \psi \rangle$$

or ‖

$$\langle \psi | R^\dagger \overleftrightarrow{a} R | \psi \rangle = \vec{e}_\ell \otimes \vec{e}_m \langle \psi | R^\dagger a_{\ell m} R | \psi \rangle$$

rotate operator

2-tensor $\Longrightarrow$ $R^\dagger a_{\ell m} R = O_{\ell j} O_{mk} a_{jk}$

$$\Longleftrightarrow \quad \underline{\underline{R a_{jk} R^\dagger = a_{\ell m} \underbrace{O_{\ell j} O_{mk}}_{\text{9-d rep}}}}$$

Each index is like a vector

Example: $\overleftrightarrow{a} = \vec{V} \otimes \vec{W}$

$a_{jk} = V_j W_k$

These <u>Cartesian</u> tensors are not matched to the properties of angular momentum eigenstates, which generate <u>irreducible</u> reps of the rotation group.

Tensor operator: A set of n operators $T_1, \dots, T_n$, such that

$$R T_\mu R^\dagger = \sum_{\nu=1}^{n} T_\nu \underbrace{D_{\nu\mu}(R)}_{\text{n-d rep of rotation group}}$$

Irreducible spherical tensor operator: T_k^q, $q = -k, \dots, k$

$$\boxed{R T_k^q R^\dagger = \sum_{q'} T_k^{q'} \underbrace{D^{(k)}_{q'q}(R)}_{\langle k q' | R | k q \rangle \ \longleftarrow \ \text{unitary}}}$$

Physical quantities have k integral: relation to Y_ℓ^m

k is <u>not</u> the other quantum number

non-negative integer or half integer

Notice that $R^\dagger T_k^{q'} R = \sum_q D^{(k)*}_{q'q} T_k^q$

conjugated rep — doesn't matter for real matrices

$$\langle\psi'|T_k^{q'}|\psi'\rangle = \sum D^{(k)*}_{q'q}\langle\psi|T_k^q|\psi\rangle$$

Commutation relations:

$$\left(1-\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}\right)T_k^q\left(1+\frac{i}{\hbar}d\alpha\,\vec{u}\cdot\vec{J}\right) = \sum_{q'} T_k^{q'}\Big(\delta_{q'q}(1-i\,d\alpha\,u_z q) - \frac{i}{2}d\alpha(u_x-iu_y)C_+(k,q)\delta_{q',q+1} - \frac{i}{2}d\alpha(u_x+iu_y)C_-(k,q)\delta_{q',q-1}\Big)$$

$$= T_k^q - \frac{i}{\hbar}d\alpha\left[\vec{u}\cdot\vec{J}, T_k^q\right]$$

$$\left[\vec{u}\cdot\vec{J}, T_k^q\right] = \hbar\Big(u_z q T_k^q + \frac{1}{2}(u_x-iu_y)C_+(k,q)T_k^{q+1} + \frac{1}{2}(u_x+iu_y)C_-(k,q)T_k^{q-1}\Big)$$

$$= u_z\left[J_z,T_k^q\right] + \frac{1}{2}(u_x-iu_y)\left[J_+,T_k^q\right] + \frac{1}{2}(u_x+iu_y)\left[J_-,T_k^q\right]$$

$$\boxed{\begin{aligned}\left[J_z,T_k^q\right] &= \hbar q T_k^q\\ \left[J_\pm,T_k^q\right] &= \hbar C_\pm(k,q)T_k^{q\pm1}\end{aligned}}$$

$$V_x = -\frac{1}{\sqrt{2}}(V^+ - V^-)$$
$$V_y = \frac{i}{\sqrt{2}}(V^+ + V^-)$$
$$\vec{V}\cdot\vec{W} = V^0W^0 - V^+W^- - V^-W^+$$

Examples:

① $k=0$: $\left[T_0^0,\vec{J}\right]=0$ scalar

② $k=+1$: $T_1^0 = V_z = V^0,\ T_1^{\pm1} = \mp\frac{1}{\sqrt{2}}(V_x \pm iV_y) = V^\pm$

③ $k=+2$:

$$a_{jk} = a_{[jk]} + \underbrace{a_{(jk)} - \frac{1}{3}\delta_{jk}a_{ll}}_{b_{jk}} + \frac{1}{3}\delta_{jk}a_{ll}$$

Quadrupole moment

vector — b_{jk} symmetric trace-free — scalar

$$T_2^0 = \frac{1}{\sqrt{6}}(2b_{zz} - b_{xx} - b_{yy})$$

$$T_2^{\pm 1} = \mp(b_{xz} \pm i b_{yz})$$

$$T_2^{\pm 2} = \frac{1}{2}(b_{xx} - b_{yy} \pm 2i b_{xy})$$

Wigner-Eckart theorem:

Use $R T_k^q R^\dagger = \sum_{q'} T_k^{q'} D_{q'q}^{(k)}(R)$

Take a matrix element

other quantum numbers

$$\langle \alpha' j' m' | R T_k^q R^\dagger | \alpha j m \rangle = \sum_{q'} \langle \alpha' j' m' | T_k^{q'} | \alpha j m \rangle D_{q'q}^{(k)}(R)$$

$$R^\dagger | \alpha j m \rangle = \sum_{\mu} | \alpha j \mu \rangle \underbrace{\langle \alpha j \mu | R^\dagger | \alpha j m \rangle}_{\langle \alpha j m | R | \alpha j \mu \rangle^* = D_{m\mu}^{(j)*}}$$

$$= \sum_{\mu} | \alpha j \mu \rangle D_{m\mu}^{(j)*}(R)$$

$$\langle \alpha' j' m' | R = \sum_{\mu'} D_{m'\mu'}^{(j')*}(R) \langle \alpha' j' \mu' |$$

$$\boxed{\sum_{\mu, \mu'} D_{m'\mu'}^{(j')*} \langle \alpha' j' \mu' | T_k^q | \alpha j \mu \rangle D_{m\mu}^{(j)*} = \sum_{q'} \langle \alpha' j' m' | T_k^{q'} | \alpha j m \rangle D_{q'q}^{(k)}}$$

Cf. $$\sum_{\mu, \mu'} D_{m'\mu'}^{(j')} \langle j k \mu q | j' \mu' \rangle D_{m\mu}^{(j)*} = \sum_{q'} \langle j k m q' | j' m' \rangle D_{q'q}^{(k)}$$

$\hookrightarrow$ Given $j k j'$, fixes value of $\langle j k \mu q | j' \mu' \rangle$ up to a constant; $\therefore$ $\langle \alpha' j' \mu' | T_k^q | \alpha j \mu \rangle = C(\alpha, \alpha', j, j', k) \langle j k \mu q | j' \mu' \rangle$

Wigner-Eckart theorem:

$$\langle \alpha' j' m' | T_k^q | \alpha j m \rangle = \underbrace{\langle j k m q | j' m' \rangle}_{\substack{\uparrow \\ \text{all dependence} \\ \text{on } m', m, \text{ and } q}} \underbrace{\langle \alpha' j' \| T_k \| \alpha j \rangle}_{\text{reduced matrix element}}$$

Geometric statement about orientation properties m, m', q'

Angular-momentum selection rules: $\langle \alpha' j' m' | T_k^q | \alpha j m \rangle$ vanishes unless

$$m + q = m' \qquad \Longleftrightarrow \qquad \boxed{q = m' - m}$$

$$|j - k| \leq j' \leq j + k \qquad \Longleftrightarrow \qquad \boxed{|j' - j| \leq k \leq j' + j}$$

Scalar operator $(k = q = 0)$: $\Delta m = m' - m = 0$

$\Delta j = j' - j = 0$

Vector operator $(k = 1)$: $|\Delta j| \leq 1,\ j' + j \geq 1 \Rightarrow \Delta j = 0, \pm 1$

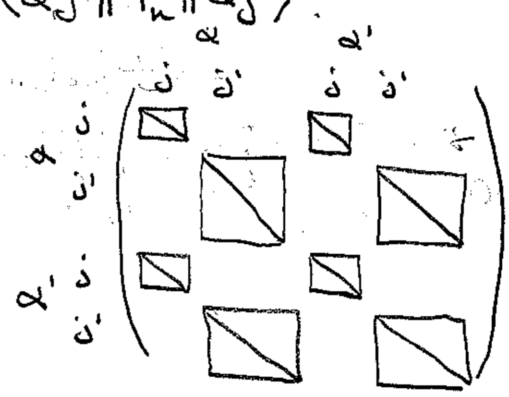
$j = j' = 0$ excluded

$\Delta m = q = 0, \pm 1$

Scalar operator:

$$\langle \alpha' j' m' | T_0^0 | \alpha j m \rangle = \underbrace{\langle j 0 m 0 | j' m' \rangle}_{\delta_{jj'} \delta_{mm'}} \langle \alpha' j' \| T_k \| \alpha j \rangle$$

Thus a scalar operator looks like a



Vector operators: The projection theorem

$$T_1^0 = V_z = V^0 \quad , \quad T_1^{\pm} = \mp\frac{1}{\sqrt{2}}(V_x \pm iV_y) = V^{\pm}$$

Apply W-E to $\vec{J}$:

$$\langle\alpha' jj | J_z | \alpha jj\rangle = \underbrace{\langle j1j0|jj\rangle}_{\sqrt{\frac{j}{j+1}}} \overbrace{\langle\alpha' j \| \vec{J} \| \alpha j\rangle}^{\text{a number, not a vector}}$$

$$\| $$

$\hbar j \delta_{\alpha'\alpha} \longleftarrow$ Special to ang mom

$$\therefore \langle\alpha' j \| \vec{J} \| \alpha j\rangle = \delta_{\alpha'\alpha}\hbar\sqrt{j(j+1)}$$

get reduced matrix element from a simple matrix element

For any vector operator, W-E $\Rightarrow$

$$\langle\alpha' jm' | V^q | \alpha jm\rangle = \langle j1mq|jm'\rangle \langle\alpha' j \| \vec{V} \| \alpha j\rangle$$

$$\Longrightarrow \langle\alpha jm' | J^q | \alpha jm\rangle = \langle j1mq|jm'\rangle \langle\alpha j \| \vec{J} \| \alpha j\rangle \longleftarrow \text{indep of } \alpha$$

$$\Longrightarrow \langle j1mq|jm'\rangle = \frac{\langle jm' | J^q | jm\rangle}{\hbar\sqrt{j(j+1)}}$$

This is a good way to get these C-G coefficients

In a subspace $|\alpha jm\rangle$ or between $|\alpha jm\rangle$ and $|\alpha' jm'\rangle$, $\vec{V}$ is proportional to $\vec{J}$.

$$\therefore \langle\alpha' jm' | V^q | \alpha jm\rangle = \langle jm' | J^q | jm\rangle \frac{\langle\alpha' j \| \vec{V} \| \alpha j\rangle}{\hbar\sqrt{j(j+1)}}$$

Clever part: (gets reduced matrix element in terms of scalar operator)

$$\langle\alpha' jm' | \overbrace{\vec{J}\cdot\vec{V}}^{\text{Scalar op}} | \alpha jm\rangle = c_{jm'm} \langle\alpha' j \| \vec{V} \| \alpha j\rangle$$

$J^0V^0 - J^+V^- - J^-V^+$

no dependence on $\vec{V}$ or on α, α'

let $\vec{V} = \vec{J}$

$$\therefore \underbrace{\langle\alpha jm' | \vec{J}^2 | \alpha jm\rangle}_{\hbar^2 j(j+1)\delta_{m'm}} = c \underbrace{\langle\alpha j \| \vec{J} \| \alpha j\rangle}_{\delta_{\alpha'\alpha}\hbar\sqrt{j(j+1)}}$$

$$\Longrightarrow c_{jm'm} = \hbar\sqrt{j(j+1)}\,\delta_{m'm}$$

$\uparrow$ Why?

$$\therefore \boxed{\langle \alpha' j m' | \vec{J}\cdot\vec{V} | \alpha j m\rangle = \hbar\sqrt{j(j+1)}\, \delta_{m'm} \langle \alpha' j \| \vec{V} \| \alpha j\rangle}$$

$$\therefore \boxed{\langle \alpha' j m' | V^q | \alpha j m\rangle = \langle j m' | J^q | j m\rangle \frac{\langle \alpha' j m | \vec{J}\cdot\vec{V} | \alpha j m\rangle}{\hbar^2 j(j+1)}}$$

projection theorem

value of m doesn't matter

Landé g-factor:

Magnetic moment $\vec{M} = -\frac{e}{2m_e}(g_L \vec{L} + g_S \vec{S})$

SI

$\frac{e}{2m_e c}$ in cgs Gaussian

always $\mu_B/\hbar$

The magnetic moment $|M| \equiv \langle \alpha j j | M_z | \alpha j j\rangle$

$$= \underbrace{\langle j 1 j 0 | j j\rangle}_{\sqrt{\frac{j}{j+1}}} \langle \alpha j \| \vec{M} \| \alpha j\rangle$$

use $|M|$ instead of reduced matrix element $\longrightarrow$ $|M| = \sqrt{\frac{j}{j+1}}\langle \alpha j \| \vec{M} \| \alpha j\rangle$

Can get any matrix element

$$\langle \alpha j m | M^q | \alpha j m'\rangle \overset{\text{W-E theorem}}{=} \underbrace{\langle j 1 m q | j m'\rangle}_{\frac{\langle j m' | J^q | j m\rangle}{\hbar\sqrt{j(j+1)}}} \langle \alpha j \| \vec{M} \| \alpha j\rangle = \underbrace{\langle j 1 m q | j m'\rangle \sqrt{\frac{j+1}{j}}}_{\frac{\langle j m' | J^q | j m\rangle}{\hbar j}} |M|$$

$$|M| = \sqrt{\frac{j}{j+1}}\langle \alpha j \| \vec{M} \| \alpha j\rangle \overset{\text{projection theorem}}{=} \frac{1}{\hbar(j+1)}\langle \alpha j j | \vec{J}\cdot\vec{M} | \alpha j j\rangle$$

$$= -\frac{e}{2m_e}\frac{1}{\hbar(j+1)}\langle \alpha j j | \underbrace{g_L \vec{J}\cdot\vec{L} + g_S \vec{J}\cdot\vec{S}}_{\frac{1}{2}\left(J^2(g_L+g_S) + (L^2-S^2)(g_L-g_S)\right)} | \alpha j j\rangle$$

$J\cdot L = L^2 + L\cdot S$

$J^2 = L^2 + S^2 + 2L\cdot S$

$J\cdot L = \frac{1}{2}(J^2 + L^2 - S^2)$

$J\cdot S = \frac{1}{2}(J^2 + S^2 - L^2)$

If $\alpha \longrightarrow \alpha l s$

$$|M| = \sqrt{\frac{j}{j+1}} \langle \alpha l s j \| \vec{M} \| \alpha l s j \rangle$$

$$= -\frac{e}{2m_e} \frac{1}{\hbar (j+1)} \frac{\hbar^2}{2} \left[(g_L + g_s) j(j+1) + (g_L - g_s)\left(l(l+1) - s(s+1) \right) \right]$$

$$= -\frac{e}{2m_e} \hbar j \underbrace{\frac{1}{2}\left(g_L + g_s + \frac{(g_L - g_s)\left(l(l+1) - s(s+1) \right)}{j(j+1)} \right)}_{g_{Lande}}$$

$$\boxed{\langle \alpha l s j m | M^q | \alpha l s j m' \rangle = -\frac{e}{2m_e} g_{Lande} \langle j m' | J^q | j m \rangle}$$

It is reasonably clear that the LHS has the same nonzero matrix elements as the RHS, but that those matrix elements are related by a constant — well, that's a nontrivial fact.