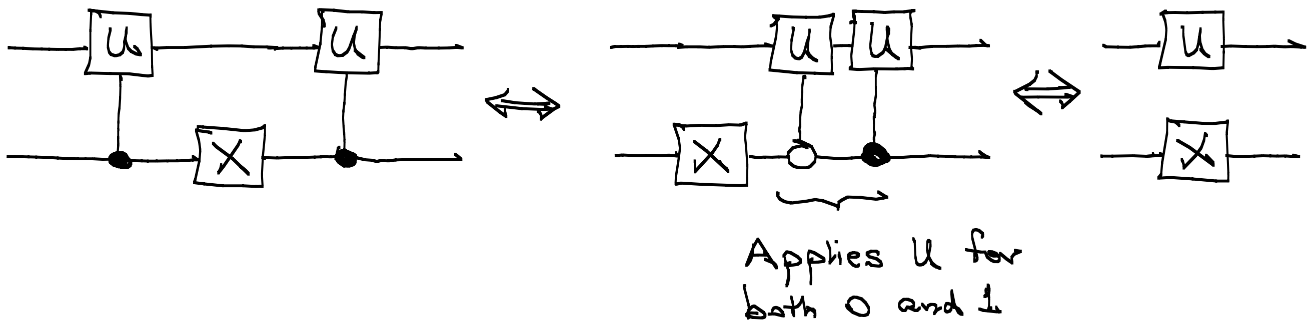
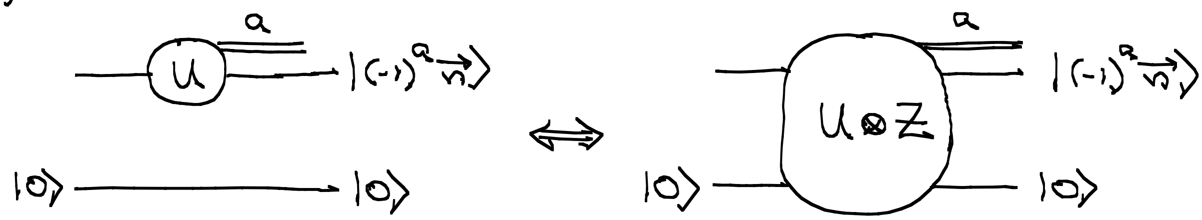
Solution 2.4

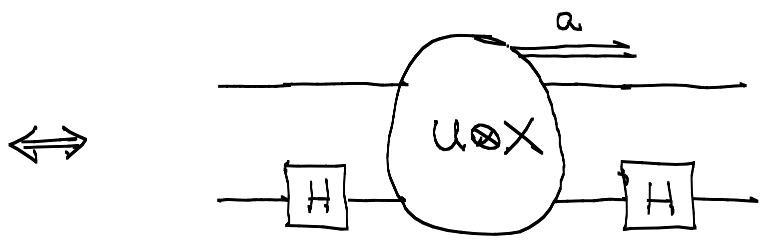
$$U = \vec{\sigma}\cdot\vec{n} = U^\dagger$$

(a)

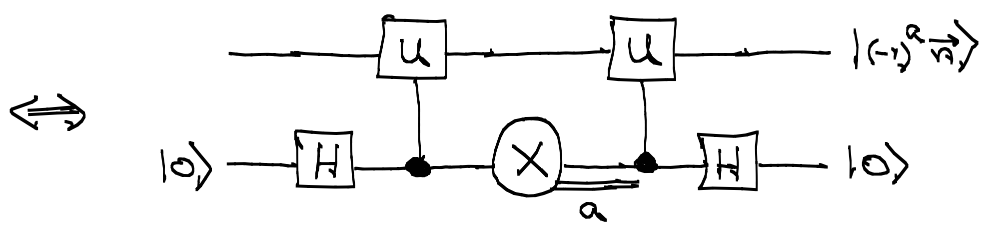


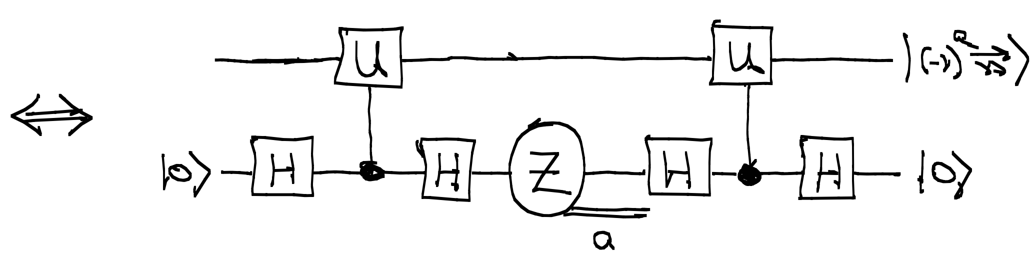
(b)





Here we use (a)





(c) State tracking in final circuit of (b):

$$|\psi\rangle\otimes|0\rangle \xrightarrow[I\otimes H]{} |\psi\rangle\otimes|\bar{0}\rangle = |\psi\rangle\otimes\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$$

$$\xrightarrow[C\text{-}U]{} \frac{1}{\sqrt{2}}\left(|\psi\rangle\otimes|0\rangle + U|\psi\rangle\otimes|1\rangle\right)$$

$$\xrightarrow[I\otimes H]{} \frac{1}{2}\Big((I+U)|\psi\rangle\otimes|0\rangle + (I-U)|\psi\rangle\otimes|1\rangle\Big)$$

$$\xrightarrow[M_z]{} \frac{1}{2\sqrt{p_a}}\big(I+(-1)^a U\big)|\psi\rangle\otimes|a\rangle$$

$$p_a = \Big\| \frac{1}{2}\big(I+(-1)^a U\big)|\psi\rangle\Big\|^2$$

$$= \frac{1}{4}\langle\psi|\big(I+(-1)^a U\big)\big(I+(-1)^a U\big)|\psi\rangle$$

$$= \frac{1}{2}\Big(1+(-1)^a\langle\psi|U|\psi\rangle\Big)$$

This has to be the answer, because $1 = p_0 + p_1$ and $\langle\psi|U|\psi\rangle = p_0 - p_1$

$$\xrightarrow[I\otimes H]{} \frac{1}{2\sqrt{p_a}}\big(I+(-1)^a U\big)|\psi\rangle\otimes|\bar{a}\rangle = \frac{1}{2\sqrt{2p_a}}\big(I+(-1)^a U\big)|\psi\rangle\otimes\big(|0\rangle+(-1)^a|1\rangle\big)$$

$$\xrightarrow[C\text{-}U]{} \frac{1}{2\sqrt{2p_a}}\Big[\big(I+(-1)^a U\big)|\psi\rangle\otimes|0\rangle + \underbrace{(-1)^a U\big(I+(-1)^a U\big)}_{I+(-1)^a U}|\psi\rangle\otimes|1\rangle\Big)$$

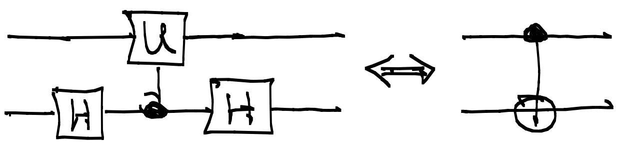
$$= \frac{1}{2\sqrt{p_a}}\big(I+(-1)^a U\big)|\psi\rangle\otimes|\bar{0}\rangle$$

$$\xrightarrow[I\otimes H]{} = \underbrace{\frac{1}{2\sqrt{p_a}}\big(I+(-1)^a U\big)|\psi\rangle}_{\equiv|\psi_a\rangle}\otimes|0\rangle$$

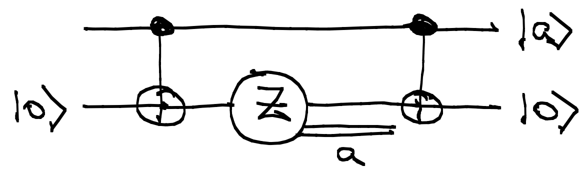
$$U|\psi_a\rangle = (-1)^a|\psi_a\rangle \Longrightarrow |\psi_a\rangle = (\text{phase})\times|(-1)^a\vec{n}\rangle$$

best we can expect because the state reduction at the measurement has an arbitrary phase

(d) When $U = Z$



so the circuit is



$$(c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle$$

$$\xrightarrow[\text{C-NOT}]{} c_0|00\rangle + c_1|11\rangle$$

$$\xrightarrow[M_Z]{} |aa\rangle \qquad p_a = |c_a|^2$$

$$\xrightarrow[\text{C-NOT}]{} |a, a \oplus a\rangle = |a\rangle \otimes |0\rangle$$