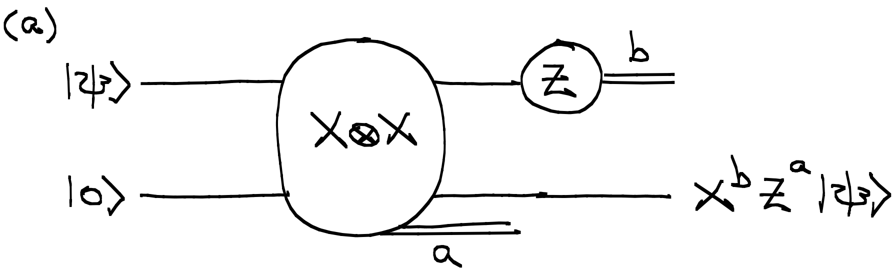
Solution 2.5

(a)



Forget about normalization.

$$|\psi\rangle\otimes|0\rangle = \sum_c C_c|c,0\rangle = \sum_{c,d,e} c_c(-1)^{cd}\,|\bar{d},\bar{e}\rangle$$

measurement of $X\otimes X$ projects out case $e=d$ for $a=0$ and $e=d\oplus 1$ for $a=1$, so $e=d\oplus a$.

$$\longrightarrow \sum_{c,d} c_c(-1)^{cd}\,|\bar{d},\overline{d\oplus a}\rangle$$

$$= \sum_{c,d,e,f} c_c(-1)^{cd}(-1)^{de}(-1)^{(d+a)f}\,|e,f\rangle$$

measurement of $Z\otimes I$ sets $e=b$ and eliminates top qubit

$$\longrightarrow \sum_{c,d,f} c_c(-1)^{cd}(-1)^{db}(-1)^{(d+a)f}\,|f\rangle$$

$$= \sum_{c,f} c_c(-1)^{af}|f\rangle\underbrace{\sum_d(-1)^{d(c+b+f)}}_{\propto\,\delta_{f,c\oplus b}}$$

forget since we're not normalizing

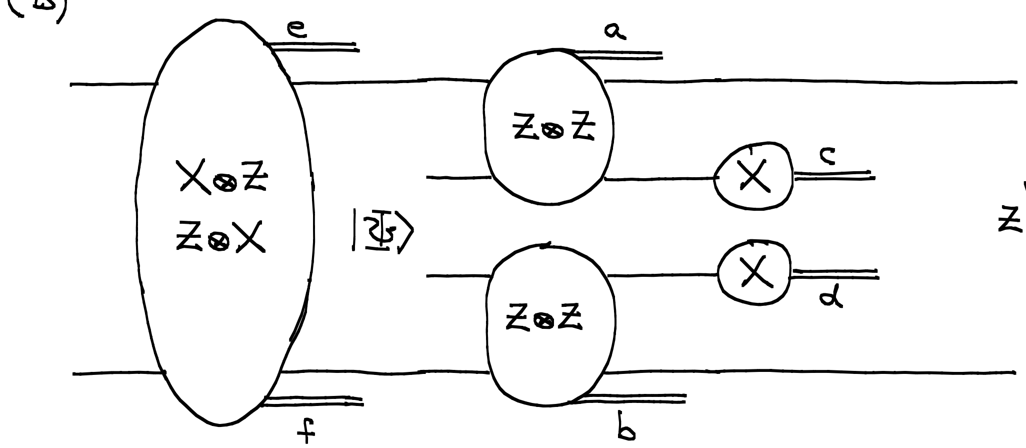
$$= \sum_c c_c(-1)^{a(c+b)}\,|c\oplus b\rangle$$

$$= (-1)^{ab}X^b\sum_c c_c(-1)^{ac}|c\rangle$$

$$= (-1)^{ab}X^bZ^a|\psi\rangle$$

irrelevant phase

(b)



$$Z^{b+c+e}X^{a}\otimes Z^{a+d+f}X^{b}\Lambda(Z)|\Psi\rangle$$

Write $|\Psi\rangle$ as $|\Psi\rangle = \sum_{g,h} c_{gh}|g,h\rangle$

First we need the simultaneous eigenstates of $X\otimes Z$ and $Z\otimes X$. But this is easy, since we can get these operators from $X\otimes X$ and $Z\otimes Z$ by conjugating with $I\otimes H$.

$$I\otimes H\; X\otimes X\; I\otimes H = X\otimes Z$$

$$I\otimes H\; Z\otimes Z\; I\otimes H = Z\otimes X$$

Since we already know that $X\otimes X|\beta_{ef}\rangle = (-1)^{e}|\beta_{ef}\rangle$ and $Z\otimes Z|\beta_{ef}\rangle = (-1)^{f}|\beta_{ef}\rangle$, we have immediately that

$$X\otimes Z\; I\otimes H|\beta_{ef}\rangle = (-1)^{e}\, I\otimes H|\beta_{ef}\rangle\,,$$

$$Z\otimes X\; I\otimes H|\beta_{ef}\rangle = (-1)^{f}\, I\otimes H|\beta_{ef}\rangle\,.$$

So the simultaneous eigenstates of $X\otimes Z$ and $Z\otimes X$ are

$$I\otimes H|\beta_{ef}\rangle = \sum_{c}(-1)^{ec}\, I\otimes H|c, c\oplus f\rangle = \sum_{c,d}(-1)^{ec+dc+df}|c,d\rangle$$

This is the phase we're going to use.

Let's forget about normalization and write the state after the first measurement as

$$I\otimes H|\beta_{ef}\rangle\otimes|\Psi\rangle = \sum_{\substack{c,d\\g,h}} (-1)^{ec+dc+df}\, c_{gh}\,\underbrace{|c,g\rangle}_{\text{top 2 qubits}}\,\underbrace{|h,d\rangle}_{\text{bottom 2 qubits}}$$

measurements of $Z\otimes Z$ pick out even and odd parity subspaces for 2 pairs of qubits, i.e.,
$c = g\oplus a$ and $d = h\oplus b$

$$\longrightarrow \sum_{g,h}(-1)^{eg+ea+gh+gb+ah+ab+hf+bf}\, c_{gh}\,|g\oplus a,g\rangle|h,h\oplus b\rangle$$

$$= (-1)^{ea+bf+ab}\sum_{g,h}(-1)^{g(e+b)}(-1)^{h(f+a)}\underbrace{(-1)^{gh}}_{\text{This is the key}} c_{gh}\,|g\oplus a,g\rangle|h,h\oplus b\rangle$$

Go to the $|\pm\rangle$ basis for the middle qubits

$$\longrightarrow\ = (-1)^{ea+bf+ab}\sum_{\substack{c,d\\g,h}}(-1)^{g(e+b)}(-1)^{h(f+a)}(-1)^{gh}c_{gh}$$
$$\times\,(-1)^{cg}|g\oplus a,\bar{c}\rangle\cdot(-1)^{dh}|\bar{d},h\oplus b\rangle$$

X measurements project out c and d; eliminate middle qubits

$$\longrightarrow (-1)^{ea+bf+ab}\sum_{g,h}(-1)^{g(e+b+c)}(-1)^{h(f+a+d)}$$
$$\times\,(-1)^{gh}c_{gh}\,\underbrace{|g\oplus a,h\oplus b\rangle}_{X^a\otimes X^b|g,h\rangle}$$

$$= (-1)^{ea+bf+ab}\,X^a\otimes X^b\sum_{g,h}(-1)^{gh}c_{gh}\,Z^{e+b+c}\otimes Z^{f+a+d}|g,h\rangle$$

$$= (-1)^{ea+bf+ab}\,X^aZ^{e+b+c}\otimes X^bZ^{f+a+d}\underbrace{\sum_{g,h}(-1)^{gh}c_{gh}|g,h\rangle}_{\Lambda(Z)|\Psi\rangle}$$

$$= \underbrace{(-1)^{ab+ac+bd}}_{\text{meaningless global phase}}\,Z^{b+c+e}X^a\otimes Z^{a+d+f}X^b\,\Lambda(Z)|\Psi\rangle$$