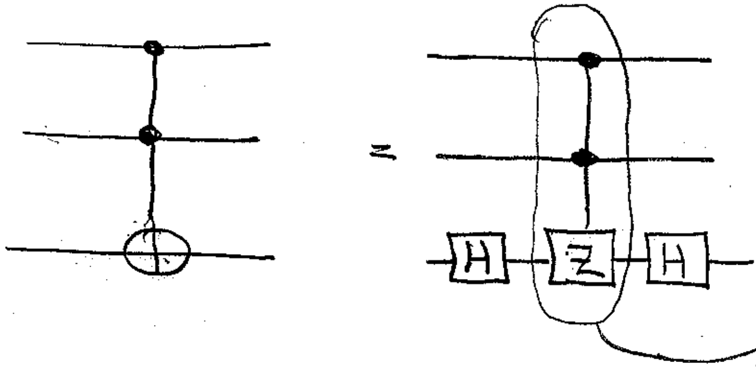
Solution

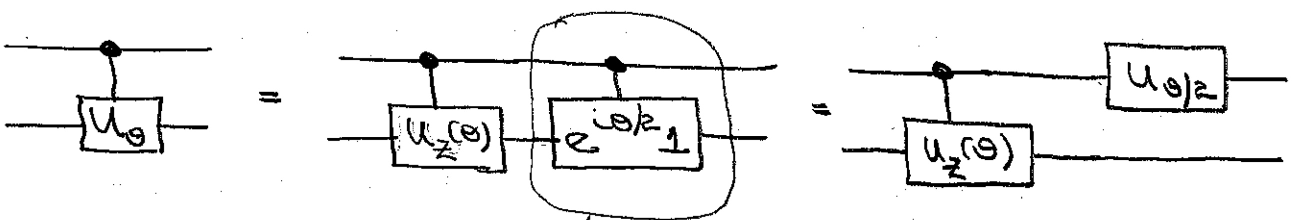
1.5. Constructing a circuit for the Toffoli gate

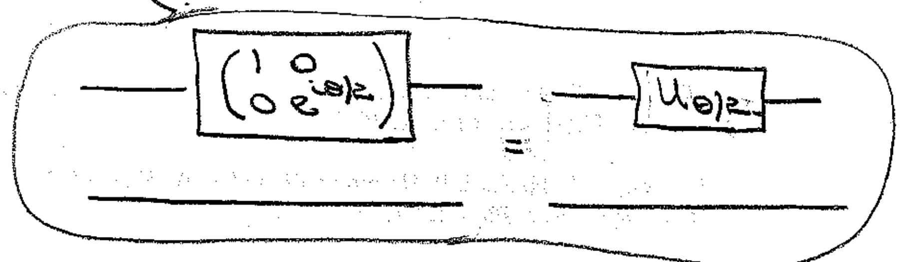


We'll construct $C^{(2)}(Z)$

$$\text{Let } U_\theta \equiv e^{i\theta/2}\,U_z(\theta) = e^{i\theta/2}\,e^{-iZ\theta/2} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

(a) First construct $C^{(1)}(U_\theta)$:





Write $U_z(\theta) = U_z^2(\theta/2)$

$$X\,U_z(\alpha)\,X = X\,e^{-iZ\alpha/2}\,X = e^{iZ\alpha/2} = U_z^\dagger(\alpha)$$

$\uparrow$

$XZX = -Z$

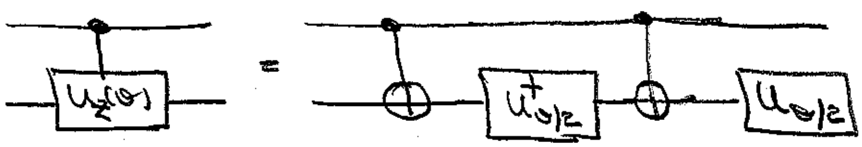
$$\boxed{U_z(\alpha) = X\,U_z^\dagger(\alpha)\,X}$$

$\longrightarrow$ This is the basic fact we're going to live on.

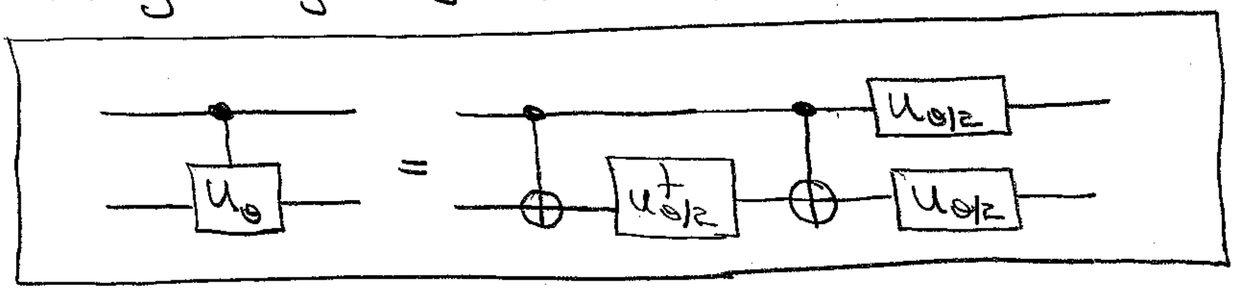
$$U_z(\theta) = U_z(\theta/2)\, X\, U_z^\dagger(\theta/2)\, X = X\, U_z^\dagger(\theta/2)\, X\, U_z(\theta/2)$$

$$= U_{\theta/2}\, X\, U_{\theta/2}^\dagger\, X \quad = X\, U_{\theta/2}^\dagger\, X\, U_{\theta/2}$$

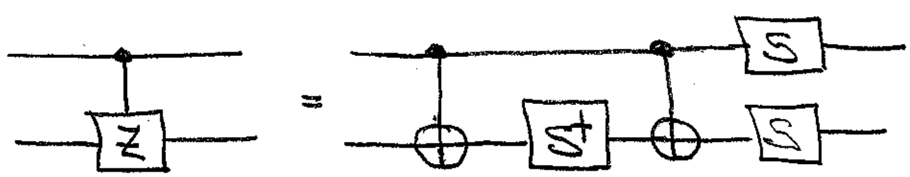
These are the decompositions of NC Corollary 4.2, simpler here because we're dealing with rotations about the z axis.



Putting things together, we get

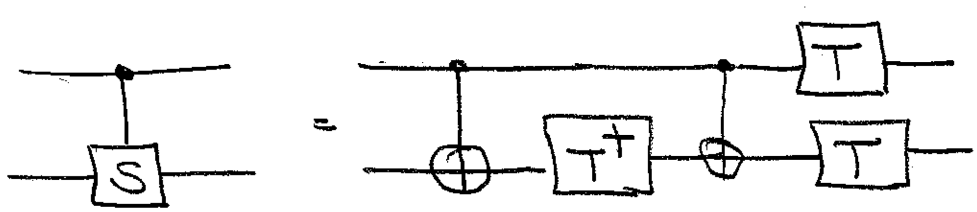


$\theta = \pi$: $U_\pi = Z$, $U_{\pi/2} = S$

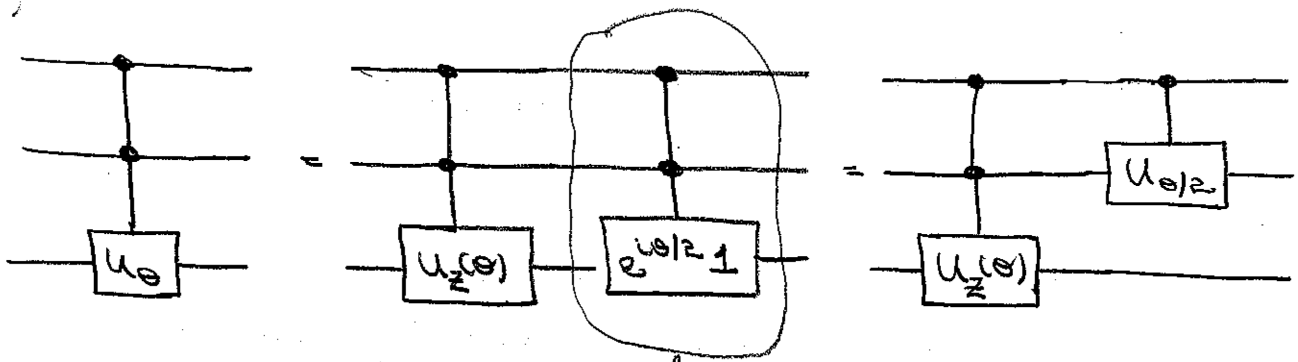


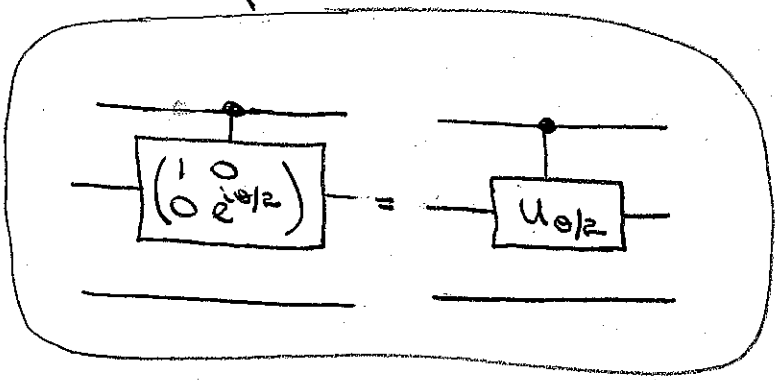
Kind of an odd decomposition for C-SIGN, but it does work.

$\theta = \pi/2$: $U_{\pi/2} = Z$, $U_{\pi/4} = T$

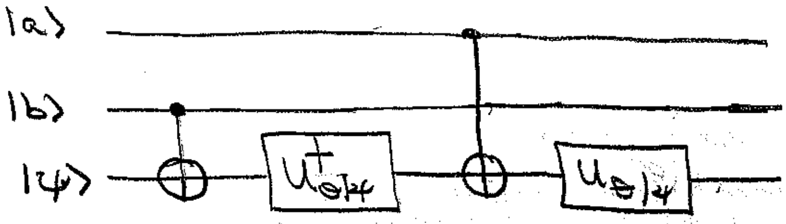


(b) Now onto $C^{(2)}(U_\theta)$:





$$U_z(\theta) = U_z^4(\theta/4) = U_z(\theta/4)\, X\, U_z^\dagger(\theta/4)\, X\, U_z(\theta/4)\, X\, U_z^\dagger(\theta/4)\, X$$
$$= U_{\theta/4}\, X\, U_{\theta/4}^\dagger\, X\, U_{\theta/4}\, X\, U_{\theta/4}^\dagger\, X$$



$|0,0\rangle \otimes |\psi\rangle \longrightarrow |0,0\rangle \otimes |\psi\rangle \longrightarrow |0,0\rangle \otimes |\psi\rangle$

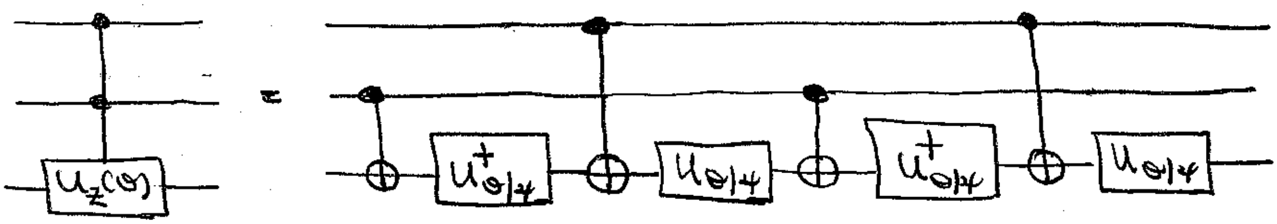
$|0,1\rangle \otimes |\psi\rangle \longrightarrow |0,1\rangle \otimes X|\psi\rangle \longrightarrow |0,1\rangle \otimes |\psi\rangle$ $\quad X^2 = 1$

$|1,0\rangle \otimes |\psi\rangle \longrightarrow |1,0\rangle \otimes U_{\theta/4} X U_{\theta/4}^\dagger |\psi\rangle \longrightarrow |1,0\rangle \otimes |\psi\rangle$ $\quad (UXU^\dagger)^2 = \mathbb{1}$

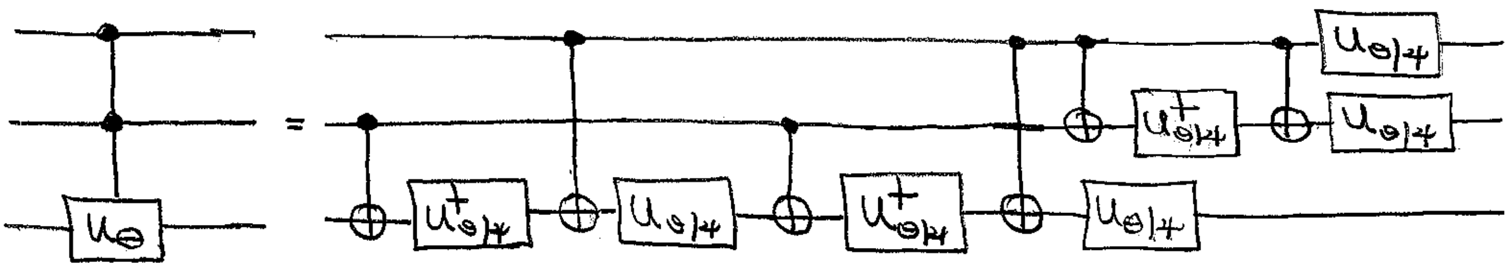
$|1,1\rangle \otimes |\psi\rangle \longrightarrow |1,1\rangle \otimes U_{\theta/4} X U_{\theta/4}^\dagger X |\psi\rangle \longrightarrow |1,1\rangle \otimes U_z(\theta) |\psi\rangle$

1st application of circuit $\qquad$ 2nd application

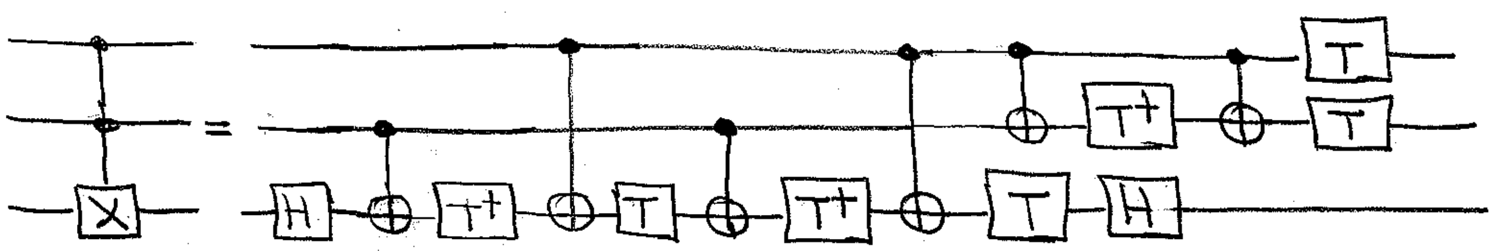
We have



Expanding the circuit at the top of p. 3, we get



$\theta=\pi$ gives $C^{(2)}(Z)$, with $U_{\pi/4}=T$. By including Hadamards at the beginning and end of the target wire, we get the Toffoli, $C^{(2)}(X)$.



This circuit differs from NC's Figure 4.9 by the final gates on the control wires. These final gates are a controlled-S, which is necessary to adjust relative phases. The textbook uses a different circuit for the controlled-S.