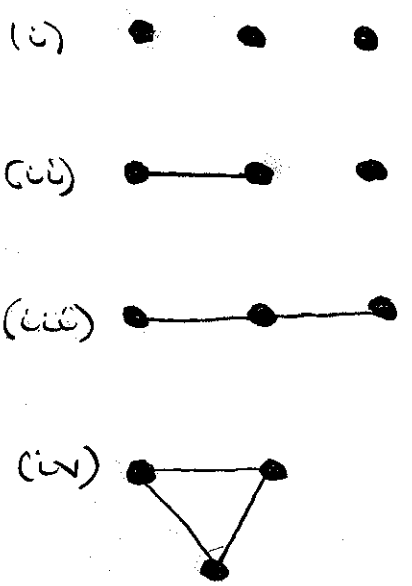
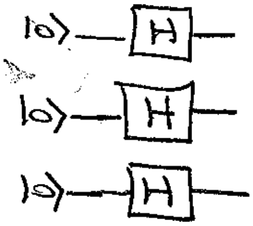
Solution 2.6.

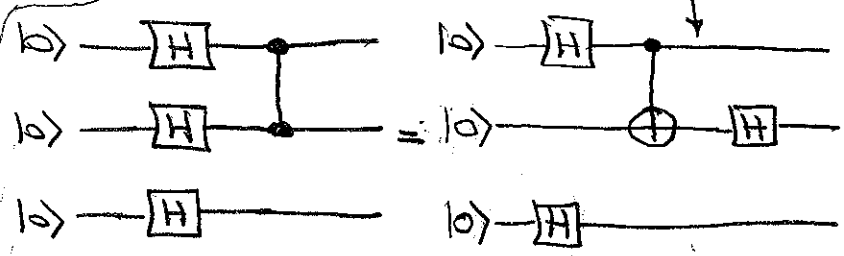
(a) 3-qubit graph states up to SWAPs.





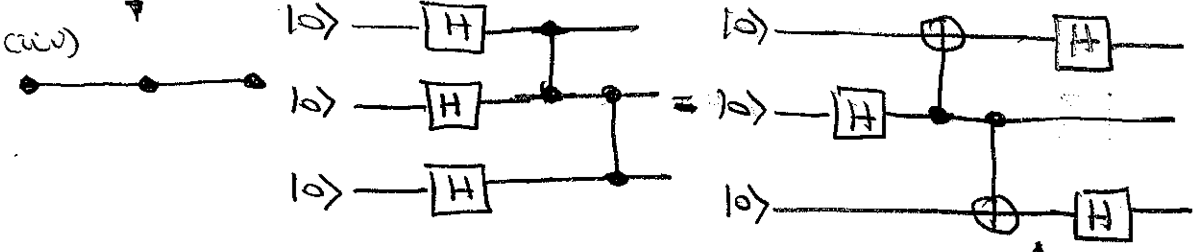
$$|\bar{0}\bar{0}\bar{0}\rangle = \frac{1}{2\sqrt{2}}\big(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle\big)$$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\otimes|\bar{0}\rangle$



$$\frac{1}{\sqrt{2}}\big(|0\bar{0}\rangle + |1\bar{1}\rangle\big)\otimes|\bar{0}\rangle$$
$$= \frac{1}{2\sqrt{2}}\big(|00\rangle + |01\rangle + |10\rangle - |11\rangle\big)\otimes(|0\rangle + |1\rangle)$$
$$= \frac{1}{2\sqrt{2}}\big(|000\rangle + |100\rangle + |010\rangle + |001\rangle - |110\rangle + |101\rangle + |011\rangle - |111\rangle\big)$$

(iv)

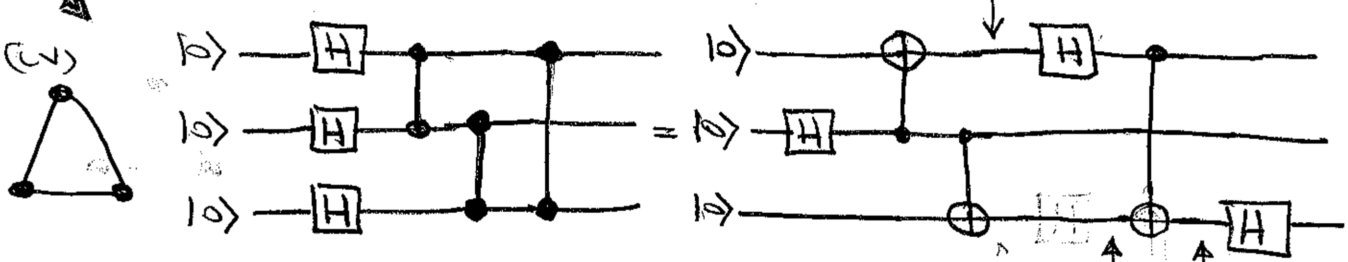


$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \equiv |\psi\rangle$

$$\boxed{\frac{1}{\sqrt{2}}\left(|\bar{0}\bar{0}\bar{0}\rangle + |\bar{1}\bar{1}\bar{1}\rangle\right) = \frac{1}{2\sqrt{2}}\Big(|000\rangle + |100\rangle + |010\rangle + |001\rangle - |110\rangle + |101\rangle - |011\rangle + |111\rangle\Big)}$$

$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

(v)



$\frac{1}{2}\big(|000\rangle + |101\rangle + |011\rangle - |110\rangle\big)$

$|\phi\rangle$ from p. 3

$$\boxed{\begin{aligned}&\frac{1}{2}\left(|\bar{0}\bar{0}\bar{0}\rangle + |\bar{1}\bar{0}\bar{1}\rangle + |\bar{0}\bar{1}\bar{1}\rangle - |\bar{1}\bar{1}\bar{0}\rangle\right)\\ &= \frac{1}{2\sqrt{2}}\Big(|000\rangle + |100\rangle + |010\rangle + |001\rangle - |110\rangle - |101\rangle - |011\rangle - |111\rangle\Big)\end{aligned}}$$

These results are consistent with the general formula

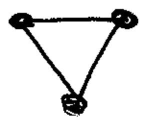
$$\left|\begin{matrix}\text{graph}\\ \text{state}\end{matrix}\right\rangle = \frac{1}{2^{N/2}} \sum_a |a\rangle \, (-1)^{\sum'_{jk} a_j a_k}$$

Sum over neighboring pairs (puts in a -1 for every pair of neighbors both of which are in $|1\rangle$)

(b) (i)  $|\bar{0}\bar{0}\bar{0}\rangle$

(ii)  $\frac{1}{\sqrt{2}}\left(|0 0 \bar{0}\rangle + |1 1 \bar{0}\rangle\right)$

(iii)  $\frac{1}{\sqrt{2}}\left(|\bar{0} 0 \bar{0}\rangle + |\bar{1} 1 \bar{1}\rangle\right)$

(iv)  $\frac{1}{2}\left(|0 0 \bar{0}\rangle + |1 0 \bar{1}\rangle + |0 1 \bar{1}\rangle - |1 1 \bar{0}\rangle\right)$

Use equivalent states for analysis ③

$|000\rangle$

$\frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right)$

$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \equiv |\psi\rangle$

$\frac{1}{2}\left(|000\rangle + |101\rangle + |011\rangle - |110\rangle\right) \equiv |\phi\rangle$

These are invariant under all SWAPs.

① Local unitaries and SWAPs preserve the structure of eigenvalues of the marginal density operators.

② These being pure states, the eigenvalues of a 1-qubit marginal density operator are the same as the eigenvalues of the density operator for the other two qubits.

③ There are three 1 vs. 2 splits of the three qubits and thus three sets of marginal eigenvalues. For SWAP-invariant states, however, all three splits have the same eigenvalues.

	Marginal eigenvalues 3 vs. 1,2	2 vs. 1,3	1 vs. 2,3
$\vert 000\rangle$	$1, 0$	$1, 0$	$1, 0$
$\frac{1}{\sqrt{2}}\left(\vert 000\rangle + \vert 110\rangle\right)$	$1, 0$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
$\frac{1}{\sqrt{2}}\left(\vert 000\rangle + \vert 111\rangle\right)$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
$\frac{1}{2}\left(\vert 000\rangle + \vert 101\rangle + \vert 011\rangle - \vert 110\rangle\right)$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
(c) $\vert W\rangle = \frac{1}{\sqrt{3}}\left(\vert 100\rangle + \vert 010\rangle + \vert 001\rangle\right)$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}, \frac{2}{3}$

W state is also invariant under all SWAPs (b)-(c) The eigenvalues show that all these states are inequivalent, except perhaps the ones coming from graphs C(iii) and C(v).

(d) To find transformations that connect C(iii) and C(v), let's look at the 1 vs. 2,3 Schmidt decompositions:

$$\text{GHZ} \longrightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle \otimes |00\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |11\rangle$$

$$|\phi\rangle = \frac{1}{2}\left(|000\rangle + |101\rangle + |011\rangle - |110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Schmidt decompositions

If these are equivalent, there is a sequence of local unitaries and SWAPs that transform $|\psi\rangle$ to $|\phi\rangle$. The SWAPs can be commuted through the local unitaries to act on $|\psi\rangle$, where they have no effect. So, if $|\psi\rangle$ and $|\phi\rangle$ are equivalent, they are connected by local unitaries. Let's assume there are local unitaries U, V, W such that

$$|\phi\rangle = U \otimes V^\dagger \otimes W^\dagger |\psi\rangle = \frac{1}{\sqrt{2}} U|0\rangle \otimes V^\dagger \otimes W^\dagger |00\rangle + \frac{1}{\sqrt{2}} U|1\rangle \otimes V^\dagger \otimes W^\dagger |11\rangle$$

Notice that

$$U \otimes I \otimes I |\psi\rangle = \frac{1}{\sqrt{2}} \sum_a U|a\rangle \otimes |aa\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{a,b} |b\rangle \otimes U_{ba} |aa\rangle$$

Define $\mathcal{U}'|bb\rangle = \sum_a |aa\rangle\, \mathcal{U}'_{ab} = \sum_a |aa\rangle\, \mathcal{U}_{ba}$

two-qubit unitary acting in subspace spanned by $|00\rangle$ and $|11\rangle$

$\Longleftrightarrow \mathcal{U}'_{ab} = \mathcal{U}_{ba}$

$\mathcal{U}'$ is the transpose of $\mathcal{U}$ in mapping the subspace spanned by $|00\rangle$ and $|11\rangle$ unitarily to itself. We extend $\mathcal{U}'$ to map the subspace spanned by $|01\rangle$ and $|10\rangle$ unitarily to itself, say, by $\mathcal{U}'|01\rangle = |01\rangle$ and $\mathcal{U}'|10\rangle = |10\rangle$.

$$\mathcal{U}\otimes I\otimes I|\psi\rangle = \frac{1}{\sqrt{2}}\sum_b |b\rangle\otimes \mathcal{U}'|bb\rangle = I\otimes \mathcal{U}'|\psi\rangle$$

Thus our assumption is that

$$|\phi\rangle = I\otimes(V^\dagger\otimes W^\dagger)\,\mathcal{U}'|\psi\rangle \quad \underline{\text{or}} \quad I\otimes V\otimes W|\phi\rangle = I\otimes \mathcal{U}'|\psi\rangle$$

$\underline{\text{or}}$, projecting onto $|0\rangle$ and $|1\rangle$ for the 1st qubit,

$$\mathcal{U}'|00\rangle = V\otimes W\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = I\otimes WV^T\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$$

transposition property of Bell state $|\beta_{00}\rangle$

$$\mathcal{U}'|11\rangle = V\otimes W\underbrace{\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)}_{I\otimes XZ\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)} = I\otimes WXZV^T\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$$

Since the states on the right are orthogonal and maximally entangled, the states on the left must be orthogonal, maximally entangled states in the subspace spanned by $|00\rangle$ and $|11\rangle$, i.e.,

This phase can be chosen to be 1.

$$\begin{cases} \mathcal{U}'|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\gamma}|11\rangle) = I\otimes WV^T\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \\ \mathcal{U}'|11\rangle = \frac{1}{\sqrt{2}}(e^{i\alpha}|00\rangle - e^{i\beta}|11\rangle) = I\otimes WXZV^T\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \end{cases}$$

Unitarity requires $e^{i\alpha} - e^{i(\beta-\gamma)} = 0$

$$\begin{cases} WV^T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{pmatrix} & \Longrightarrow \quad V^T = W^\dagger \begin{pmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{pmatrix} \\ WXZV^T = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & -e^{i\beta} \end{pmatrix} & \Longrightarrow \quad WXZW^\dagger \begin{pmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & -e^{i\beta} \end{pmatrix} \end{cases}$$

So $$\underset{-iWYW^\dagger}{\underset{\shortparallel}{WXZW^\dagger}} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & \underbrace{-e^{i(\beta-\gamma)}}_{-e^{i\alpha}} \end{pmatrix} = e^{i\alpha} Z$$

$$\Longrightarrow WYW^\dagger = ie^{i\alpha} Z$$

Unitarity of W implies that $ie^{i\alpha} = \pm 1$

Let's make a particular choice, realizing that $U, V,$ and W are not unique. Choose $\alpha = \pi/2$, $\beta = \pi/2$, $\gamma = 0$. So we have $WYW^\dagger = -Z$, which is satisfied by $W = HS^\dagger H$ (W is not unique, but we'll use this one).

$S^\dagger Y S = X$

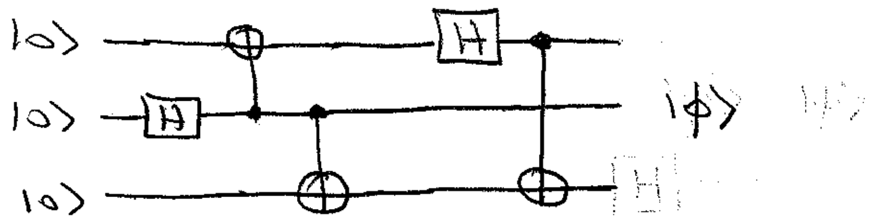
$$W = HS^\dagger H, \quad W^\dagger = HSH$$

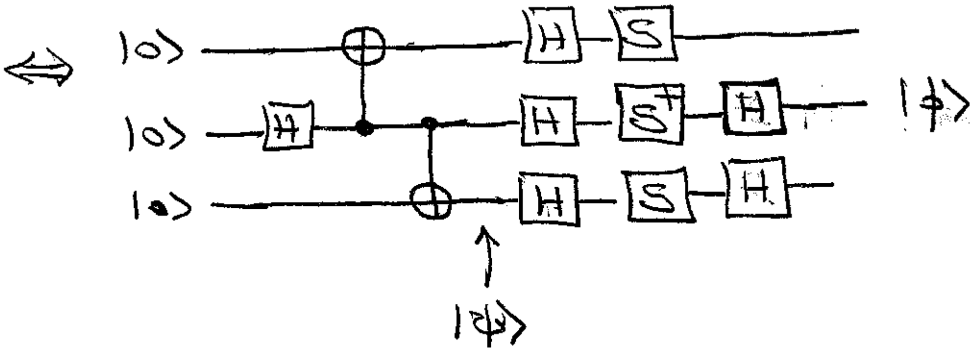
$$V = \begin{pmatrix} 1 & 0 \\ 0 & \underset{1}{\underset{\shortparallel}{e^{i\gamma}}} \end{pmatrix} W^* = H^* S^T H^* = HSH, \quad V^\dagger = HS^\dagger H$$

$$U' = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{pmatrix} 1 & ie^{i\alpha} \\ e^{i\gamma} & -e^{i\beta} \end{pmatrix} \begin{matrix} |00\rangle \\ |11\rangle \end{matrix} \\ \begin{matrix} |00\rangle & |11\rangle \end{matrix} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = SH$$

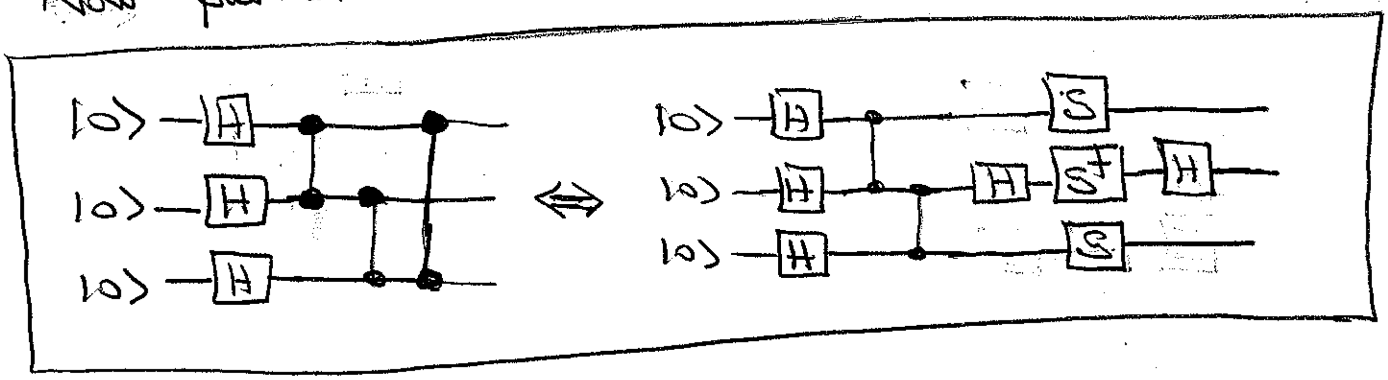
$$\boxed{|\phi\rangle = U \otimes V^\dagger \otimes W^\dagger |\psi\rangle = SH \otimes HS^\dagger H \otimes HSH |\psi\rangle}$$

Circuit diagram:





Now put an H at the end of the bottom wire:



But this is a slightly weaker version of the Eastin identity proved on pp. A-B of the lectures on cluster-state quantum computation. Indeed, the result here is the reason I asked Bryan to show the identity directly. If one uses the Eastin identity, one gets the equivalence of (iii) and (iv) immediately, without bothering with all the work on pp. 4-7.