4.1.

$$M_1 = \sum_\alpha A_\alpha |e_1\rangle\langle e_1| A_\alpha^\dagger \qquad tr(M_1) = \sum_\alpha \underbrace{\langle e_1 | A_\alpha^\dagger A_\alpha | e_1\rangle}_{\mu^2 m_{\alpha\alpha}} = \mu^2$$

M_1 is a positive operator, so can be diagonalized,

$$M_1 = \sum_\beta \lambda_\beta |\tilde{e}_\beta\rangle\langle\tilde{e}_\beta|. \qquad M_1 \geq 0 \Rightarrow \lambda_\beta > 0 \qquad tr(M_1) = \mu^2 = \sum_\beta \lambda_\beta \Rightarrow \lambda_\beta = \mu d_\beta,\ \sum_\beta d_\beta = 1$$

eigenvalues (λ_β), orthonormal eigenvectors ($|\tilde{e}_\beta\rangle$)

Define the action of the canonical operators $\tilde{A}_\beta$ on $|e_1\rangle$ by

$$\tilde{A}_\beta |e_1\rangle \equiv \sqrt{\lambda_\beta}\,|\tilde{e}_\beta\rangle = \mu\sqrt{d_\beta}\,|\tilde{e}_\beta\rangle$$

Because all the row subspaces have the same pairwise inner products, we can define unitary operators that map the 1st row to each of the other rows, i.e.,

$$U_j A_\alpha |e_1\rangle = A_\alpha |e_j\rangle \longleftarrow \text{preserves inner products} \quad \mu^2 m_{\alpha\beta} = \langle e_j | A_\beta^\dagger A_\alpha | e_j\rangle = \langle e_1 | A_\beta^\dagger A_\alpha | e_1\rangle$$

Now extend the action of the canonical operators to $|e_j\rangle$ by

$$\tilde{A}_\beta |e_j\rangle = U_j \tilde{A}_\beta |e_1\rangle = \mu\sqrt{d_\beta}\, U_j |\tilde{e}_\beta\rangle.$$

The canonical operators satisfy

$$\langle e_j | \tilde{A}_\beta^\dagger \tilde{A}_\alpha | e_k\rangle = \mu^2\sqrt{d_\alpha d_\beta}\,\underbrace{\langle\tilde{e}_\beta | U_j^\dagger U_k | \tilde{e}_\alpha\rangle}_{} = \mu^2 d_\alpha \delta_{\alpha\beta}\delta_{jk}$$

$$= \delta_{jk}\langle\tilde{e}_\beta|\tilde{e}_\alpha\rangle$$

because the different U_j's map to $\perp$ subspaces

$$= \delta_{jk}\delta_{\alpha\beta}$$

because the eigenvectors are orthonormal

We need to show that the $\tilde{A}_\alpha$ are Kraus operators for Q.

$$M_1 = \sum_\alpha A_\alpha |e_1\rangle\langle e_1| A_\alpha^\dagger = \sum_\beta \lambda_\beta |\tilde{e}_\beta\rangle\langle \tilde{e}_\beta|$$

Two decompositions of the positive operator M_1, so by the HJW theorem, there is a unitary matrix s.t.

$$A_\alpha |e_1\rangle = \sum_\beta u_{\alpha\beta} \sqrt{\lambda_\beta}\, |\tilde{e}_\beta\rangle$$

(add zero vectors to the set $\sqrt{\lambda_\beta}|\tilde{e}_\beta\rangle$ to make $u_{\alpha\beta}$ square)

$$\sim \sqrt{\lambda_\beta}\, \langle e_1| A_\alpha^\dagger |\tilde{e}_\beta\rangle$$

you can easily show this is unitary, but that's precisely what HJW has already done.

$$\Rightarrow \quad A_\alpha |e_j\rangle = U_j A_\alpha |e_1\rangle$$

$$= \sum_\beta u_{\alpha\beta} \underbrace{\sqrt{\lambda_\beta}\, U_j |\tilde{e}_\beta\rangle}_{\tilde{A}_\beta |e_j\rangle}$$

$$= \Big(\sum_\beta u_{\alpha\beta} \tilde{A}_\beta \Big) |e_j\rangle$$

The difference between what's done here and in the notes is that here we use HJW on a positive operator whereas the notes use the comparable property for a CP map.

So $A_\alpha = \sum_\beta u_{\alpha\beta} \tilde{A}_\beta$ within $\mathcal{C}$. We can extend this relation outside $\mathcal{C}$, because the extension doesn't matter, and this gives

$$Q = \sum_\alpha A_\alpha \odot A_\alpha^\dagger = \sum_\beta \tilde{A}_\beta \odot \tilde{A}_\beta^\dagger .$$